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International Mathematics for the Middle Years 5

PEARSON

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The publisher's policy is to use **paper manufactured from sustainable forests**

Contents

Interactive Student CD

You can access this material by clicking on the links provided on the Interactive Student CD. Go to the Home Page for information about these links.

Student Coursebook

Appendixes

Foundation Worksheets

Technology Applications

The material below is found in the Companion Website which is included on the Interactive Student CD as both an archived version and a fully featured live version.

Activities and Investigations

Drag and Drops

Animations

Chapter 10: The box, Greatest volume
Chapter 11: Scale it
Chapter 12: Unit circle
Chapter 13: Spin graphs

Chapter Review Questions

These can be used as a diagnostic tool or for revision. They include multiple choice, pattern-matching and fill-in-the-gaps style questions.

Destinations

Links to useful websites that relate directly to the chapter content.

Features of *International Mathematics for the Middle Years*

International Mathematics for the Middle Years is organised with the international student in mind. Examples and exercises are not restricted to a particular syllabus and so provide students with a global perspective.

Each edition has a review section for students who may have gaps in the Mathematics they have studied previously. Sections on the language of Mathematics and terminology will help students for whom English is a second language.

Areas of Interaction are given for each chapter and Assessment Grids for Investigations provide teachers with aids to assessing Analysis and Reasoning, Communication, and Reflection and Evaluation as part of the International Baccalaureate Middle Years Program (IBMYP). The Assessment Grids will also assist students in addressing these criteria and will enhance students' understanding of the subject content.

How is *International Mathematics for the Middle Years* organised?

As well as the student coursebook, additional support for both students and teachers is provided:

- Interactive Student CD — **free** with each coursebook
- Companion Website
- Teacher's Resource — printout and CD.

Coursebook

Chapter-opening pages summarise the key content and present the learning outcomes addressed in each chapter.

Areas of Interaction references are included in the chapter-opening pages to make reporting easier. For example, Homo faber.

prep quiz

Prep Quizzes review skills needed to complete a topic. These anticipate problems and save time in the long run. These quizzes offer an excellent way to start a lesson.

Well-graded exercises — Within each exercise, **levels of difficulty** are indicated by the colour of the question number.

1 green . . . foundation   4 blue . . . core   9 red . . . extension

2 a An equilateral triangle has a side of length 4.68 m. What is its perimeter?

7 Solve the following equations.

a $\frac{x}{2} + \frac{x}{3} = 5$ b $\frac{p}{6} + \frac{p}{2} = 8$

8 a A radio on sale for \$50 is to be reduced in price by 30%. Later, the discounted price is increased by 30%. What is the final price? By what percentage (to the nearest per cent) must the first discounted price be increased to give the original price?

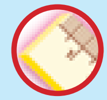
Worked examples are used extensively and are easy for students to identify.

worked examples

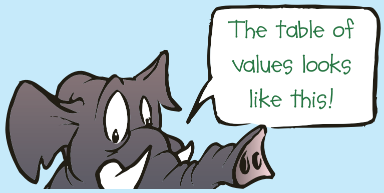
1 Express the following in scientific notation.

a 243   b 60 000   c 93 800 000

Important rules and concepts are clearly highlighted at regular intervals throughout the text.



Cartoons are used to give students friendly advice or tips.



Foundation Worksheets provide alternative exercises for students who need to consolidate earlier work or who need additional work at an easier level. Students can access these on the CD by clicking on the Foundation Worksheet icons. These can also be copied from the Teacher's Resource CD or from the Teacher's Resource Centre on the Companion Website.

Foundation Worksheet 4:01A

Grouping symbols

1 a $(3 + 2) \times 10$
2 a $(8 - 2) \times 3$
3 a $10 - (4 + 3)$

Challenge activities and worksheets provide more difficult investigations and exercises. They can be used to extend more able students.

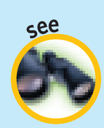
Fun Spots provide amusement and interest, while often reinforcing course work. They encourage creativity and divergent thinking, and show that Mathematics is enjoyable.

Investigations and **Practical Activities** encourage students to seek knowledge and develop research skills. They are an essential part of any Mathematics course. Where applicable, investigations are accompanied by a set of assessment criteria to assist teachers in assessing criteria B, C and D as prescribed by the MYP.

Diagnostic Tests at the end of each chapter test students' achievement of outcomes. More importantly, they indicate the weaknesses that need to be addressed by going back to the section in the text or on the CD listed beside the test question.

Assignments are provided at the end of each chapter. Where there are two assignments, the first revises the content of the chapter, while the second concentrates on developing the student's ability to work mathematically.

The **See** cross-references direct students to other sections of the coursebook relevant to a particular section.



The **Algebra Card** (see p xx) is used to practise basic algebra skills. Corresponding terms in columns can be added, subtracted, multiplied or divided by each other or by other numbers. This is a great way to start a lesson.

The Language of Mathematics

Within the coursebook, Mathematics literacy is addressed in three specific ways:

ID Cards (see pp xiv–xix) review the language of Mathematics by asking students to identify common terms, shapes and symbols. They should be used as often as possible, either at the beginning of a lesson or as part of a test or examination.

Mathematical Terms met during the chapter are defined at the end of each chapter. These terms are also tested in a **Drag and Drop** interactive that follows this section.

Reading Mathematics help students to develop maths literacy skills and provide opportunities for students to communicate mathematical ideas. They present Mathematics in the context of everyday experiences.

An **Answers** section provides answers to all the exercises in the coursebook, including the ID Cards.

Interactive Student CD

This is provided at the back of the coursebook and is an important part of the total learning package.

Bookmarks and links allow easy navigation within and between the different electronic components of the CD that contains:

- A copy of the student coursebook.
- Appendixes A–D for enrichment and review work, linked from the coursebook.
- Printable copies of the Foundation Worksheets and Challenge Worksheets, linked from the coursebook.
- An archived, offline version of the Companion Website, including:
  - Chapter Review Questions and Quick Quizzes
  - All the Technology Applications: activities and investigations and drag-and-drops
  - Destinations (links to useful websites)

All these items are clearly linked from the coursebook via the Companion Website.

- A link to the live Companion Website.

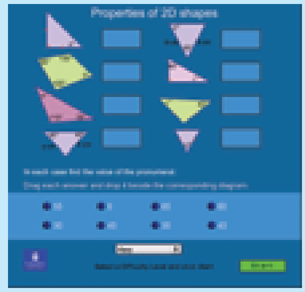
Companion Website

The Companion Website contains a wealth of support material for students and teachers:

- **Chapter Review Questions** which can be used as a diagnostic tool or for revision. These are self-correcting and include multiple-choice, pattern-matching and fill-in the-gaps-style questions. Results can be emailed directly to the teacher or parents.
- **Quick Quizzes** for most chapters.
- **Destinations** — links to useful websites which relate directly to the chapter content.
- **Technology Applications** — activities that apply concepts covered in most chapters and are designed for students to work independently:

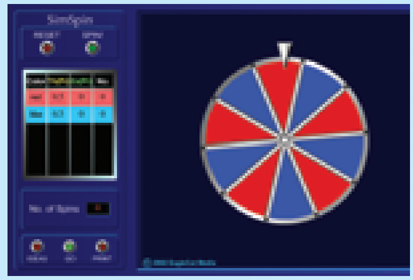
Activities and investigations using technology, such as Excel spreadsheets and The Geometer's Sketchpad.

Drag and Drop interactives to improve mastery of basic skills.



Sample Drag and Drop

Animations to develop key skills by manipulating visually stimulating and interactive demonstrations of key mathematical concepts.



Sample Animation

- **Teacher's Resource Centre** — provides a wealth of teacher support material and is password protected:
  - — Coursebook corrections
  - — Topic Review Tests and answers
  - — Foundation and Challenge Worksheets and answers

Teacher's resource

This material is provided as both a printout and as an electronic copy on CD:

- Electronic copy of the complete Student Coursebook in PDF format
- Teaching Program, including treatment of learning outcomes, in both PDF and editable Microsoft Word formats
- Practice Tests and Answers
- Foundation and Challenge Worksheets and answers
- Answers to some of the Technology Application Activities and Investigations

Most of this material is also available in the Teacher's Resource Centre of the Companion Website.

Using this Book for Teaching MYP for the IB

- **Holistic Learning**
- **Intercultural Awareness**
- **Communication**

These elements of the MYP Mathematics course are integrated throughout the text. Links are made possible between subjects, and different methods of communicating solutions to problems through investigations allow students to explore their own ideas.

The Areas of Interaction

- **Approaches to Learning**
- **Community and Service**
- **Health and Social Education**
- **Environment**
- **Homo Faber**

Areas of Interaction covered are outlined at the start of each chapter, allowing teachers to develop links between subjects and formulate their own Interdisciplinary Units with additional assistance in the Teacher's Resource.

Addressing the Objectives

Assessment grids are provided for Investigations throughout the text to not only help teachers assess criteria B, C and D of the MYP, but also to assist students in addressing the criteria. The assessment grids should be modified to suit the student where necessary.

A Knowledge and Understanding
This criterion is addressed in the Diagnostic Tests and Revision Assignments that accompany each chapter. Teachers can also use the worksheets from the CD to add to material for this criterion.

B Investigating Patterns
It is possible to address this criterion using the Working Mathematically sections accompanying each chapter, and also using the Investigations throughout the text.

C Communication
This can be assessed using the Investigations throughout the book.

D Reflection in Mathematics
This can be assessed using the Investigations throughout the book.

Fulfilling the Framework for Mathematics

The content of the text covers the five broad areas required to fulfil the Framework:

- **Number**
- **Algebra**
- **Geometry**
- **Statistics**
- **Discrete Mathematics**

Although the material in the text is not exhaustive, it covers the required areas in sufficient depth. Teachers can use the text as a resource to build on as they develop their own scheme of work within their school.

Metric Equivalents

Length
1 m = 1000 mm
= 100 cm
= 10 dm
1 cm = 10 mm
1 km = 1000 m

Area
1 m^2 = 10 000 cm^2
1 ha = 10 000 m^2
1 km^2 = 100 ha

Mass
1 kg = 1000 g
1 t = 1000 kg
1 g = 1000 mg

Volume
1 m^3 = 1 000 000 cm^3
= 1000 dm^3
1 L = 1000 mL
1 kL = 1000 L
1 m^3 = 1 kL
1 cm^3 = 1 mL
1000 cm^3 = 1 L

Time
1 min = 60 s
1 h = 60 min
1 day = 24 h
1 year = 365 days
1 leap year = 366 days

Months of the year

30 days each has September,
April, June and November.
All the rest have 31, except February alone,
Which has 28 days clear and 29 each leap year.

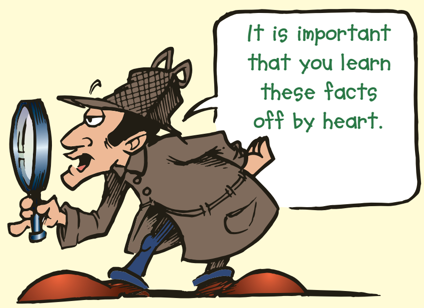
Seasons

Southern Hemisphere

Summer:	December, January, February
Autumn/Fall:	March, April, May
Winter:	June, July, August
Spring:	September, October, November

Northern Hemisphere

Summer:	June, July, August
Autumn/Fall:	September, October, November
Winter:	December, January, February
Spring:	March, April, May



The Language of Mathematics

You should regularly test your knowledge by identifying the items on each card.

ID Card 1 (Metric Units)

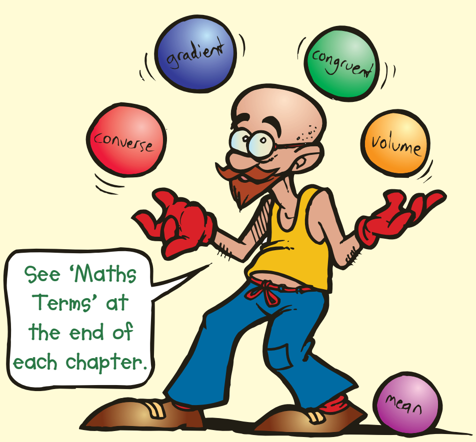
1 m	2 dm	3 cm	4 mm
5 km	6 m^2	7 cm^2	8 km^2
9 ha	10 m^3	11 cm^3	12 s
13 min	14 h	15 m/s	16 km/h
17 g	18 mg	19 kg	20 t
21 L	22 mL	23 kL	24 °C

See page 602 for answers.

ID Card 2 (Symbols)

1 $=$	2 $\doteqdot$ or $\approx$	3 $\neq$	4 $<$
5 $\leqslant$	6 $\nless$	7 $>$	8 $\geqslant$
9 4^2	10 4^3	11 $\sqrt{2}$	12 $\sqrt[3]{2}$
13 $\perp$	14 $\parallel$	15 $\equiv$	16 $\|\|\|$
17 %	18 $\therefore$	19 eg	20 ie
21 π	22 Σ	23 $\bar{x}$	24 P(E)

See page 602 for answers.



ID Card 3 (Language)			
1 6 minus 2	**2** the sum of 6 and 2	**3** divide 6 by 2	**4** subtract 2 from 6
5 the quotient of 6 and 2	**6** $\begin{array}{r}3\\2\overline{)6}\end{array}$ the divisor is	**7** $\begin{array}{r}3\\2\overline{)6}\end{array}$ the dividend is	**8** 6 lots of 2
9 decrease 6 by 2	**10** the product of 6 and 2	**11** 6 more than 2	**12** 2 less than 6
13 6 squared	**14** the square root of 36	**15** 6 take away 2	**16** multiply 6 by 2
17 average of 6 and 2	**18** add 6 and 2	**19** 6 to the power of 2	**20** 6 less 2
21 the difference between 6 and 2	**22** increase 6 by 2	**23** share 6 between 2	**24** the total of 6 and 2

See page 602 for answers.

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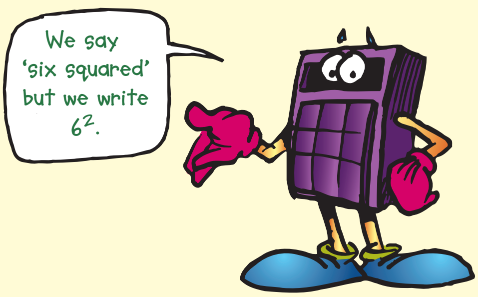
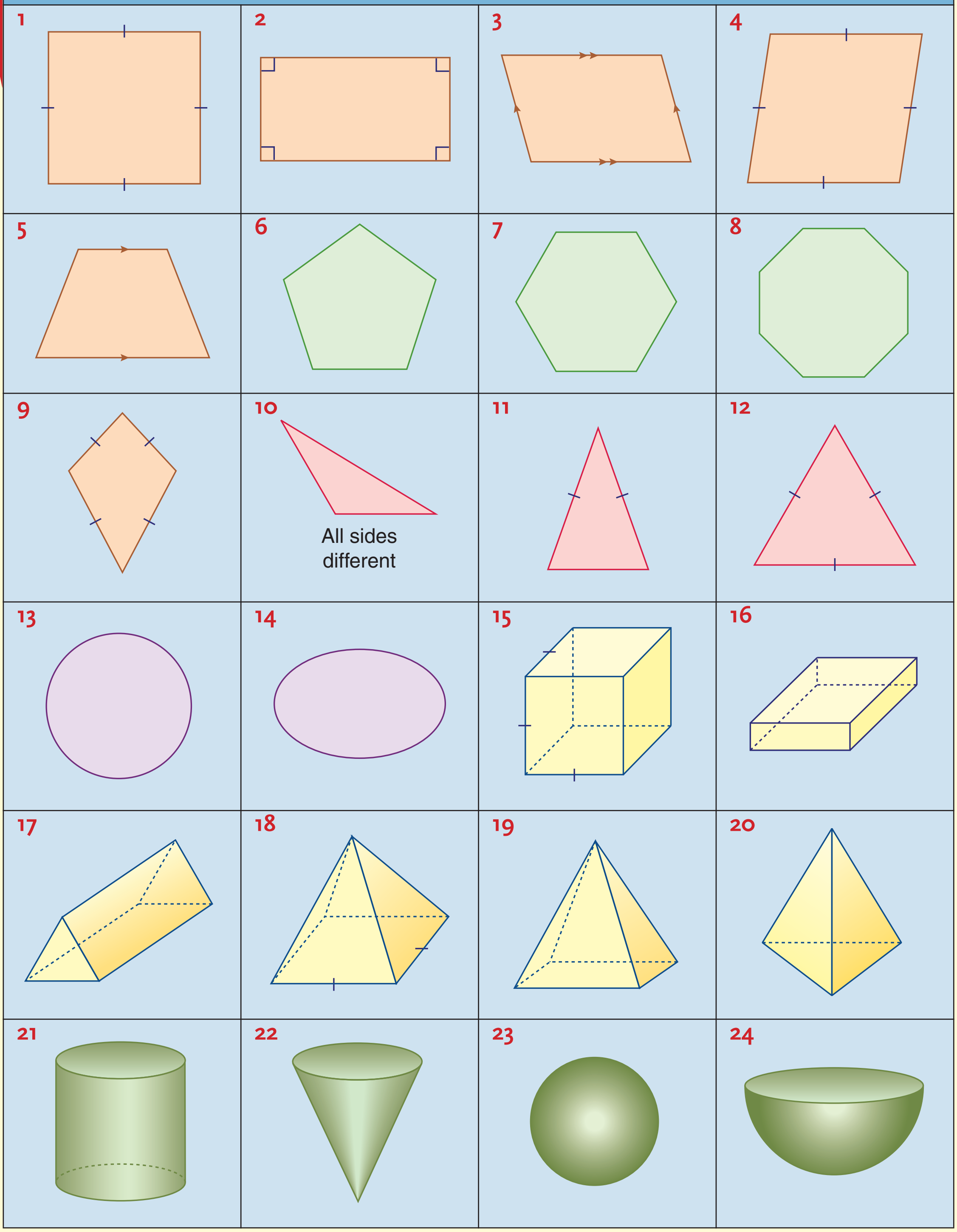
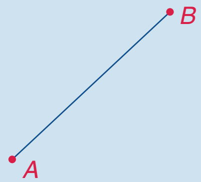
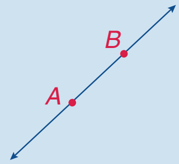
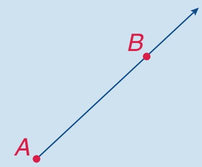
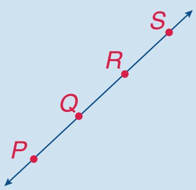
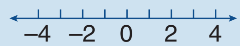
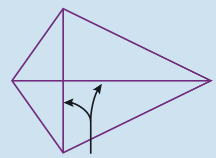
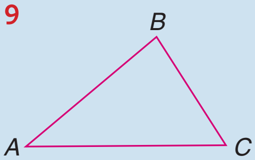
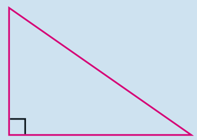
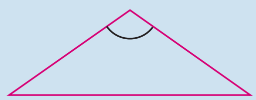
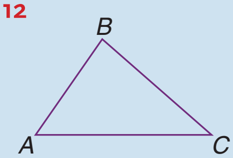
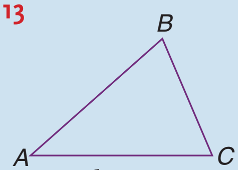
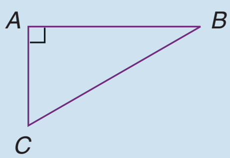
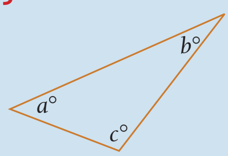
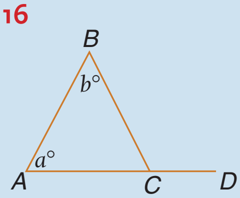
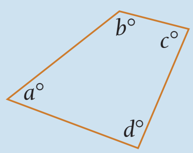
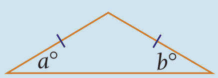
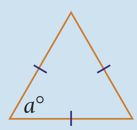
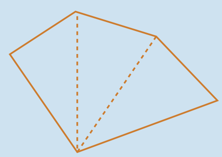
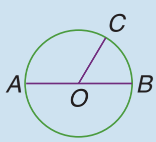
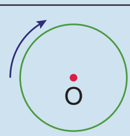
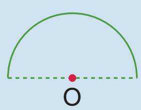
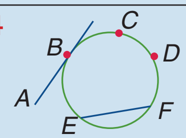


ID Card 4 (Language)



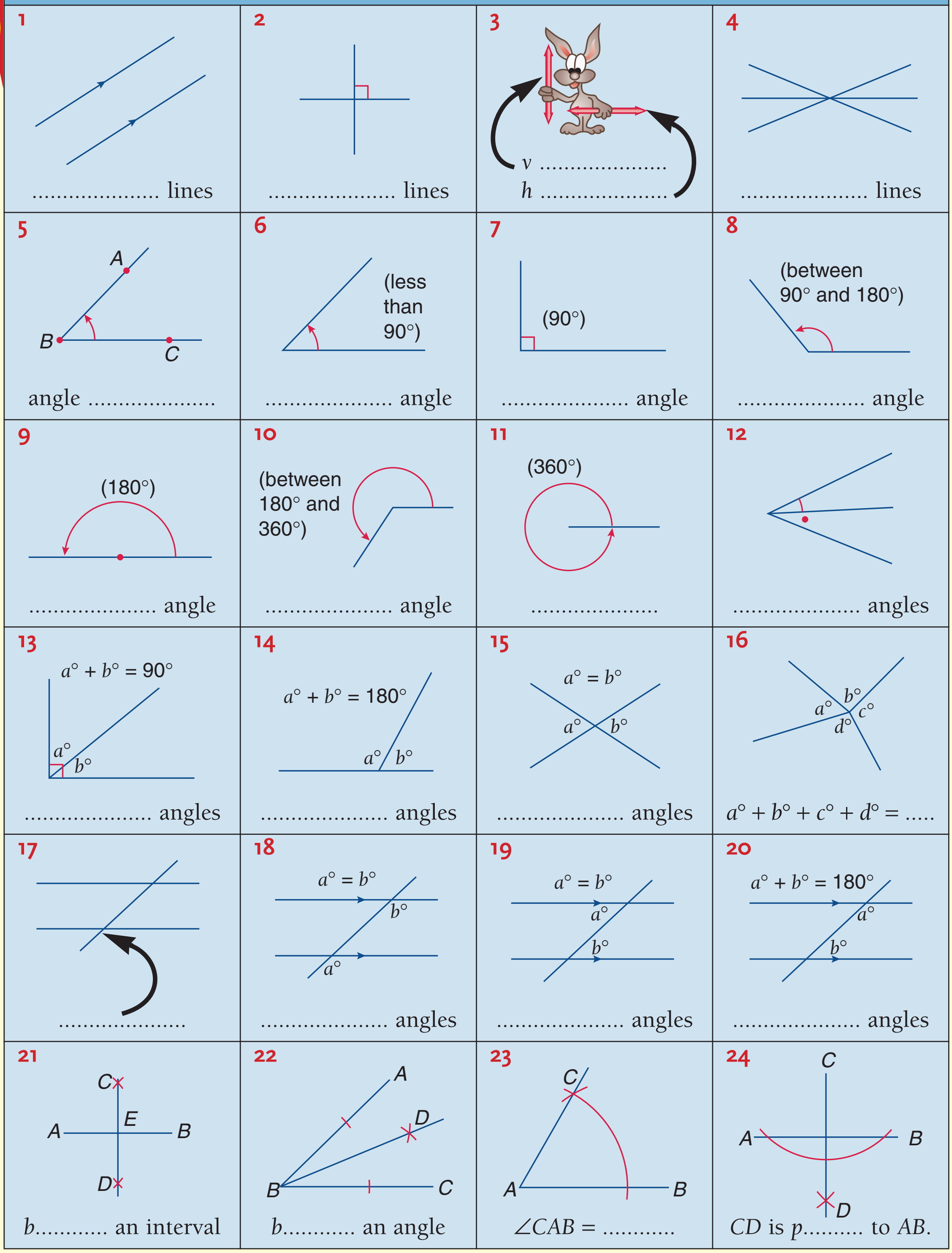
See page 602 for answers.

ID Card 5 (Language)

1 	**2** 	**3** 	**4**  ....
5  .... points	**6**  *C* is the	**7** 	**8**  ....
9  all angles less than 90°	**10**  one angle 90°	**11**  one angle greater than 90°	**12**  *A*, *B* and *C* are .... of the triangle.
13  Use the vertices to name the Δ.	**14**  *BC* is the .... of the right-angled Δ.	**15**  $a° + b° + c° =$	**16**  $\angle BCD =$ ....
17  $a° + b° + c° + d° =$	**18**  Which is true? (a) $a° < b°$ (b) $a° = b°$ (c) $a° > b°$	**19**  $a° =$	**20**  Angle sum = ...
21  *AB* is a .... *OC* is a	**22**  Name of distance around the circle.	**23** 	**24**  *AB* is a .... *CD* is an .... *EF* is a..............

See page 602 for answers.

ID Card 6 (Language)



See page 602 for answers.

ID Card 7 (Language)

1. **AD**

   a........... *D*...........

2. **BC**

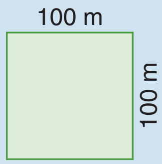
   b........... *C*...........

3. **am**

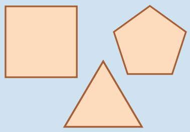
   a........... *M*...........

4. **pm**

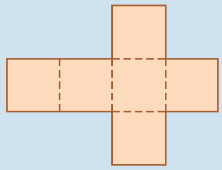
   p........... *m*...........

5. 

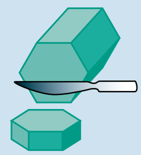
   area is 1 ....

6. 

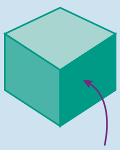
   r........... shapes

7. 

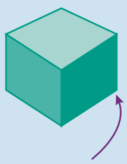
   .... of a cube

8. 

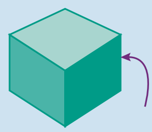
   c...........-*s*...........

9. 

   f...........

10. 

    v...........

11. 

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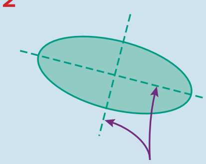
12. 

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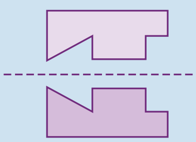
13. 

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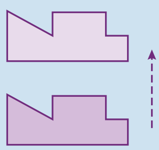
14. 

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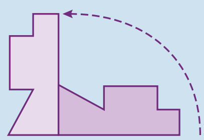
15. 

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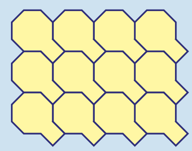
16. 

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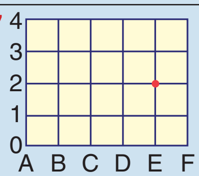
17. 

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    of the dot are E2.

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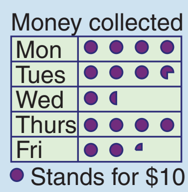
19. 

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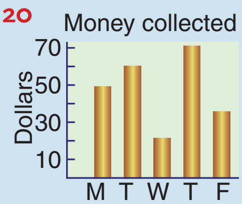
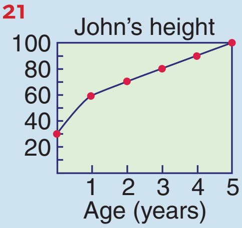
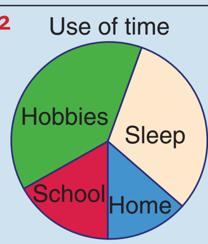
20. 

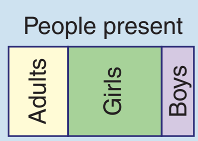
    c........... graph

21. 

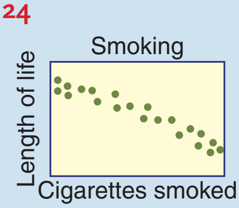
    l........... graph

22. 

    s........... graph

23. 

    b........... graph

24. 

    s........... *d*...........

See page 602 for answers.

Algebra Card

	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O
1	3	2·1	$\frac{1}{4}$	m	$\frac{2m}{3}$	$-3m$	$5m^2$	$-5x$	$\frac{x}{6}$	$-3x$	$-\frac{x}{2}$	$x + 2$	$x - 3$	$2x + 1$	$3x - 8$
2	−1	−0·4	$\frac{1}{8}$	$-4m$	$\frac{m}{4}$	$2m$	$-2m^3$	$3x$	$-\frac{x}{3}$	$5x^2$	$\frac{x}{4}$	$x + 7$	$x - 6$	$4x + 2$	$x - 1$
3	5	0·8	$\frac{1}{3}$	$10m$	$-\frac{m}{4}$	$-5m$	$8m^5$	$10x$	$-\frac{2x}{7}$	$-8x$	$\frac{2x}{5}$	$x + 5$	$x + 5$	$6x + 2$	$x - 5$
4	−2	1·5	$\frac{1}{20}$	$-8m$	$-\frac{3m}{2}$	$7m$	$6m^2$	$-15x$	$\frac{x}{10}$	$-4x^4$	$-\frac{x}{5}$	$x + 1$	$x - 9$	$3x + 3$	$2x + 4$
5	−8	−2·5	$\frac{3}{5}$	$2m$	$-\frac{m}{5}$	$10m$	m^2	$7x$	$\frac{2x}{3}$	$2x^3$	$\frac{x}{3}$	$x + 8$	$x + 2$	$3x + 8$	$3x + 1$
6	10	−0·7	$\frac{2}{7}$	$-5m$	$-\frac{3m}{7}$	$-6m$	$-9m^3$	$9x$	$-\frac{2x}{5}$	x^2	$\frac{3x}{5}$	$x + 4$	$x - 7$	$3x + 1$	$x + 7$
7	−6	−1·2	$\frac{3}{8}$	$8m$	$-\frac{m}{6}$	$9m$	$2m^6$	$-6x$	$\frac{5x}{6}$	$5x^2$	$\frac{2x}{3}$	$x + 6$	$x - 1$	$x + 8$	$2x - 5$
8	12	0·5	$\frac{9}{20}$	$20m$	$\frac{2m}{5}$	$-4m$	$-3m^3$	$-12x$	$\frac{3x}{4}$	$4x^3$	$-\frac{x}{7}$	$x + 10$	$x - 8$	$5x + 2$	$x - 10$
9	7	0·1	$\frac{3}{4}$	$5m$	$\frac{3m}{5}$	$-10m$	m^7	$5x$	$-\frac{3x}{7}$	$-3x^5$	$-\frac{3x}{7}$	$x + 2$	$x + 5$	$2x + 4$	$2x - 4$
10	−5	−0·6	$\frac{7}{10}$	$-9m$	$-\frac{4m}{5}$	$-7m$	$-8m^4$	$-3x$	$-\frac{x}{6}$	$-7x^5$	$\frac{2x}{9}$	$x + 1$	$x - 7$	$5x + 4$	$x + 7$
11	−11	−1·8	$\frac{1}{10}$	$-7m$	$\frac{m}{5}$	$-8m$	$-4m$	$-4x$	$\frac{x}{5}$	$-x^3$	$\frac{x}{3}$	$x + 9$	$x + 6$	$2x + 7$	$x - 6$
12	4	−1·4	$\frac{2}{5}$	$3m$	$\frac{m}{3}$	$12m$	$7m^2$	$-7x$	$-\frac{3x}{4}$	x^{10}	$\frac{x}{6}$	$x + 3$	$x - 10$	$2x + 3$	$2x + 3$

How to use this card

If the instruction is 'column D + column F', then you add corresponding terms in columns D and F.

eg

1 $m + (-3m)$ **2** $(-4m) + 2m$ **3** $10m + (-5m)$

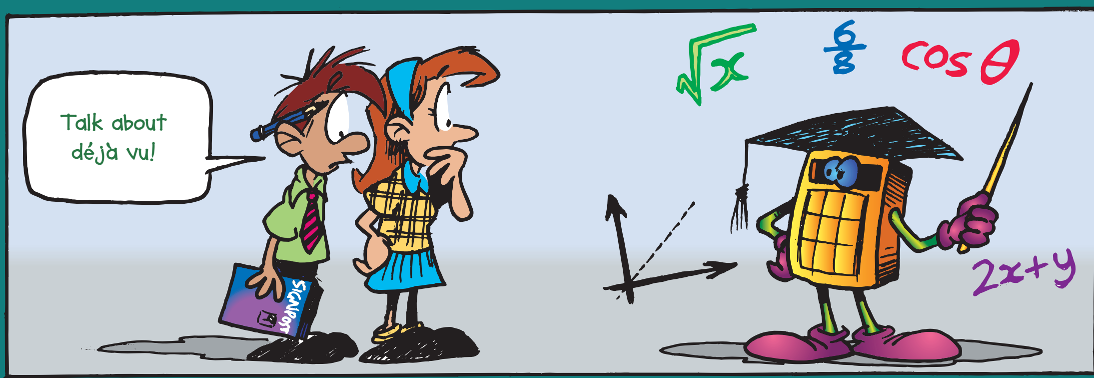
4 $(-8m) + 7m$ **5** $2m + 10m$ **6** $(-5m) + (-6m)$

7 $8m + 9m$ **8** $20m + (-4m)$ **9** $5m + (-10m)$

10 $(-9m) + (-7m)$ **11** $(-7m) + (-8m)$ **12** $3m + 12m$

1

Basic Skills and Number — Review of Books 1 to 4



Chapter Contents

Learning Outcomes

Students will be able to:

- Compare, order and calculate with integers.
- Operate with fractions, decimals, percentages, ratios and rates.
- Manipulate algebraic expressions.
- Solve simple probability problems.
- Operate with indices and surds.
- Use and substitute into formulae.
- Solve money problems.
- Use the Cartesian plane to solve coordinate geometry problems.
- Perform simple data analysis.
- Solve problems using equations.
- Solve problems using basic trigonometry.
- Interpret graphs.
- Round to a specified number of significant figures, express recurring decimals as fractions and convert rates between units.
- Identify special angles and make use of the relationships between them.
- Classify, construct and determine the properties of triangles and quadrilaterals.

Areas of Interaction

Approaches to Learning (Knowledge Acquisition, Problem Solving, Logical Thinking, Reflection)

This chapter is a summary of the work covered in *New Signpost Mathematics 9, Stage 5.1–5.3*. For an explanation of the work, refer to the cross-reference on the right-hand side of the page which will direct you to the Appendix on the Interactive Student CD.

1:01 | Basic Number Skills

Rational numbers: Integers, fractions, decimals and percentages (both positive and negative) are rational numbers. They can all be written as a terminating or recurring decimal. The following exercises will remind you of the skills you should have mastered.

A | Order of operations

Exercise 1:01A

CD Appendix

Answer these questions without using a calculator.

1 (A:01A)
- a $4 - (5 - 3)$ b $6 - (9 - 4)$ c $-4 + (3 + 1)$
- d $6 + 4 \times 2$ e $9 - 3 \times 4$ f $16 + 4 \div 4$
- g $10 \times 4 - 4 \times 7$ h $30 \div 3 + 40 \div 2$ i $5 \times 8 + 6 \times 5$
- j 5×2^2 k 3×10^2 l $3^2 + 4^2$
- m $6 + 3 \times 4 + 1$ n $8 + 4 \div 2 + 1$ o $6 - (-6 - 6)$

2 (A:01A)
- a $6 \times (5 - 4) + 3$ b $27 \div (3 + 6) - 3$ c $16 - [10 - (6 - 2)]$
- d $\dfrac{30 + 10}{30 - 10}$ e $\dfrac{15 + 45}{45 + 5}$ f $\dfrac{14}{14 - 7}$
- g $(6 + 3)^2$ h $(10 + 4)^2$ i $(19 - 9)^2$

B | Fractions

Exercise 1:01B

CD Appendix

1 Change to mixed numerals. (A:01B$_1$)
- a $\frac{7}{4}$ b $\frac{49}{6}$ c $\frac{15}{4}$ d $\frac{11}{8}$

2 Change to improper fractions. (A:01B$_2$)
- a $5\frac{1}{2}$ b $3\frac{1}{7}$ c $8\frac{3}{4}$ d $66\frac{2}{3}$

3 Simplify the fractions. (A:01B$_3$)
- a $\frac{48}{80}$ b $\frac{70}{150}$ c $\frac{200}{300}$ d $\frac{250}{450}$

4 Complete the following equivalent fractions. (A:01B$_4$)
- a $\frac{3}{4} = \frac{\square}{24}$ b $\frac{2}{5} = \frac{\square}{50}$ c $\frac{2}{7} = \frac{\square}{28}$ d $\frac{1}{3} = \frac{\square}{120}$

5 (A:01B$_5$)
- a $\frac{7}{15} + \frac{1}{15}$ b $\frac{13}{20} - \frac{2}{5}$ c $\frac{5}{8} + \frac{3}{10}$ d $\frac{6}{7} - \frac{3}{5}$

6 (A:01B$_6$)
- a $6\frac{1}{2} + 2\frac{3}{5}$ b $4\frac{3}{4} - 2\frac{3}{10}$ c $4\frac{3}{4} + 6\frac{1}{10}$ d $5\frac{3}{8} - 1\frac{9}{10}$

7 (A:01B$_7$)
- a $\frac{3}{5} \times \frac{4}{7}$ b $\frac{18}{25} \times \frac{15}{16}$ c $\frac{4}{9} \times \frac{3}{10}$ d $\frac{7}{10}$ of $\frac{2}{3}$

8 (A:01B$_8$)
- a $6 \times \frac{3}{4}$ b $2\frac{1}{2} \times 1\frac{4}{5}$ c $1\frac{1}{3} \times 15$ d $10\frac{1}{2} \times 1\frac{3}{7}$

9 (A:01B$_9$)
- a $\frac{9}{10} \div \frac{2}{3}$ b $\frac{3}{8} \div \frac{3}{5}$ c $\frac{4}{5} \div 6$ d $2\frac{3}{4} \div 1\frac{1}{2}$

C | Decimals

Exercise 1:01C

CD Appendix

1 Put in order, smallest to largest. (A:01C$_1$)
- **a** {0·606, 0·6, 0·66, 0·066}
- **b** {1·53, 0·153, 1·053}
- **c** {0·7, 0·017, 7, 0·77}
- **d** {3·5, 3·45, 3·05, 3·4}

2 Do not use your calculator to do these. (A:01C$_2$)
- **a** 7·301 + 2
- **b** 3·05 + 0·4
- **c** 0·004 + 3·1
- **d** 6 + 0·3 + 0·02
- **e** 8·67 − 6·7
- **f** 9·12 − 1·015
- **g** 8 − 3·112
- **h** 162·3 − 3

3 **a** $0{\cdot}012 \times 3$ **b** $0{\cdot}03 \times 0{\cdot}2$ **c** $0{\cdot}45 \times 1{\cdot}3$ **d** $(0{\cdot}05)^2$ (A:01C$_3$)

4 **a** $3{\cdot}14 \times 10$ **b** $0{\cdot}5 \times 1000$ **c** $0{\cdot}0003 \times 100$ **d** $3{\cdot}8 \times 10^4$ (A:01C$_4$)

5 **a** $0{\cdot}15 \div 5$ **b** $1{\cdot}06 \div 4$ **c** $15{\cdot}35 \div 5$ **d** $0{\cdot}01 \div 4$ (A:01C$_5$)

6 **a** $1{\cdot}3 \div 3$ **b** $9{\cdot}1 \div 11$ **c** $14 \div 9$ **d** $6 \div 7$ (A:01C$_6$)

7 **a** $48{\cdot}04 \div 10$ **b** $1{\cdot}6 \div 100$ **c** $0{\cdot}9 \div 1000$ **d** $6{\cdot}5 \div 10^4$ (A:01C$_7$)

8 **a** $8{\cdot}4 \div 0{\cdot}4$ **b** $0{\cdot}836 \div 0{\cdot}08$ **c** $7{\cdot}5 \div 0{\cdot}005$ **d** $1{\cdot}4 \div 0{\cdot}5$ (A:01C$_8$)

9 Express as a simplified fraction or mixed numeral. (A:01C$_9$)
- **a** 3·017 **b** 0·04 **c** 0·86 **d** 16·005

10 Express as a decimal. (A:01C$_{10}$)
- **a** $\frac{4}{5}$ **b** $\frac{7}{200}$ **c** $\frac{5}{8}$ **d** $\frac{8}{11}$

11 Express these recurring decimals as fractions. (A:01C$_{11}$)
- **a** 0·5555… **b** 0·257 257 2… **c** $0{\cdot}\dot{7}\dot{2}$ **d** $0{\cdot}\dot{6}4\dot{2}$

12 Express these recurring decimals as fractions. (A:01C$_{11}$)
- **a** 0·8333… **b** 0·915 151 5… **c** $0{\cdot}43\dot{5}$ **d** $0{\cdot}8\dot{9}4\dot{2}$

D | Percentages

Exercise 1:01D

CD Appendix

1 Express as a fraction. (A:01D$_1$)
- **a** 54% **b** 203% **c** $12\frac{1}{4}\%$ **d** 9·1%

2 Express as a percentage. (A:01D$_2$)
- **a** $\frac{11}{20}$ **b** $\frac{4}{9}$ **c** $1\frac{1}{4}$ **d** $\frac{2}{3}$

3 Express as a decimal. (A:01D$_3$)
- **a** 16% **b** 8·6% **c** 3% **d** $18\frac{1}{4}\%$

4 Express as a percentage. (A:01D$_4$)
- **a** 0·47 **b** 0·06 **c** 0·375 **d** 1·3

5 (A:01D$_5$)
- **a** 36% of 400 m **b** 9% of 84 g **c** $8\frac{1}{2}\%$ of \$32
- **d** At the local Anglican church, the offertories for 2005 amounted to \$127 000. If 68% of this money was used to pay the salary of the two full-time ministers, how much was paid to the ministers?

6 a 9% of Luke's money was spent on fares. If \$5.40 was spent on fares, how much money did Luke have?
b 70% of Alana's weight is 17·5 kg. How much does Alana weigh?
c Lyn bought a book for a reduced price of 70 cents. This was 14% of the book's recommended retail price. What was the recommended retail price?
d 54 minutes of mathematics lesson time was lost in one week because of other activities. If this represents 30% of the allocated weekly time for mathematics, what is this allocated time?

A:01D$_6$

7 a Express 85 cents as a percentage of \$2.
b 4 kg of sugar, 9 kg of flour and 7 kg of mixed fruit were mixed. What is the percentage (by weight) of flour in the mixture?
c Of 32 birds in Rachel's aviary, 6 are canaries. What percentage of her birds are canaries?
d When Steve Waugh retired from test cricket in 2003, he had scored 32 centuries from 260 innings. In what percentage of his innings did he score centuries?

A:01D$_7$

E | Ratio

Exercise 1:01E

CD Appendix

1 a Simplify each ratio.
i \$15 : \$25 ii 9 kg : 90 kg iii 75 m : 35 m iv 120 m^2 : 40 m^2
b Find the ratio in simplest terms of 5·6 m to 40 cm.
c Naomi spends \$8 of \$20 she was given by her grandparents and saves the rest. What is the ratio of money spent to money saved?
d Three-quarters of the class walk to school while $\frac{1}{5}$ ride bicycles. Find the ratio of those who walk to those who ride bicycles.
e At the end of their test cricket careers, Steve Waugh had scored 50 fifties and 32 hundreds from 260 innings, while Mark Waugh had scored 47 fifties and 20 hundreds from 209 innings.
i Find the ratio of the number of hundreds scored by Steve to the number scored by Mark.
ii Find the ratio of the number of times Steve scored 50 or more to the number of innings.
f Express each ratio in the form $X : 1$.
i 3 : 5 ii 2 : 7 iii 10 : 3 iv 25 : 4
g Express each ratio in **f** in the form $1 : Y$.

A:01E$_1$

2 a If x : 15 = 10 : 3, find the value of x.
b If the ratio of the populations of Africa and Europe is 5 : 4, find the population of Africa if Europe's population is 728 million.
c The ratio of the average population density per km^2 of Asia to that of Australia is 60 : 1. If the average in Asia is 152 people per km^2, what is the average in Australia?
d The ratio of the population of Sydney to the population of Melbourne is 7 : 6. If 4.2 million people live in Sydney, how many people live in Melbourne?

A:01E$_2$

3 a If 84 jellybeans are divided between Naomi and Luke in the ratio 4 : 3, how many jellybeans does each receive? A:01E_3

b The sizes of the angles of a triangle are in the ratio 2 : 3 : 4. Find the size of each angle.

c A total of 22 million people live in the cities of Tokyo and Moscow. If the ratio of the populations of Tokyo and Moscow is 6 : 5, what is the population of each city?

d At Christ Church, Cobargo, in 1914, there were 60 baptisms. The ratio of males to females who were baptised was 3 : 2. How many of each were baptised?

F | Rates

Exercise 1:01F

CD Appendix

1 Complete these equivalent rates. A:01F

a 5 km/min = . . . km/h
b 8 km/L = . . . m/mL
c 600 kg/h = . . . t/day
d 2·075 cm^3/g = . . . cm^3/kg

2 a At Cobargo in 1915, the Rector, H. E. Hyde, travelled 3396 miles by horse and trap. Find his average speed (to the nearest mile per hour) if it took a total of 564 hours to cover the distance. A:01F

b Over a period of 30 working days, Adam earned $1386. Find his average daily rate of pay.

c Sharon marked 90 books in 7 hours. What rate is this in minutes per book?

d On a hot day, our family used an average of 36 L of water per hour. Change this rate to cm^3 per second (cm^3/s).

G | Significant figures

Exercise 1:01G

CD Appendix

1 State the number of significant figures in each of the following. A:01G

a 21
b 4·6
c 2·52
d 0·616
e 16·32
f 106
g 3004
h 2·03
i 1·06
j 50·04
k 0·5
l 0·003
m 0·000 32
n 0·06
o 0·006
p 3·0
q 25·0
r 2·60
s 13·000
t 6·40
u 41 235
v 600 (to nearest hundred)
w 482 000 (to nearest thousand)
x 700 (to nearest ten)
y 1600
z 16 000

2 State the number of significant figures in each of the following. A:01G

a 3·0
b 3·00
c 0·3
d 0·03
e 0.030
f 0.0030
g 0.0300
h 3.0300

H | Approximations

Exercise 1:01H

CD Appendix

1 Approximate each of the following correct to one decimal place. A:01H

a 4·63	**b** 0·81	**c** 3·17	**d** 0·062
e 15·176	**f** 8·099	**g** 0·99	**h** 121·62
i 0·119	**j** 47·417	**k** 0·35	**l** 2·75

2 Approximate each of the following correct to two decimal places. A:01H

a 0·537	**b** 2·613	**c** 7·134	**d** 1·169
e 12·0163	**f** 8·399	**g** 412·678	**h** 0·0756
i 0·4367	**j** 100·333	**k** 0·015	**l** 0·005

3 Approximate each number correct to: **i** 1 sig. fig. **ii** 2 sig. figs. A:01H

a 7·31	**b** 84·9	**c** 0·63	**d** 2·58
e 4·16	**f** 0·0073	**g** 0·0828	**h** 3·05
i 0·009 34	**j** 0·0098	**k** 7·52	**l** 0·0359

4 Approximate each of the following numbers correct to the number of significant figures indicated. A:01H

a 2·3 (1 sig. fig.)	**b** 14·63 (3 sig. figs.)	**c** 2·15 (2 sig. figs.)
d 0·93 (1 sig. fig.)	**e** 4·07 (2 sig. figs.)	**f** 7·368 94 (3 sig. figs.)
g 0·724 138 (3 sig. figs.)	**h** 5·716 (1 sig. fig.)	**i** 31·685 (4 sig. figs.)
j 0·007 16 (1 sig. fig.)	**k** 0·78 (1 sig. fig.)	**l** 0·007 16 (2 sig. figs.)

5 Approximate each of the following numbers correct to the number of decimal places indicated. A:01H

a 5·61 (1 dec. pl.)	**b** 0·16 (1 dec. pl.)	**c** 0·437 (2 dec. pl.)
d 15·37 (1 dec. pl.)	**e** 8·333 (2 dec. pl.)	**f** 413·789 (1 dec. pl.)
g 71·98 (1 dec. pl.)	**h** 3·0672 (3 dec. pl.)	**i** 9·99 (1 dec. pl.)
j 4·7998 (3 dec. pl.)	**k** 0·075 (2 dec. pl.)	**l** 0·0035 (3 dec. pl.)

I | Estimation

Exercise 1:01I

CD Appendix

1 Give estimates for each of the following. A:01I

a $12{\cdot}7 \times 5{\cdot}8$ **b** $0{\cdot}55 \times 210$ **c** $17{\cdot}8 \times 5{\cdot}1 \times 0{\cdot}336$

d $15{\cdot}6 \div 2{\cdot}165$ **e** $(4{\cdot}62 + 21{\cdot}7) \times 4{\cdot}21$ **f** $7{\cdot}8 \times 5{\cdot}2 + 21{\cdot}7 \times 0{\cdot}89$

g $(0{\cdot}93 + 1{\cdot}72)(8{\cdot}5 - 1{\cdot}7)$ **h** $\dfrac{43{\cdot}7 + 18{\cdot}2}{7{\cdot}8 + 2{\cdot}9}$ **i** $\dfrac{101{\cdot}6 - 51{\cdot}7}{21{\cdot}3 - 14{\cdot}8}$

j $\dfrac{0{\cdot}68 \times 51}{0{\cdot}25 \times 78}$ **k** $\dfrac{11{\cdot}6 - 3{\cdot}92}{12{\cdot}7 + 6{\cdot}58}$ **l** $3{\cdot}52^2 \times \sqrt{17{\cdot}9}$

m $\sqrt{41{\cdot}7 \times 5{\cdot}6}$ **n** $\sqrt{4{\cdot}26} \times \sqrt{105{\cdot}6}$ **o** $3.1^3 \times 1{\cdot}8^4$

p $\dfrac{4{\cdot}1 \times \sqrt{48{\cdot}12}}{26{\cdot}23}$ **q** $\dfrac{15{\cdot}7^2}{11{\cdot}3 \times 3{\cdot}1}$ **r** $\dfrac{16{\cdot}7}{2{\cdot}15} + \dfrac{41{\cdot}6}{4{\cdot}7}$

s $\dfrac{0{\cdot}65}{0{\cdot}01} - \dfrac{0{\cdot}75 \times 3{\cdot}6}{0{\cdot}478}$

1:02 | Algebraic Expressions

Being able to use algebra is often important in problem-solving. Below is a reminder of the skills you have met up to Year 9.

Generalisation

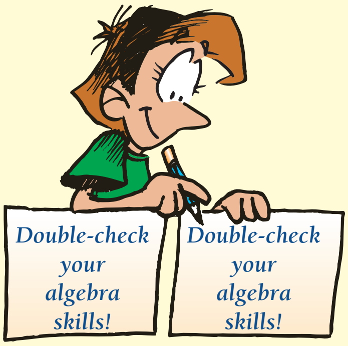
What is the average of a and b?

Answer: Average $= \dfrac{a+b}{2}$

Substitution

Find the value of $2x + y^2$ if $x = 3$, $y = -2$.

Answer: $2(3) + (-2)^2$
$= 10$



Fractions

1 $\dfrac{2x}{3} + \dfrac{x}{5}$

$= \dfrac{5 \times 2x + 3 \times x}{15}$

$= \dfrac{13x}{15}$

2 $\dfrac{{}^{1}\cancel{5a}}{{}_{1}\cancel{6b}} \times \dfrac{\cancel{18}^{3}\,\cancel{b}^{2b}}{{}_{2}\cancel{10a}}$

$= \dfrac{3b}{2}$

Simplifying expressions

1 $3x^2 + 5x + x^2 - 3x$
$= 4x^2 + 2x$

2 $12xy \div 8xz$

$= \dfrac{\cancel{12}^{3}\cancel{x}y}{{}_{2}\cancel{8}\cancel{x}z}$

$= \dfrac{3y}{2z}$

Products

1 $5(x + 3) - 2(x - 5)$
$= 5x + 15 - 2x + 10$
$= 3x + 25$

2 $(3x - 1)(x + 7)$
$= 3x^2 + 20x - 7$

Factorisation

1 $5a^2b - 10a$
$= 5a(ab - 2)$

2 $x^2 + 3x - 10$
$= (x + 5)(x - 2)$

3 $ab - 3a + xb - 3x$
$= a(b - 3) + x(b - 3)$
$= (b - 3)(a + x)$

Exercise 1:02

CD Appendix
A:02A

1 Write an expression for:

- **a** the sum of $3a$ and $4b$
- **b** the product of $3a$ and $4b$
- **c** the difference between k and m, if $k > m$
- **d** the difference between k and m, if $k < m$
- **e** the average of x, y and z
- **f** twice the sum of m and 5
- **g** the square of the difference between a and b
- **h** the square root of the sum of $5m$ and $4n$
- **i** the next even number after n, if n is even
- **j** the sum of three consecutive integers, if the first one is m

2 If $a = 3$, $b = 5$ and $c = -6$, find the value of: A:02B

a $2a + 3b$ b $a + b + c$ c $2b - c$
d $ab + bc$ e $ac - b^2$ f $a^2 + c^2$
g $\frac{4a}{c}$ h $\frac{3ab}{3 - c}$ i $\frac{3b - c}{2a}$
j $\sqrt{ab + c}$ k $\sqrt{\frac{a + b + c}{2}}$ l $\sqrt{\frac{-3c}{b - a}}$

3 Simplify these expressions. A:02C

a $5a + 3b - a + b$ b $5ab - 2ba$ c $3x^2 + x - x^2 + x$
d $5x \times 3y$ e $6ab \times 3a$ f $-2m \times 5mn$
g $15a \div 5$ h $24m \div 12m$ i $10a^2b \div 5ab$
j $n \div 3n$ k $15m \div 10n$ l $12xy^2 \div 8x^2y$
m $6a \times 7 \div 2a$ n $20y - 2 \times 5y$ o $7x + 2 \times 4x - 10x$

4 Simplify these fractions. A:02D

a $\frac{2a}{5} + \frac{4a}{5}$ b $\frac{6x}{7} - \frac{4x}{7}$ c $\frac{3}{y} + \frac{4}{y}$
d $\frac{a}{3} + \frac{a}{4}$ e $\frac{2m}{3} - \frac{m}{5}$ f $\frac{3}{2n} - \frac{4}{3n}$
g $\frac{a}{3} \times \frac{b}{4}$ h $\frac{m}{5} \times \frac{2m}{3}$ i $\frac{9a^2}{2x} \times \frac{xy}{3a}$
j $\frac{5}{m} \div \frac{2}{m}$ k $\frac{ab}{5} \div \frac{a}{10}$ l $\frac{xy}{z} \div \frac{y}{z^2}$

5 Expand and simplify these products. A:02E A:02F

a $3(2a + 1) - 5a$ b $10m - 2(m + 5)$ c $6a - (a - 5) + 10$
d $3(2n - 1) + 2(n + 5)$ e $4(2a - 1) - 3(a + 5)$ f $6(1 - 2x) - (3 - 10x)$
g $(x + 3)(x + 7)$ h $(y - 4)(y - 1)$ i $(k - 7)(k + 9)$
j $(2p + 3)(p - 5)$ k $(6x + 1)(3x - 2)$ l $(3m - 1)(2m - 5)$
m $(m - 7)(m + 7)$ n $(3a - 4)(3a + 4)$ o $(10 - 3q)(10 + 3q)$
p $(a + 8)^2$ q $(2m - 1)^2$ r $(4a + 5)^2$
s $(x + y)(x - 2y)$ t $(a + 2b)(a - 2b)$ u $(m - 3n)^2$

6 Factorise: A:02G

a $15a - 10$ b $3m^2 - 6m$ c $4n + 6mn$
d $6mn - 4m$ e $10y^2 + 5y$ f $6a^2 - 2a + 4ab$
g $x^2 - 49$ h $100 - a^2$ i $16a^2 - 9b^2$
j $x^2 + 8x + 12$ k $x^2 - x - 12$ l $x^2 - 6x + 8$
m $a^2 + 6a + 9$ n $y^2 - 10y + 25$ o $1 - 4m + 4m^2$
p $2x^2 + 7x + 3$ q $3m^2 + 7m - 6$ r $6a^2 - 11a + 4$
s $4n^2 + 12n + 9$ t $25x^2 - 10x + 1$ u $9 - 24m + 16m^2$
v $ab - 4a + xb - 4x$ w $x^2 + ax - 2x - 2a$ x $2m^2 + 6mn - m - 3n$

7 Factorise these expressions completely. A:02G

a $2x^2 - 18$ **b** $4x^2 + 4x - 24$ **c** $3a^2 - 6a - 3ab + 6b$

d $8n^2 - 8n + 2$ **e** $9 - 9q^2$ **f** $m^4 - m^2$

g $k^4 - 16$ **h** $y^3 + y^2 + y + 1$ **i** $x^3 - x^2 - x + 1$

8 Factorise and simplify: A:02H

a $\dfrac{3a+12}{3}$ **b** $\dfrac{5x-15}{x-3}$ **c** $\dfrac{a+5}{a^2+7a+10}$

d $\dfrac{m^2-m}{m^2-1}$ **e** $\dfrac{n^2-n-6}{n^2+5n+6}$ **f** $\dfrac{2x^2-x-3}{4x^2-9}$

9 Simplify: A:02H

a $\dfrac{3x+15}{2} \times \dfrac{4x}{x+5}$ **b** $\dfrac{a^2-9}{a-3} \times \dfrac{a+1}{a+3}$

c $\dfrac{3x+6}{10x} \div \dfrac{x+2}{5x}$ **d** $\dfrac{m^2-25}{m^2-5m} \div \dfrac{m+5}{5m}$

e $\dfrac{a^2+7a+12}{a^2+5a+4} \times \dfrac{a^2+6a+5}{a^2+12a+35}$ **f** $\dfrac{n^2-3n-4}{3n^2-48} \div \dfrac{n^3-n}{n^2+4n}$

10 Simplify each of the following. A:02I

a $\dfrac{1}{a+4} + \dfrac{1}{a+3}$ **b** $\dfrac{3}{2x-1} - \dfrac{5}{4x+3}$

c $\dfrac{3}{(x+1)(x+2)} + \dfrac{2}{x(x+2)}$ **d** $\dfrac{5}{(x+3)(x+5)} - \dfrac{3}{(x+3)(x+4)}$

11 Factorise each denominator where possible and then simplify. A:02I

a $\dfrac{1}{a^2-1} + \dfrac{1}{a+1}$ **b** $\dfrac{2}{3x+6} - \dfrac{1}{x^2-4}$

c $\dfrac{2}{x^2+x-6} + \dfrac{3}{x^2+4x+3}$ **d** $\dfrac{6}{x^2-x-2} - \dfrac{3}{x^2-2x-3}$

e $\dfrac{x+1}{x^2-9} + \dfrac{x-1}{x^2-5x+6}$ **f** $\dfrac{n+5}{2n^2+n-1} - \dfrac{n-3}{2n^2+5n-3}$

fun spot 1:02

Fun Spot 1:02 | How do mountains hear?

Work out the answer to each question and put the letter for that part in the box that is above the correct answer.

Simplify:

T $7x + x$ **T** $7x - x$

E $7x \times x$ **I** $7x \div x$

I $3(x + 1) - (x + 3)$

O $(x - 1)^2 + 2x - 1$

Solve:

N $5x + 1 = 13 - x$

E $\dfrac{x+2}{5} = \dfrac{x-1}{3}$

H $5x - 2(x + 3) = 12$

Find the value of $b^2 - 4ac$ if:

H $a = 4,\ b = 10,\ c = 2$ **I** $a = -1,\ b = 5,\ c = 7$

R $a = 6,\ b = -9,\ c = 3$ **E** $a = -2,\ b = -3,\ c = 5$

Simplify:

R $3x^5 \times 2x^3$ **A** $15x^5 \div 5x^4$ **W** $(3x)^2$ **T** $5x^0 \times (3x)^0$

Factorise:

M $x^2 - 81$ **S** $x^2 - 8x - 9$ **U** $x^2 - 9x$ **N** $x^2 - xy - 9x + 9y$

$9x^2$	7	5	68	$8x$	6	$7x^2$	53	$6x^8$

$(x-9)(x+9)$	x^2	$x(x-9)$	2	$6x$	$3x$	$2x$	$(x-9)(x-y)$	49	$5\frac{1}{2}$	9	$(x-9)(x+1)$

1:03 | Probability

Exercise 1:03

CD Appendix

1 Using the figures shown in the table, find the probability of selecting at random a matchbox containing: A:03A

Number of matches	48	49	50	51	52
Number of boxes	3	6	10	7	4

a 50 matches **b** 48 matches **c** more than 50 **d** at least 50

2 A single dice is rolled. What is the probability of getting: A:03B

a a five? **b** less than 3? **c** an even number? **d** less than 7?

3 A bag contains 3 red, 4 white and 5 blue marbles. If one is selected from the bag at random, find the probability that it is: A:03B

a white **b** red or white **c** *not* red **d** pink

4 A pack of cards has four suits, hearts and diamonds (both red), and spades and clubs (both black). In each suit there are 13 cards: Ace, 2, 3, 4, 5, 6, 7, 8, 9, 10, Jack, Queen and King. The Jack, Queen and King are called *court* cards. A:03C

A card is drawn from a standard pack. What is the probability that the card is:

a red? **b** not red? **c** a six? **d** not a six?
e a court card? **f** a red Ace? **g** a spade? **h** a red thirteen?
i either a red five or a ten? **j** either a heart or a black Ace?
k either a blue five or a seven? **l** either a heart or a black card?

In each of these cases, the events may not be mutually exclusive.

m either a court card or a diamond?
n either a number larger than two or a club?
o either a heart or a five?
p either a Queen or a black court card?
q either a number between two and eight or an even-numbered heart?

> Since there are 4 suits with 13 cards in each suit, the number of cards in a standard pack is 52. (In some games a Joker is also used.)

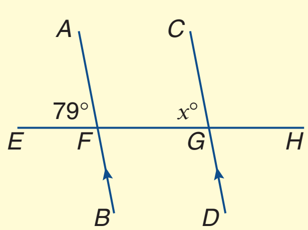
1:04 | Geometry

Exercise 1:04

CD Appendix

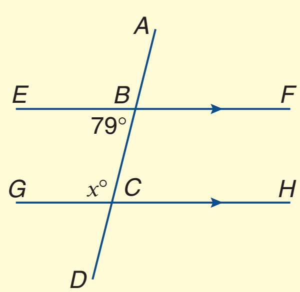
1 A:04A

a



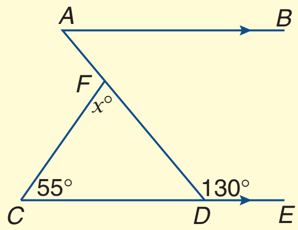
Find x. Give reasons.

b



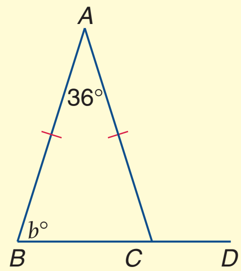
Find the size of x. Give reasons.

c



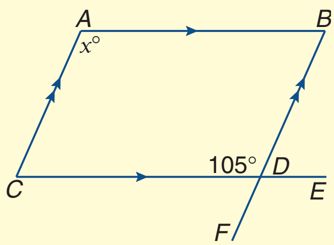
Find the size of x.
Give reasons.

d



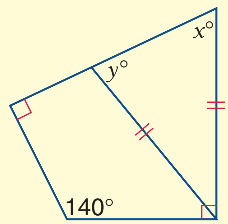
Find the value of b.
Give reasons.

e



ABDC is a parallelogram. Find the size of x. Give reasons.

f



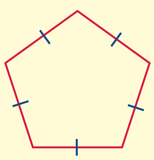
Find the value of x and y.
Give reasons.

2 **a** What is the sum of the interior angles of: A:04B

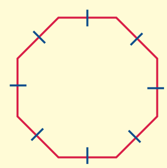
i a hexagon? **ii** a decagon?

b What is the size of each interior angle in these regular polygons?

i



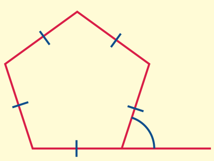
ii



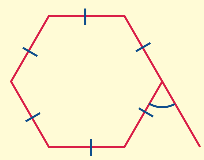
c What is the sum of the exterior angles of an octagon?

d Find the size of each exterior angle of these regular polygons.

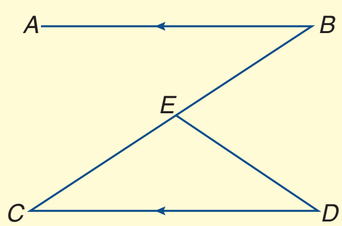
i



ii

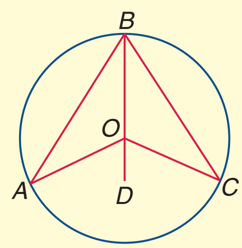


3 **a** A:04C



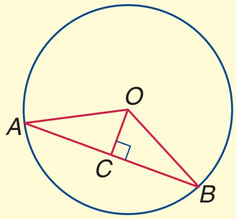
Prove $\angle BED = \angle ABC + \angle CDE$.

b



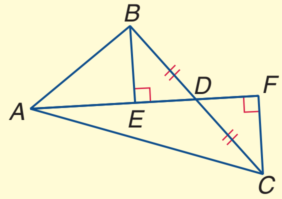
O is the centre of the circle.
Prove that $\angle AOC = 2 \times \angle ABC$.
(*Hint:* $AO = BO = CO$ (radii).)

4 **a**



O is the centre and $OC \perp AB$.
Prove that $\Delta OCA \equiv \Delta OBC$ and hence that $AC = BC$.

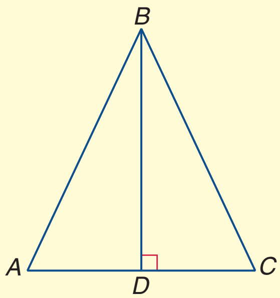
b



ΔABC is any triangle. D is the midpoint of BC, and BE and CF are perpendiculars drawn to AD, produced if necessary.
Prove that $\Delta BED \equiv \Delta CFD$ and hence that $BE = CF$.

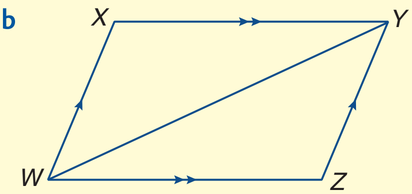
A:04D
A:04E

5 **a**



In ΔABC, a perpendicular drawn from B to AC bisects $\angle ABC$.
Prove that ΔABC is isosceles.

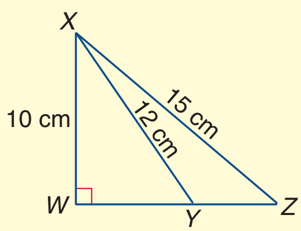
b



$WXYZ$ is a parallelogram, ie $WX \parallel ZY$ and $WZ \parallel XY$.
Prove $\angle WXY = \angle YZW$
(*Hint:* Use congruent triangles.)

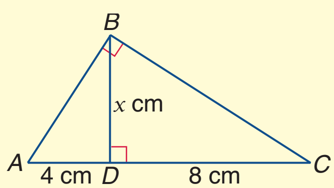
A:04F

6 **a**



Find the value of YZ.

b



i Find an expression for AB.
ii Find an expression for BC.
iii Hence, find the value of x.

A:04G

1:05 | Indices

Exercise 1:05

CD Appendix

1 Write in index form. — A:05A

a $a \times a \times a$ b $2 \times 2 \times 2 \times 2$
c $n \times n \times n \times n \times n$ d $10 \times 10 \times 10$

2 Simplify, giving your answers in index form. — A:05B

a $2^4 \times 2^5$ b $a^3 \times a^2$ c $m \times m^4$ d $10^6 \times 10^2$
e $a^{10} \div a^2$ f $y^4 \div y^3$ g $b^3 \div b$ h $10^5 \div 10^2$
i $(m^3)^4$ j $(a^2)^3$ k $(x^4)^2$ l $(10^5)^2$
m $a^0 \times 3$ n $b^0 + c^0$ o $6y^0$ p $e^6 \div e^6$
q $6a^2 \times 5$ r $6m^3 \div 3$ s $6a \times 5a$ t $(4x^4)^2$

3 Simplify. — A:05B

a $6a^4 \times 5ab^3$ b $7a^2b^2 \times 8a^3b$ c $4a^2b^3 \times 6a^2b^4$
d $10a^7 \times a^3b^3$ e $(7x^3)^2$ f $(2m^2)^4$
g $(x^2y^3)^3$ h $(5xy^2)^4$ i $30a^5 \div 5a^3$
j $100x^4 \div 20x$ k $36a^3b^4 \div 12a^2b^4$ l $8y^7z^2 \div y^7z^2$

4 Rewrite without a negative index. — A:05C

a 4^{-1} b 10^{-1} c x^{-1} d $2a^{-1}$
e 5^{-2} f 2^{-3} g m^{-3} h $5x^{-2}$

5 Rewrite each of the following with a negative index. — A:05C

a $\frac{1}{3}$ b $\frac{1}{8}$ c $\frac{1}{a}$ d $\frac{3}{x}$

e $\frac{1}{2^4}$ f $\frac{1}{10^6}$ g $\frac{1}{y^4}$ h $\frac{5}{n^3}$

6 Find the value of the following. — A:05D

a $9^{\frac{1}{2}}$ b $36^{\frac{1}{2}}$ c $8^{\frac{1}{3}}$ d $27^{\frac{1}{3}}$

7 Rewrite, using fractional indices. — A:05D

a $\sqrt{a}$ b $\sqrt[3]{y}$ c $5\sqrt{m}$ d $\sqrt{16x}$

8 Simplify these expressions. — A:05C A:05D

a $x^4 \times x^{-2}$ b $5m^3 \div m^{-2}$ c $4n^{-2} \times 3n^3$

d $6y^4 \times 3y^{\frac{1}{2}}$ e $12x^{\frac{3}{2}} \div 6x^{\frac{1}{2}}$ f $(27x^6)^{\frac{1}{3}}$

g $\dfrac{5a^4 \times 4a^5}{10a^8}$ h $\dfrac{6m^4 \times (2m)^3}{3m^2 \times 8m^5}$ i $\left[\dfrac{9x^3 \times (2x^3)^3}{6x^6 \div 3x^{-2}}\right]^{\frac{1}{2}}$

9 Write these numbers in scientific (or standard) notation. — A:05E

a 148 000 000 b 68 000 c 0·000 15 d 0·000 001 65

10 Write these as basic numerals. — A:05E

a $6{\cdot}2 \times 10^4$ b $1{\cdot}15 \times 10^6$ c $7{\cdot}4 \times 10^{-3}$ d $6{\cdot}91 \times 10^{-5}$

1:06 | Surds

Exercise 1:06

CD Appendix

1 Indicate whether each of the following is *rational* or *irrational*. A:06A

a $6\frac{3}{5}$ **b** $1{\cdot}31$ **c** $\sqrt{3}$ **d** $5{\cdot}162$

e $\sqrt[3]{2}$ **f** π **g** $0{\cdot}\dot{7}$ **h** $\sqrt{49}$

2 Evaluate each of the following to one decimal place. A:06A

a $\sqrt{7}$ **b** $\sqrt{5}+\sqrt{2}$ **c** $\sqrt{11}-3$ **d** $\frac{\sqrt{3}}{2}$

3 Simplify: A:06B

a $5\times\sqrt{2}$ **b** $\sqrt{5}\times\sqrt{7}$ **c** $\sqrt{3}\times\sqrt{2}$ **d** $\sqrt{3}\times 6$

e $\frac{\sqrt{20}}{\sqrt{2}}$ **f** $\frac{\sqrt{42}}{\sqrt{6}}$ **g** $\sqrt{130}\div\sqrt{5}$ **h** $\sqrt{\frac{49}{81}}$

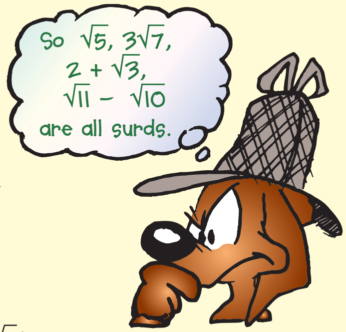
i $(\sqrt{7})^2$ **j** $\sqrt{2}\times 3\sqrt{2}$ **k** $(5\sqrt{3})^2$ **l** $8\sqrt{6}\div\sqrt{2}$

4 Simplify: A:06B

a $\sqrt{75}$ **b** $3\sqrt{8}$ **c** $\sqrt{180}$

d $4\sqrt{3}-7\sqrt{3}$ **e** $6\sqrt{5}-\sqrt{5}$ **f** $\sqrt{2}+\sqrt{2}$

g $\sqrt{8}+\sqrt{18}$ **h** $5\sqrt{32}-\sqrt{50}$ **i** $\sqrt{24}-2\sqrt{54}$



5 Simplify: A:06C

a $3\sqrt{2}\times 5\sqrt{2}$ **b** $4\sqrt{7}\times 9\sqrt{5}$ **c** $\sqrt{96}\div\sqrt{12}$

d $(7\sqrt{5})^2\times\sqrt{2}$ **e** $\frac{4\sqrt{3}\times\sqrt{18}}{\sqrt{12}}$ **f** $\sqrt{3}(2\sqrt{3}-\sqrt{5})$

6 Expand and simplify: A:06D

a $(\sqrt{2}+1)(\sqrt{2}+5)$ **b** $(\sqrt{5}-3)(\sqrt{5}-2)$

c $(2+\sqrt{3})(5-\sqrt{3})$ **d** $(\sqrt{5}+\sqrt{3})(\sqrt{5}+\sqrt{2})$

e $(2\sqrt{3}-1)(\sqrt{3}+7)$ **f** $(5\sqrt{2}-2\sqrt{3})(3\sqrt{2}+5\sqrt{3})$

g $(\sqrt{3}+2)^2$ **h** $(\sqrt{5}-3)^2$

i $(2\sqrt{3}+3\sqrt{2})^2$ **j** $(\sqrt{5}-2)(\sqrt{5}+2)$

k $(7+\sqrt{3})(7-\sqrt{3})$ **l** $(5\sqrt{3}-2\sqrt{2})(5\sqrt{3}+2\sqrt{2})$

7 Rewrite each fraction with a rational denominator. A:06E

a $\frac{1}{\sqrt{3}}$ **b** $\frac{5}{\sqrt{5}}$ **c** $\frac{6}{\sqrt{2}}$

d $\frac{1}{3\sqrt{2}}$ **e** $\frac{3}{2\sqrt{6}}$ **f** $\frac{2+\sqrt{5}}{2\sqrt{5}}$

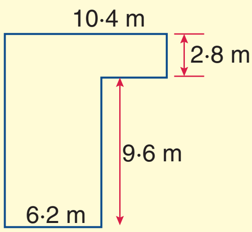
1:07 | Measurement

Exercise 1:07

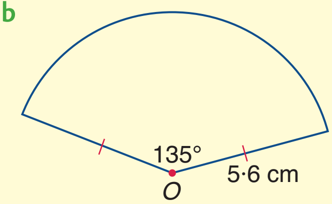
CD Appendix

1 Find the perimeter of the following figures. (Answer to 1 dec. pl.) A:07A

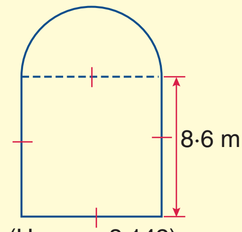
a



b



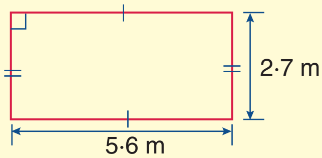
c



(Use π = 3·142)

2 Find the area of each plane shape. (Answer to 2 dec. pl.) A:07B

a



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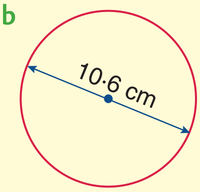
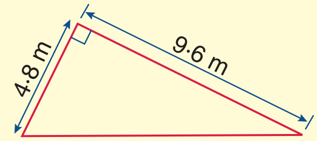
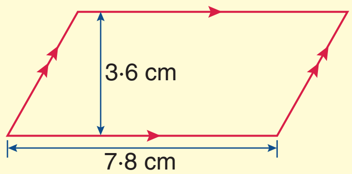


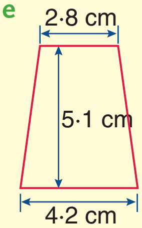
c



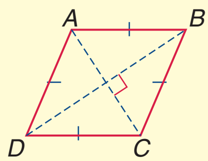
d



e



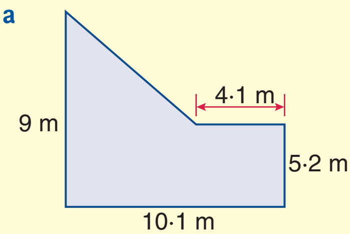
f



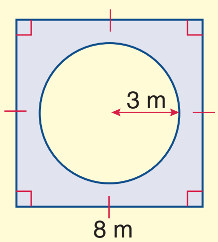
AC = 3·6 cm
BD = 6·4 cm

3 Find the area of the following shaded figures (correct to 3 sig. figs.). A:07B

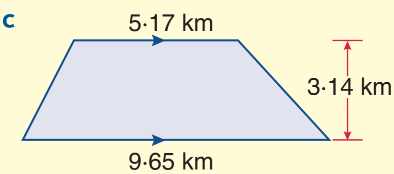
a



b

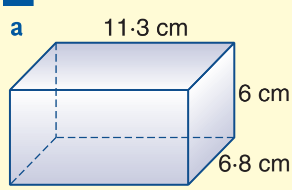


c



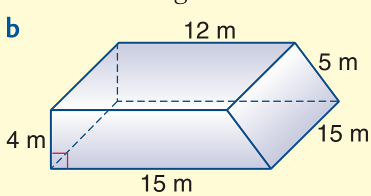
4 Find the surface area of the following solids. A:07C

a



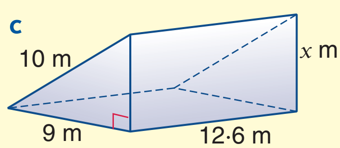
Rectangular prism

b



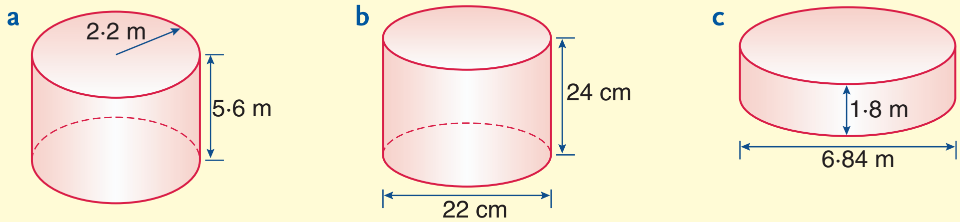
Trapezoidal prism

c



Triangular prism
(*Note:* Use Pythagoras' theorem to find x).

5 For each of the following cylinders, find **i** the curved surface area, **ii** the area of the circular ends, and **iii** the total surface area. (Give answers correct to two decimal places.) A:07D



6 Find the volume of each prism in question **4**. A:07E

7 Find the volume of each cylinder in question **5**. A:07E

1:08 | Equations, Inequations and Formulae

Exercise 1:08

CD Appendix

1 Solve the following. A:08A

a $a + 7 = 25$ **b** $m - 6 = -1$ **c** $5x = 75$ **d** $10 - y = 12$

e $3p = 7$ **f** $\frac{n}{4} = 3$ **g** $2x + 3 = 7$ **h** $8m + 5 = 21$

i $5y + 2 = 3$ **j** $9k - 1 = 5$ **k** $5 + 3x = 11$ **l** $15 - 2q = 8$

2 Solve the following. A:08A

a $5m + 2 = 4m + 7$ **b** $3x - 7 = 2x - 3$ **c** $5x + 2 = 6x - 5$

d $2a + 3 = 3a - 5$ **e** $3m - 2 = 5m - 10$ **f** $q + 7 = 8q + 14$

g $10 - 2x = x + 4$ **h** $3z + 7 = z + 10$ **i** $13 - 2m = 9 - 5m$

3 Solve these equations involving grouping symbols. A:08B

a $5(a + 1) = 15$ **b** $4(x - 3) = 16$ **c** $3(2x + 5) = 33$

d $3(5 - 2a) = 27$ **e** $4(3 - 2x) = 36$ **f** $3(2m - 5) = 11$

g $3(a + 2) + 2(a + 5) = 36$ **h** $2(p + 3) + p + 1 = 31$

i $4(2b + 7) = 2(3b - 4)$ **j** $4(2y + 3) + 3(y - 1) = 2y$

k $3(m - 4) - (m + 2) = 0$ **l** $2m - 3(1 - m) = 22$

m $5(y - 3) - 3(1 - 2y) = 4$ **n** $4(2x - 1) - 2(x + 3) = 5$

4 Solve these equations. A:08C

a $\frac{5x}{2} = 10$ **b** $\frac{2a}{3} = 6$ **c** $\frac{3m}{5} = 4$ **d** $\frac{n+1}{5} = 2$ **e** $\frac{x-4}{2} = 1$ **f** $\frac{2p+5}{3} = 1$

5 Solve these equations involving fractions. A:08C

a $\frac{a}{3} + \frac{a}{3} = 4$ **b** $\frac{2x}{5} - \frac{x}{5} = 3$ **c** $\frac{5p}{3} - \frac{p}{3} = 8$

d $\frac{q}{2} - \frac{q}{3} = 6$ **e** $\frac{2k}{3} - \frac{k}{4} = 10$ **f** $\frac{3x}{4} - \frac{x}{2} = 15$

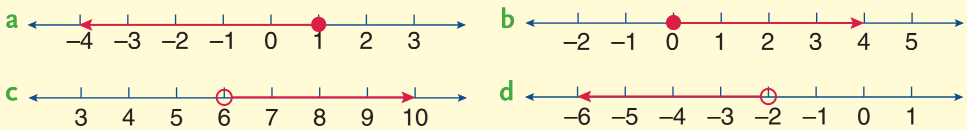
g $\frac{m+6}{3} = \frac{2m+4}{4}$ **h** $\frac{n-3}{2} = \frac{3n-5}{4}$ **i** $\frac{5x-1}{3} = \frac{3-x}{2}$

j $\frac{x+3}{2} + \frac{x+5}{5} = 8$ **k** $\frac{m+2}{5} - \frac{m+3}{6} = 1$ **l** $\frac{3a+4}{2} - \frac{a-1}{3} = \frac{2a+3}{4}$

6 **a** Translate these into an equation, using n as the unknown number. A:08D
- **i** A certain number is multiplied by 8, then 11 is added and the result is 39.
- **ii** I think of a number, double it, add 7 and the result is 5.
- **iii** I think of a number, add 4 and then divide the result by 10. The answer is 7.

b Solve each of the following problems by first forming an equation.
- **i** If 5 is added to 3 times a certain number, the result is 38. What is the number?
- **ii** If one quarter of a certain number is added to half the same number, the result is 6. What is the number?
- **iii** A rectangle is four times as long as it is wide. If it has a perimeter of 340 m, what are its dimensions?

7 Write the set of x that has been graphed below. A:08E



8 Solve these inequations and show the solution to each on a number line. A:08E

a $x + 7 > 11$	**b** $a - 5 < 3$	**c** $10 - y \geqslant 8$
d $3m \leqslant 21$	**e** $15 < 4x$	**f** $\frac{m}{4} < 1$
g $2x + 1 > 5$	**h** $7 - 3n > 4$	**i** $5x + 6 > x + 18$
j $3x - 5 < x + 6$	**k** $3 - a < 5 - 2a$	**l** $3(m + 4) < 2(m + 6)$
m $\frac{x}{2} + 1 < 6$	**n** $\frac{3x}{4} - 5 > 1$	**o** $5 - \frac{2y}{3} < 6$
p $\frac{p-1}{4} < 2$	**q** $\frac{2p+3}{2} > 7$	**r** $\frac{4-x}{3} > 1$
s $\frac{x}{2} + \frac{x}{3} > 5$	**t** $\frac{a}{4} + \frac{a}{2} < 6$	**u** $\frac{x}{2} - \frac{2x}{3} < 3$

9 **a** If $s = ut + \frac{1}{2}at^2$, find s if $u = 9$, $t = 4$ and $a = 7$. A:08F

b Given $F = p + qr$, find F if $p = 2{\cdot}3$, $q = 3{\cdot}9$ and $r = 0{\cdot}9$.

c For the formula $T = a + (n - 1)d$, find T if $a = 9{\cdot}2$, $n = 6$ and $d = 1{\cdot}3$.

10 **a** Given that $V = LBH$, evaluate B when $V = 4{\cdot}32$, $L = 1{\cdot}2$ and $H = 0{\cdot}9$. A:08F

b It is known that $S = \frac{a}{1-r}$. Find a when $S = 25$ and $r = 0{\cdot}6$.

c $F = 32 + \frac{9C}{5}$. Find C if $F = 77$.

d If $v^2 = u^2 + 2as$, find a if $v = 2{\cdot}1$, $u = 1{\cdot}6$ and $s = 0{\cdot}3$.

e Given that $T = a + (n - 1)d$, find d if $T = 24{\cdot}6$, $a = 8{\cdot}8$ and $n = 4$.

11 Change the subject of each formula to y. A:08G

a $\frac{x}{a} + \frac{y}{b} = 1$ **b** $ay^2 = x$

c $T = \sqrt{\frac{B}{y}}$ **d** $ay = by - 1$

1:09 | Consumer Arithmetic

Exercise 1:09

CD Appendix

1 **a** Michelle is paid \$8.40 per hour and time-and-a-half for overtime. If a normal day's work is 7 hours, how much would she be paid for 10 hours' work in one day? A:09A

b Jake receives a holiday loading of $17\frac{1}{2}$% on four weeks' normal pay. If he works 37 hours in a normal week and is paid \$9.20 per hour, how much money does he receive as his holiday loading?

c In a week, a saleswoman sells \$6000 worth of equipment. If she is paid \$150 plus 10% commission on sales in excess of \$4000, how much does she earn?

d A waiter works from 5:00 pm till 1:30 am four days in one week. His hourly rate of pay is \$14.65 and he gets an average of \$9.20 as tips per working night. Find his income for the week.

2 **a** Find the net pay for the week if Saransh earns \$420.80, is taxed \$128.80, pays \$42.19 for superannuation and his miscellaneous deductions total \$76.34. What percentage of his gross pay did he pay in tax? (Answer correct to decimal place of 1 per cent.) A:09B

b Find the tax payable on a taxable income of \$40 180 if the tax is \$2380 plus 30 cents for each \$1 in excess of \$20 000.

c James received a salary of \$18 300 and from investments an income of \$496. His total tax deductions were \$3050. What is his taxable income?

d Tom's taxable income for the year was \$13 860. Find the tax which must be paid if it is 17 cents for each \$1 in excess of \$6000.

3 **a** An item has a marked price of \$87.60 in two shops. One offers a 15% discount, and the other a discount of \$10.65. Which is the better buy, and by how much? A:09C

b Emma bought a new tyre for \$100, Jade bought one for \$85 and Diane bought a retread for \$58. If Emma's tyre lasted 32 000 km, Jade's 27 500 km, and Diane's 16 000 km, which was the best buy? (Assume that safety and performance for the tyres are the same.)

c Alice wants to get the best value when buying tea. Which will she buy if Pa tea costs \$1.23 for 250 g, Jet tea costs \$5.50 for 1 kg and Yet tea costs \$3.82 for 800 g?

4 **a** Find the GST (10%) that needs to be added to a base price of: A:09D

i \$75 **ii** \$6.80 **iii** \$18.75

b For each of the prices in part **a**, what would the retail price be? (Retail price includes the GST.)

c How much GST is contained in a retail price of:

i \$220? **ii** \$8.25? **iii** \$19.80?

5 **a** What is meant by the expression 'buying on terms'? A:09E

b Find the amount Jason will pay for a fishing line worth \$87 if he pays \$7 deposit and \$5.69 per month for 24 months. How much extra does he pay in interest charges?

c Nicholas was given a discount of 10% on the marked price of a kitchen table. If the discount was \$22, how much was the marked price?

d A factory's machinery depreciates at a rate of 15% per annum. If it is worth \$642 000, what will be its value after one year?

e The price of a book was discounted by 20%. A regular customer was given a further discount of 15%. If the original price was \$45, what was the final price of the book?

6 **a** The cost price of a DVD player was \$180 and it was sold for \$240. What was: A:09F

- **i** the profit as a percentage of the cost price?
- **ii** the profit as a percentage of the selling price?

b A new car worth \$32 000 was sold after two years for \$24 000. What was:

- **i** the loss as a percentage of the original cost price?
- **ii** the loss as a percentage of the final selling price?

1:10 | Coordinate Geometry

Exercise 1:10

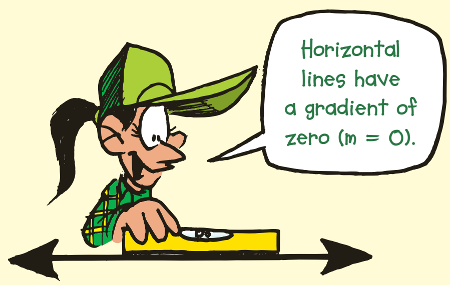
CD Appendix

1 Find the gradient of the line that passes through the points: A:10A

- **a** (1, 2) and (1, 3)
- **b** (1, 7) and (0, 0)
- **c** (−3, −2) and (5, −2)

2 Find the midpoint of the interval joining: A:10B

- **a** (2, 6) and (8, 10)
- **b** (−3, 5) and (4, −2)
- **c** (0, 0) and (7, 0)



3 **a** Find the distance between (1, 4) and (5, 2). A:10C

b A is the point (−5, 2). B is the point (−2, −6). Find the distance AB.

c Find the distance between the origin and (6, 8).

d Find the distance AB between A(−2, 1) and B(5, 3).

4 Sketch each of these lines on a number plane. A:10D

a $y = 3x + 4$ **b** $2x + 3y = 12$ **c** $y = -3x$ **d** $x = 2$

5 Find the gradient and y-intercept of the lines: A:10E

a $y = 3x + 5$ **b** $y = -x - 2$ **c** $y = -2x + 5$

6 Write each equation in question **4** in the *general form*. A:10E

7 Write the equation of the line that has: A:10F

a gradient 5 and y-intercept -2
b gradient 0 and y-intercept 4
c gradient 2 and passes through $(0, 5)$
d gradient -1 and passes through $(-2, 3)$.

8 Write the equation of the line that passes through: A:10G

a $(2, 1)$ and $(4, 2)$
b $(-1, 5)$ and $(3, 1)$

9 A:10H

a Which of the lines $y = 3x$, $2x + y = 3$ and $y = 3x - 1$ are parallel?
b Are the lines $y = 2x - 1$ and $y = 2x + 5$ parallel?
c Show that the line passing through $(1, 4)$ and $(4, 2)$ is parallel to the line passing through $(-4, 0)$ and $(-1, -2)$.
d Which of these lines are parallel to the y-axis?
$\{y = 4, y = x + 1, x = 7, y = 5x - 5, x = -2\}$

10 A:10H

a Are $y = 4x$ and $y = \frac{1}{4}x - 2$ perpendicular?
b Are $y = 3$ and $x = 4$ perpendicular?
c Which of the lines $y = -x + 1$, $y = \frac{1}{2}x - 1$ and $y = x - 7$ are perpendicular?
d Show that the line passing through $(0, -5)$ and $(-3, -4)$ is perpendicular to $y = 3x - 8$

11 Sketch the regions corresponding to the inequations. A:10I

a $x > 2$ **b** $y \leqslant -1$ **c** $x + y < 2$
d $y \geqslant 2x$ **e** $y < x - 1$ **f** $2x + 3y \geqslant 6$

12 A:10I

a Graph the region described by the *intersection* of $y > x + 1$ and $y < 3$.
b Graph the region described by the *union* of $y > x + 1$ and $y < 3$.

1:11 | Statistics

Exercise 1:11

CD Appendix

1 In a game, a dice was rolled 50 times, yielding the results below. Organise these results into a frequency distribution table and answer the questions. A:11A

5	4	1	3	2	6	2	1	4	5
5	1	3	2	6	3	2	4	4	1
6	2	5	1	6	6	6	5	3	2
6	3	4	2	4	1	4	2	4	4
2	3	1	5	4	2	2	3	2	1

a Which number on the dice was rolled most often?
b Which number had the lowest frequency?
c How often did a 3 appear?
d For how many throws was the result an odd number?
e On how many occasions was the result greater than 3?

2 Use the information in question **1** to draw, on separate diagrams: A:11B
a a frequency histogram **b** a frequency polygon

3 **a** For the scores 5, 1, 8, 4, 3, 5, 5, 2, 4, find: A:11C
  i the range **ii** the mode **iii** the mean **iv** the median
b Use your table from question **1** to find, for the scores in question **1**:
  i the range **ii** the mode **iii** the mean **iv** the median
c Copy your table from question **1** and add a cumulative frequency column.
  i What is the cumulative frequency of the score 4?
  ii How many students scored 3 or less?
  iii Does the last figure in your cumulative frequency column equal the total of the frequency column?

4 Use your table in question **3** to draw on the same diagram: A:11D
a a cumulative frequency histogram **b** a cumulative frequency polygon

5 The number of cans of drink sold by a shop each day was as follows: A:11E

30	28	42	21	(54)	47	36	37	22	(18)
25	26	43	50	23	29	30	19	28	20
40	33	35	31	27	42	26	44	53	50
29	20	32	41	36	51	46	37	42	27
28	31	29	32	41	36	32	41	35	41
29	39	46	36	36	33	29	37	38	25
27	19	28	47	51	28	47	36	35	40

The highest and lowest scores are circled.
a Tabulate these results using classes of 16–22, 23–29, 30–36, 37–43, 44–50, 51–57. Make up a table using these column headings: Class, Class centre, Tally, Frequency, Cumulative frequency.
b What was the mean number of cans sold?
c Construct a cumulative frequency histogram and cumulative frequency polygon (or ogive) and find the median class.
d What is the modal class?
e Over how many days was the survey held?

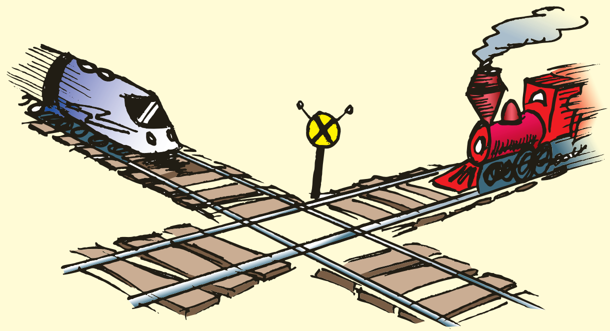
1:12 | Simultaneous Equations

Exercise 1:12

CD Appendix

1 Find the value of y if: A:12A

a $x + y = 12$ and $x = -3$
b $2x - 4y = 1$ and $x = 4$



2 Find the value of x if: A:12A

a $y = 5x - 4$ and $y = 21$
b $3x + y = 12$ and $y = -6$

3 **a** Does the line $2x - 4y = 12$ pass through the point (14, 4)? A:12A

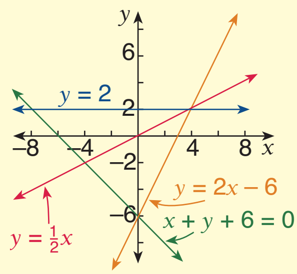
b Does the point (4, 8) lie on the line $6x - 2y = 7$?

4 Use the graph to solve these pairs of simultaneous equations. A:12A

a $\begin{cases} y = 2 \\ y = 2x - 6 \end{cases}$ **b** $\begin{cases} y = 2 \\ x + y + 6 = 0 \end{cases}$

c $\begin{cases} y = \frac{1}{2}x \\ x + y + 6 = 0 \end{cases}$ **d** $\begin{cases} y = \frac{1}{2}x \\ y = 2x - 6 \end{cases}$

e $\begin{cases} y = 2x - 6 \\ x + y + 6 = 0 \end{cases}$ **f** $\begin{cases} y = 2 \\ y = \frac{1}{2}x \end{cases}$



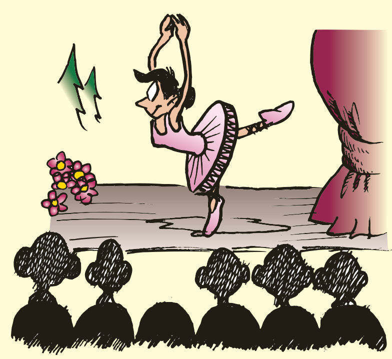
5 Solve these simultaneous equations by the substitution method. A:12B

a $\begin{cases} 2x + y = 12 \\ 3x + 2y = 22 \end{cases}$ **b** $\begin{cases} 4x - 3y = 13 \\ 2x = y + 9 \end{cases}$ **c** $\begin{cases} y = x - 2 \\ 2x + y = 7 \end{cases}$ **d** $\begin{cases} 4a - b = 3 \\ 2a + 3b = 11 \end{cases}$

6 Solve these simultaneous equations by the elimination method. A:12C

a $\begin{cases} 5x - 3y = 20 \\ 2x + 3y = 15 \end{cases}$ **b** $\begin{cases} 4a - 3b = 11 \\ 4a + 2b = 10 \end{cases}$ **c** $\begin{cases} 3c + 4d = 16 \\ 7c - 2d = 60 \end{cases}$ **d** $\begin{cases} 2x + 7y = 29 \\ 3x + 5y = 16 \end{cases}$

7 A theatre has 2100 seats. All of the rows of seats in the theatre have either 45 seats or 40 seats. If there are three times as many rows with 45 seats as there are with 40 seats, how many rows are there? A:12D



8 Fiona has three times as much money as Jessica. If I give Jessica \$100, she will have twice as much money as Fiona. How much did Jessica have originally? A:12D

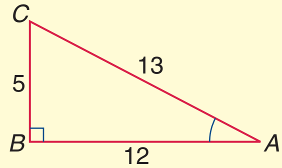
1:13 | Trigonometry

Exercise 1:13

CD Appendix

1 Write down the formula for: A:13A
a $\sin \theta$ **b** $\cos \theta$ **c** $\tan \theta$

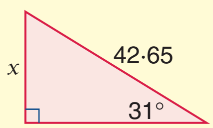
2 Use the triangle on the right to find the value of each ratio. Give each answer as a fraction. A:13A
a $\sin A$ **b** $\cos A$ **c** $\tan A$



3 Use the triangle on the right to give, correct to three decimal places, the value of: A:13A
a $\sin A$ **b** $\cos A$ **c** $\tan A$

4 Use your calculator to find (correct to three decimal places) the value of: A:13B
a $\sin 14°$ **b** $\sin 8°$ **c** $\sin 85°30'$ **d** $\sin 30°27'$
e $\cos 12°$ **f** $\cos 6°$ **g** $\cos 88°15'$ **h** $\cos 60°50'$
i $\tan 45°$ **j** $\tan 7°$ **k** $\tan 87°07'$ **l** $\tan 35°27'$

5 Find the value of x (correct to two decimal places) for each triangle. (All measurements are in metres.) A:13C

a


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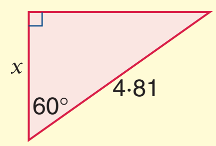

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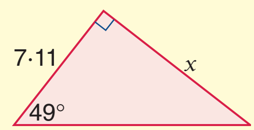

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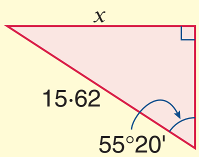

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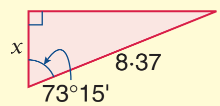

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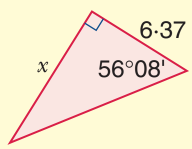

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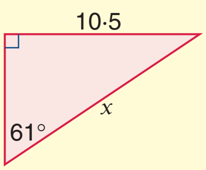

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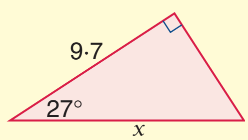
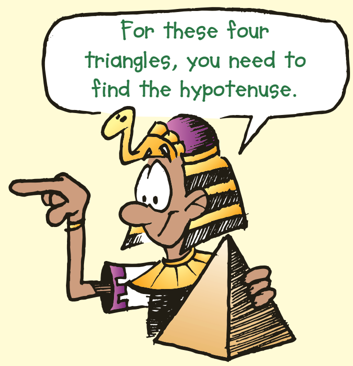
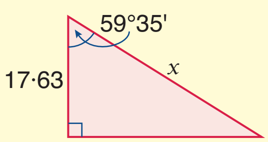




i


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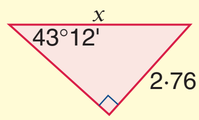
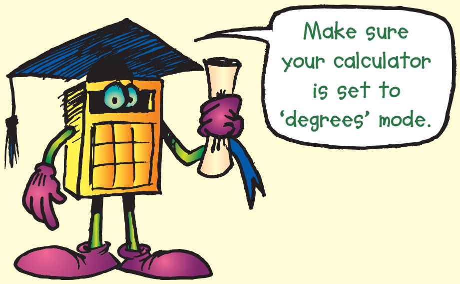
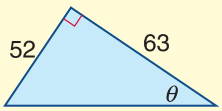


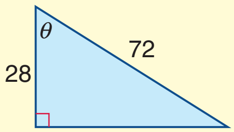
6 For each figure, find the size of angle θ. (Answer to the nearest minute. Measurements are in centimetres.) A:13D



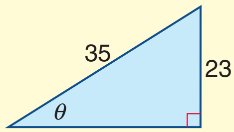
a



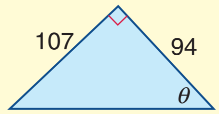
b



c

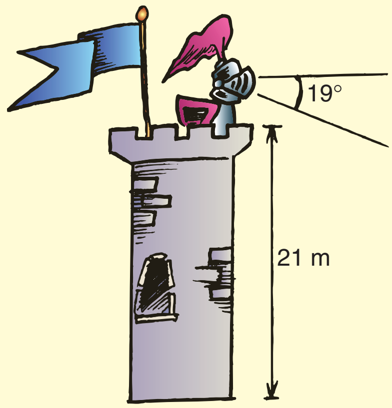


d



7 **a** The angle of depression of an object on a level plain is observed to be 19° from the top of a 21 m tower. How far from the foot of the tower is the object? A:13E

b The angle of elevation of the top of a vertical cliff is observed to be 23° from a boat 180 m from the base of the cliff. What is the height of the cliff?



8 **a** A ship sails south for 50 km, then 043° until it is due east of its starting point. How far is the ship from its starting point (to the nearest metre)? A:13E

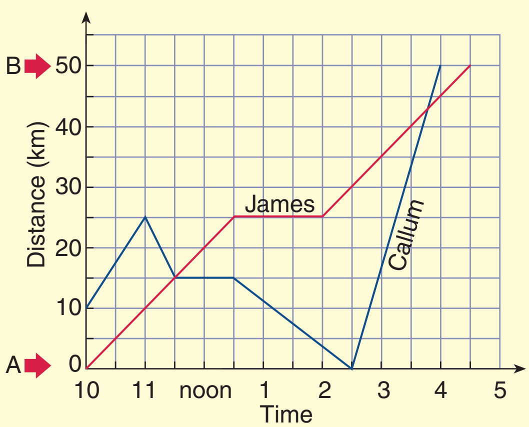
b If the town of Buskirk is 15 km north and 13 km east of Isbister, find the bearing of Buskirk from Isbister.

1:14 | Graphs of Physical Phenomena

Exercise 1:14

CD Appendix

1



The travel graph shows the journeys of James and Callum between town A and town B. (They travel on the same road.) A:14A

a How far from A is Callum when he commences his journey?

b How far is James from B at 2:30 pm?

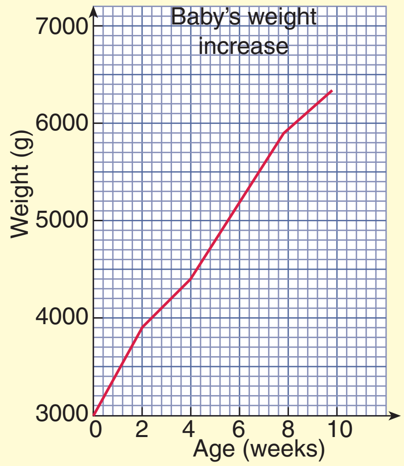
c When do James and Callum first meet?

d Who reaches town B first?

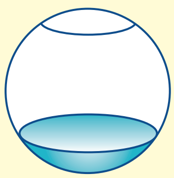
e At what time does Callum stop to rest?

f How far does James travel?
g How far apart are James and Callum when Callum is at town A?
h How far does Callum travel?

2 **a** What did the baby weigh at birth? A:14B
b What was the baby's weight at 4 weeks of age?
c By how much did the baby's weight increase in the first two weeks of age?
d By how much did the baby's weight increase from 2 weeks of age to 4 weeks of age?
e Considering your answer to parts **c** and **d**, in which period, (0–2) or (2–4) was the baby's rate of growth the greatest?



3 Water is poured into each container at a constant rate. The graphs indicate the height of the water in each container against time. Match each graph with the correct container. A:14B

a


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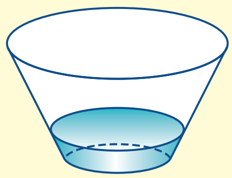
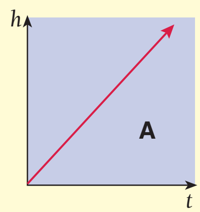
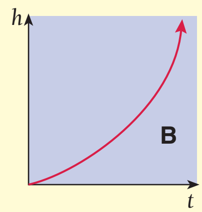
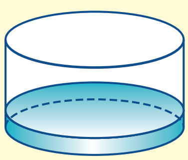






c


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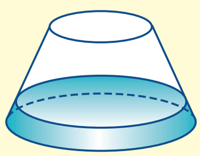
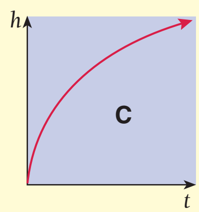
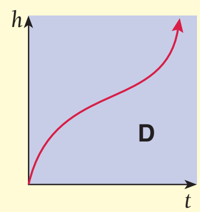




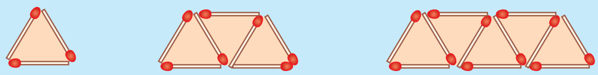


Chapter 1 | Working Mathematically

assignment 1

1 Complete a table of values for each matchstick pattern below, and hence find the rule for each, linking the number of coloured triangles (t) to the number of matches (m).

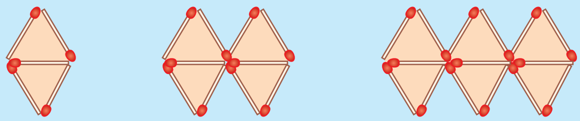
a



t	1	3	5
m			

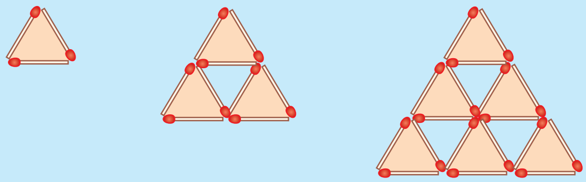
Appendix B

b



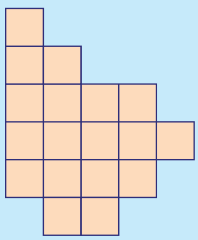
t	2	4	6
m			

c



t	1	3	6
m			

2 Divide this shape into three pieces that have the same shape.



3 Ryan answered all 50 questions in a maths competition in which he received 4 marks for each correct answer but lost one mark for each incorrect answer.

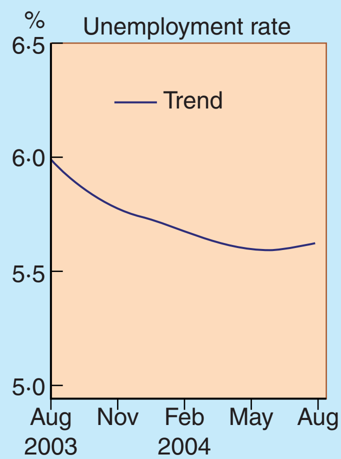
a What is Ryan's score if he answered 47 questions correctly?

b How many answers did he get right if his score was 135?

4 It takes 3 min 15 s to join two pieces of pipe. How long would it take to join 6 pieces of pipe into one length?

5 A number of cards can be shared between 4 people with no remainder. When shared between 5 or 6 people, there are two cards left over. If there are fewer than 53 cards, how many are there?

6 From August 2003 to August 2004, the unemployment rate fell from 6·0% to 5·6%.



Source: Australian Bureau of Statistics, Labour Force, October 2004.

1 Surd magic square
2 Algebraic fractions
Technology Applications Activities
Chapter Review Questions

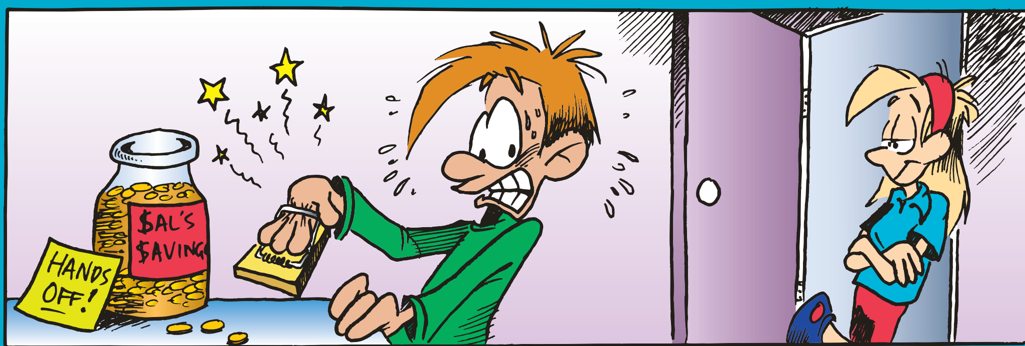
a If the number of unemployed in August 2004 was 576 400, how many were unemployed in August 2003? Answer correct to four significant figures.

b If 576 400 represents 5·6% of the total workforce, what is the size of the total workforce? Answer correct to four significant figures.

c If the rate of 5·6% is only correct to one decimal place, the rate could really be from 5·55% to 5·65%. How many people does this approximation range of 0·1% represent?

2

Consumer Arithmetic



Chapter Contents

Learning Outcomes

Students will be able to:

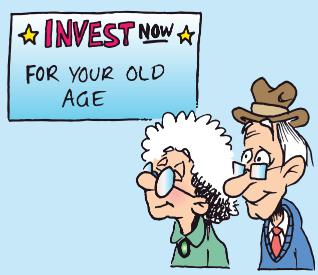
- Solve problems involving earning, spending and saving money.
- Solve problems involving simple and compound interest, depreciation and successive discounts.

Areas of Interaction

Approaches to Learning (Knowledge Acquisition, Reflection), Human Ingenuity, Environments, Community and Service

In Book 4, we concentrated on aspects of earning and spending money. Now we will consider aspects of saving and borrowing.

2:01 | Saving Money

SAVING MONEY		
Savings accounts	*Target and award saver accounts*	*Term deposit accounts*
Meaning		
Usually a transaction card is used to deposit or withdraw your money.	A savings account from which you cannot withdraw money without forfeiting interest.	Your money is invested for a fixed period of time, usually at an agreed interest rate.
Advantages		
You can deposit and withdraw without notice. Safe, encourages saving and may help you get a loan.	Better interest rate than savings account. Encourages saving by reducing the temptation to withdraw your money.	Higher interest rates than savings accounts and this rate is usually fixed for the period of the investment. No extra bank charges.
Disadvantages		
Lower interest rates are offered for savings accounts. Bank fees apply.	If you withdraw your money early you do not receive the higher rate of interest, but receive a lower rate instead. This causes inconvenience if some of the money is needed urgently.	It usually requires the investment of a minimum amount. The fixed amount invested cannot be withdrawn before the end of the agreed period (or term) without reducing the interest rate for the investment.
		

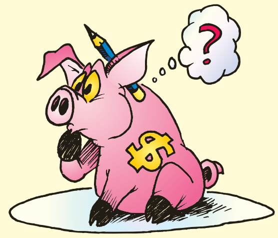
When saving or investing money:

1 Compare rates of interest (they are continually changing).
2 Consider the terms of the investment carefully.
3 Ask questions about terms, services and safety.
4 Higher interest rates are often given where there is inconvenience or some risk to your money.

Exercise 2:01

1 Write answers in your own words. Refer to the previous page if necessary.

a What are the advantages of a normal *savings* account?
b What is a *term deposit* account?
c What are the advantages of a *term deposit* account?
d What are the disadvantages of a *term deposit* account?
e What are the disadvantages of a normal *savings* account?
f What are the advantages of a *target savings* account?
g What are the disadvantages of an *award saver* account?



2 Allan decided to open an *award saver* account. He chose to bank \$200 each month, starting with a balance of \$200. The money will earn 2·65% pa interest calculated daily and paid monthly as long as no withdrawals are made during that month. If a withdrawal is made, the interest rate for that month falls to 0·5% pa.

a What rate of interest applies if a withdrawal is made?
b What rate of interest applies if no withdrawal is made?
c How often is interest calculated?
d How often is interest paid?
e If 2·65% interest is given for one year (365 days), what percentage interest would be given for one day? (Give your answer correct to three decimal places.)

3 Tomeka was given the following information when asking about a *term deposit* account.

Term rates — Interest paid at maturity — 1 to 12 months						
	1 < 3 mths	*3 < 4 mths*	*4 < 6 mths*	*6 mths*	*6 < 12 mths*	*12 mths*
\$1000 – \$4999	4·00% pa	4·75% pa	5·00% pa	5·00% pa	5·00% pa	5·95% pa
\$5000 – \$19 999	5.50% pa	6·00% pa	6·60% pa	6·90% pa	6·90% pa	7·20% pa
\$20 000 +	5·50% pa	6·25% pa	6·75% pa	7·15% pa	7·15% pa	7·50% pa

a If Tomeka has \$4000 to invest, what is the least investment time necessary to receive 5% pa for the investment?
b What interest rate is given on \$12 000 invested for 7 months?
c What is the least amount that must be invested to receive 7·5% pa for a 12-month investment?

4 Emma opened a special *cash management* account that offered interest of 2·5% pa for a balance under €5000, 4·25% pa for a balance between €5000 and €10 000, 4·75% pa for a balance between €10 000 and €20 000, and 6% pa for a balance over €20 000. Her first five cash or personal cheque withdrawals each month are free of bank transaction fees. After that she pays €3 per transaction. She needed €5000 to open this account.

a How many transactions are allowed each month before bank transaction fees are charged? What are these transaction fees?
b What interest rate would Emma receive if her balance is:
i €8150? **ii** €1809? **iii** €17 080?
c How much did Emma need to open this account?

Reading mathematics 2:01 | Financial spreadsheets

Below are two versions of a spreadsheet produced using a computer program.

- The first table shows the income earned by a student over 4 years.
- The numbers down the left and the letters across the top allow us to use **coordinates** to name any **cell** of the spreadsheet.
- The second table shows the **formula** used to obtain the cells in column F and row 10.

	A	B	C	D	E	F
1	**Income**	**2003 (Y7)**	**2004 (Y8)**	**2005 (Y9)**	**2006 (Y10)**	**Total**
2	Odd jobs	212	264	220	160	**$856**
3	Selling newspapers	364	380	0	0	**$744**
4	Mowing lawns	60	260	180	45	**$545**
5	Baby-sitting	0	140	235	380	**$755**
6	Typing	0	0	20	110	**$130**
7	McDonald's cashier	0	0	1654	1840	**$3494**
8	Washing cars	104	260	86	52	**$502**
9						
10	**Total**	**$740**	**$1304**	**$2395**	**$2587**	**$7026**

	A	B	C	D	E	F
1	**Income**	**2003 (Y7)**	**2004 (Y8)**	**2005 (Y9)**	**2006 (Y10)**	**Total**
2	Odd jobs	212	264	220	160	**=SUM(B2..E2)**
3	Selling newspapers	364	380	0	0	**=SUM(B3..E3)**
4	Mowing lawns	60	260	180	45	**=SUM(B4..E4)**
5	Baby-sitting	0	140	235	380	**=SUM(B5..E5)**
6	Typing	0	0	20	110	**=SUM(B6..E6)**
7	McDonald's cashier	0	0	1654	1840	**=SUM(B7..E7)**
8	Washing cars	104	260	86	52	**=SUM(B8..E8)**
9						
10	**Total**	**=SUM(B2..B8)**	**=SUM(C2..C8)**	**=SUM(D2..D8)**	**=SUM(E2..E8)**	**=SUM(F2..F8)**

1 What is referred to in cell:
a D6? **b** B3? **c** A5? **d** F8? **e** C10?

2 What is meant by:
a =SUM(B2..E2)? **b** =SUM(C2..C8)?

3 What would be the answer to '=SUM(B10..E10)'?

4 Why do '=SUM(B10..E10)' and '=SUM(F2..F8)' have the same value?

Make a spreadsheet of your own on a subject of your choice.

2:02 | Simple Interest

prep quiz 2:02

What fraction of a year is:

1 1 month? **2** 5 months? **3** 11 months?

4 1 day? **5** 7 days? **6** 128 days?

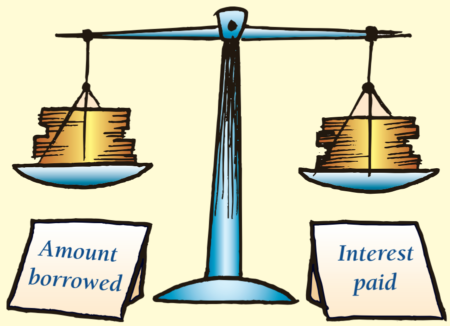
To find 9·5% of \$800 we press:

9·5 [÷] 100 [×] 800 [=]

Find:

7 9% of \$650 **8** 13·6% of \$800 **9** $37\frac{1}{2}$% of \$40 000 **10** 95% of \$7·80

- **Interest** is the payment made for the use of money invested (or borrowed).
- The money invested (or borrowed) is called the **principal**.
- **Simple interest** is interest paid on the original principal. The same interest is paid for each time period.



What is the simple interest on \$750 invested at 8% pa for 3 years?

'pa' means per annum or per year

Interest for one year is 8% of \$750
= \$60

For simple interest, the same amount of interest is earned each year.
Simple interest = (interest for one year) × (number of years)

Interest for 3 years is (8% of \$750) × 3
= \$60 × 3
= \$180

If I invest \$100 at a rate of 15% pa simple interest for 1 year,
the interest = \$100 × 15% × 1
= \$100 × 0·15 × 1 or $\$\frac{100}{1} \times \frac{15}{100} \times \frac{1}{1}$

If I invest \$100 at a rate of 15% pa simple interest for 7 months,
the interest = \$100 × 15% × $\frac{7}{12}$ (7 months = $\frac{7}{12}$ of a year.)

Simple interest

$$I = PRT$$

- **If *R* is given as % pa, then *T* must be in years.**
- **If *R* is given as % per month then *T* must be in months.**

Where *I* is the simple *interest*
***P* is the *principal* invested**
R* is the *rate of interest
***T* is the *time* involved**

worked examples

1 Find the simple interest paid on \$860 invested at $6\frac{1}{2}$% pa for 5 years.

2 What is the simple interest on \$2400 at 8% pa for 5 months?

3 Find the simple interest on \$900 for 240 days at a rate of 11% pa.

4 What is the simple interest paid on \$1950 invested for 7 months at 0·75% per month?

Solutions

1 $I = PRT$

$= 860 \times 6\frac{1}{2}\% \times 5$

$= 860 \times 0{\cdot}065 \times 5$

$= \$279.50$

$\therefore$ The simple interest is \$279.50.

2 $I = PRT$

$= 2400 \times 8\% \times \frac{5}{12}$

$= 2400 \times 0{\cdot}08 \times 5 \div 12$

$= \$80$

$\therefore$ The simple interest is \$80.

■ 5 months is $\frac{5}{12}$ years

3 $I = PRT$

$= 900 \times 11\% \times \frac{240}{365}$

$= 900 \times 0{\cdot}11 \times 240 \div 365$

$\doteqdot \$65.10$

$\therefore$ The simple interest is \$65.10.

4 $I = PRT$

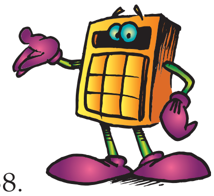
$= 1950 \times 0{\cdot}75\% \times 7$

$= 1950 \times (0{\cdot}75 \div 100) \times 7$

$= 1950 \times 0{\cdot}0075 \times 7$

$\doteqdot \$102.38$

$\therefore$ The simple interest is \$102.38.



■ To change 0·75% to a decimal, press ·75 [÷] 100 [=] on your calculator.

eg 1950 × 0·75% × 7 becomes 1950 [×] ·75 [÷] 100 [×] 7 [=]

Exercise 2:02

Foundation Worksheet 2:02

Simple interest

1 Find:
a 15% of \$60 b 9% of \$80

2 Find simple interest on:
a \$200 at 10% pa for 5 years
b \$500 at 8% pa for 10 years

1 Find the simple interest charged for a loan of:

a \$620 at 18% pa for 4 years

b \$4500 at 26% pa for 5 years

c \$15.60 at 15% pa for 3 years

d \$391 at 9% pa for 7 years

e \$90 000 at 7% pa for 25 years

f \$90 000 at 11% pa for 25 years

g \$500 for 2 years at 14% pa

h \$25 000 for 6 years at 8% pa

i \$16 860 for 10 years at 9% pa

j \$2758 for 5 years at 10% pa

2 Find the simple interest earned on:

a \$98 at 6·4% pa for 5 years

b \$9500 at 7·2% pa for 3 years

c ¥12 080 at $6\frac{1}{2}$% pa for 2 years

d €8.40 at $5\frac{1}{2}$% pa for 7 years

e \$600 at 5·8% pa for 4 years

f \$850 at 7·1% pa for 6 years

g €138 at $8\frac{1}{4}$% pa for 2 years

h \$140 at $10\frac{3}{4}$% pa for 14 years

3 Find the simple interest charged for a loan of:

a ¥200 000 for 3 months at 1·1% per month

b \$8550 for 2 months at 1·6% per month

c \$96 for 7 months at 0·9% per month

d €900 for 18 months at 0·7% per month

e \$280 000 for 6 months at 1·2% per month

f ¥897 500 for 1 month at 0·9% per month

g $697.80 for 5 months at 1·25% per month
h $896.40 for 3 months at 0·25% per month
i $30 465 for 7 months at 1·3% per month
j $806 for 9 months at 1·125% per month

4 Find the simple interest on:

a $6200 at 12% pa for 6 months
b ¥115 000 at 10% pa for 1 month
c €1150 at 10% pa for 7 months
d $20 000 at 7% pa for 11 months
e €8400 at 17% pa for 9 months
f $9516 at 13% pa for 11 months
g $28.20 at 12% pa for 5 months
h $708 at 8% pa for 1 month
i ¥184 800 for 5 months at 14% pa
j $540 for 7 months at 4% pa

> 11 months is $\frac{11}{12}$ years

5 Find the simple interest (to the nearest cent) earned on:

a $490 for 76 days at 16% pa
b $1096 for 207 days at 26% pa
c $740 for 9 days at 37% pa
d €9700 for 304 days at 9% pa
e €6000 for 40 days at $10\frac{1}{4}$% pa
f $50 000 for 30 days at $6\frac{1}{2}$% pa
g $3865 for 23 days at $9\frac{3}{4}$% pa
h $65 for 203 days at $3\frac{1}{2}$% pa
i $961.80 for 407 days at 11·8% pa
j €4150 for 15 days at 12·75% pa

> $10\frac{1}{4}\% = 10{\cdot}25 \div 100$
> $= 0{\cdot}1025$

reading mathematics 2:02

Reading mathematics 2:02 | Why not buy a tent?

The advertisements shown are for a family tent and a smaller cabin tent. Read the information carefully and answer the questions below.

ADD-ON FAMILY TENT!

Add-on living to suit your needs!
Trailmaster 3·5 x 2·43 m tent with water-resistant poly/cottton roof, walls, awning.
Sewn-in floor and screened windows with storm flaps.
IF MORE SPACE NEEDED add-on a poly/cotton annex for $119 extra or a mesh sunroom for $89.95 extra.

TENT $ 299
$2.90 WEEKLY

CABIN TENT – MORE ROOM INSIDE!
Trailmaster is 3 x 2.36 m.
Water-resistant poly/cotton roof, walls, awning. Frame outside – more room! Sewn in floor, screen windows.

TENT $ 199
$1.90 WEEKLY

Terms are after 10% deposit over 3 years.
Sizes quoted are approximate.

1 What are the dimensions of the family tent? Use these to determine the floor area inside the tent.
2 Calculate the floor area of the cabin tent.
3 The family tent may be purchased by paying only $2.90 a week. But how much deposit must first be paid and how many payments must be made?
4 Calculate the total cost of the family tent if it is bought on terms.
5 How much extra above the cash price is paid if the cabin tent is bought on terms?
6 By comparing the area of each tent with its cash price, which tent provides the most shelter per dollar?

2:03 | Solving Simple Interest Problems

prep quiz 2:03

Write each of these percentages as a decimal.

1 18% **2** 5% **3** 5·4% **4** $5\frac{1}{2}\%$ **5** $5\frac{1}{4}\%$

If $I = PRT$:

6 find I when $P = 2000$, $R = 5\%$ and $T = 2$ **7** find I when $P = 100$, $R = 0{\cdot}1$ and $T = 3$

8 find P when $I = 5$, $R = 0{\cdot}1$ and $T = 10$

9 find R when $I = 32$, $P = 80$ and $T = 4$

10 find T when $I = 120$, $P = 500$ and $R = 0{\cdot}08$.

Simple interest formula: $I = PRT$

worked examples

1 Rhonda borrowed \$78 000 for 4 months at 12·75% pa simple interest to pay for her new house while her old one was being sold. How much interest did she pay?

2 I paid \$8000 to borrow \$50 000 for 2 years. What was the rate of simple interest charged?

3 Greg borrowed \$1200 at 1·5% simple interest per month. Which is the best estimate of the interest charges for six months: \$20, \$100, \$200 or \$1800?

Solutions

1 $P = \$78\,000$

$T = 4$ months or $\frac{4}{12}$ years

$R = 12{\cdot}75\%$ or $0{\cdot}1275$

$I = \ldots$

Now $I = PRT$

$= 78\,000 \times 12{\cdot}75\% \times \frac{4}{12}$

$= 78\,000 \times 0{\cdot}1275 \times \frac{4}{12}$

$= \$3315$

∴ Rhonda paid \$3315 in interest.

2 $I = \$8000$

$P = \$50\,000$

$T = 2$ years

$R = \ldots$

Now $I = PRT$

$8000 = 50\,000 \times R \times 2$

$= 100\,000\,R$

Divide both sides by 100 000.

$0{\cdot}08 = R$

$\therefore R = 8\%$

∴ I was charged 8% interest.

3 Here we are asked to make an estimate. Your reasoning could be like this.

'Each month Greg pays \$1.5 for every \$100 borrowed (ie 1·5%). That's \$15 for \$1000, so it's more than \$15 for \$1200. He'd pay 6 times as much for 6 months, so the interest must be more than \$90.'

The best of these estimates is \$100.

Exercise 2:03

1 **a** Juan borrowed \$4695 at 8% pa simple interest for 5 years so that he could buy a car. At the end of the 5 years, both interest and loan had been paid.

i How much interest was charged? **ii** How much was paid back altogether?

b Scott borrowed \$4000 from Mona to be repaid after 3 years, along with 9% pa simple interest. How much will Mona receive after 3 years?

c Erika invested \$8400 in the credit union for 11 months at 1·2% per month simple interest. How much interest did she earn?

d Jenny invests \$4000 for 3 years at 6·5% pa simple interest, while Robert invests \$4000 for 3 years at 5·8% pa simple interest. How much more interest does Jenny receive than Robert?

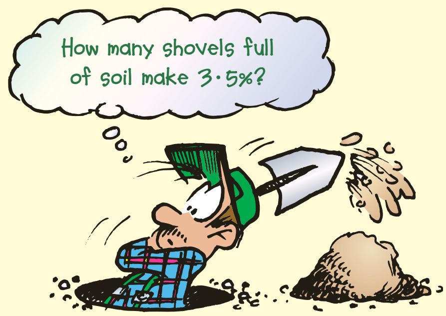
e Sonny borrowed \$6000 and paid it back over 5 years. During this time he was charged 9·5% pa simple interest on the original amount borrowed. How much interest did he pay?

f What simple interest would \$8000 earn in 3 months if the rate of interest is $11\frac{1}{4}$% pa?

g A building society pays interest daily. If an amount of \$1564 were invested at 5·5% pa, how much simple interest would be earned in:
i 1 day? ii 3 days? iii 29 days?

2 a Each year 350 people are allowed to join a club. Of these an average of 46% are women. How many women would you expect to join the club in 4 years?

b Twenty-five loads of soil, each 9·6 tonnes when loaded, were taken to the sports complex to top-dress the new oval. An average of 3·5% of each load was lost in transit. How much soil was lost?



c When James got a job, he decided to give his parents 22% of his weekly net pay. This was his contribution to family expenses. If his weekly net pay was €316.80, how much did he give his parents in the first 15 weeks?

d At church, Julia heard that a famine was causing great suffering to many people in Africa. She decided that for the next 7 months she would give 15% of her savings of €124 per month towards relief for famine victims. How much did she give?

Do not use a calculator for question 3.

3 a Morgan borrowed £7000 at 8·25% pa simple interest. Which is the best estimate of the interest charged for 6 years: £35, £350, £550 or £3500?

b Luke borrowed \$880 at 11% pa simple interest for 9 months. Which is the best estimate of the interest charged: \$8, \$80, \$800 or \$8000?

c Tess borrowed \$140 000 at 9·75% pa interest over 20 years. Which is the best estimate of the interest charged: \$2800, \$28 000, \$280 000 or \$2 800 000?

d Rajiv invested ¥4 000 000 at 4·9% pa interest for 7 months. Which is the best estimate of the interest earned: ¥10 000, ¥100 000, ¥1 000 000 or ¥10 000 000?

4 When I invest money, at the end of the time of investment I am given the principal invested plus the interest.

a How much will I receive after 3 years if I invest \$1000 at 7% pa simple interest?

b How much will I receive after 18 months if I invest \$3700 at 10% pa simple interest?

c How much will I receive after 7 months if I invest \$30 000 at 1·1% pa per month simple interest?

d How much will I receive after 10 months if I invest \$14 600 at 9·5% pa simple interest?

5 When an amount of money is borrowed over a period of time, the amount to be repaid is equal to the amount borrowed plus the interest charged.

a How much must I pay back if I borrow \$10 000 for 3 years at 10% pa simple interest?

b How much must I pay back if I borrow \$108 000 for 25 years at a simple interest rate of 6·25% pa?

c How much must I pay back if I borrow \$5000 at $8\frac{1}{2}$% simple interest for 6 years?

d How much must I pay back if I borrow \$30 000 at 12% pa simple interest for 3 months?

6 **a** Find the simple interest rate (to the nearest per cent) which will allow €2500 to earn €675 interest in 3 years.

b What sum of money would you have to invest for 5 years at 11% simple interest to produce interest at €3080?

c For how many years would \$8000 have to be invested at $11\frac{1}{2}$% simple interest to produce €5520 interest?

d What simple interest rate would allow €6000 to grow to an amount of €14 550 in 10 years?

Discussion

If the interest paid on \$50 000 is \$8000, what might be the interest rate and time period?

2:04 | Compound Interest

prep quiz 2:04

What amount would be returned after 1 year if the rate of simple interest is 10% pa and the principal invested is:

1 \$10 000? **2** \$11 000? **3** \$12 100? **4** \$13 310? **5** \$14 641?

What amount would be returned after 1 month if the rate of simple interest is 12% pa and the principal invested is:

6 \$100 000? **7** \$101 000? **8** \$102 010? **9** \$103 030.10?

10 \$2000 is invested for 1 year at 10% pa simple interest. The money received is then invested for 1 year at 10% pa simple interest. How much is received at the end of the second year?

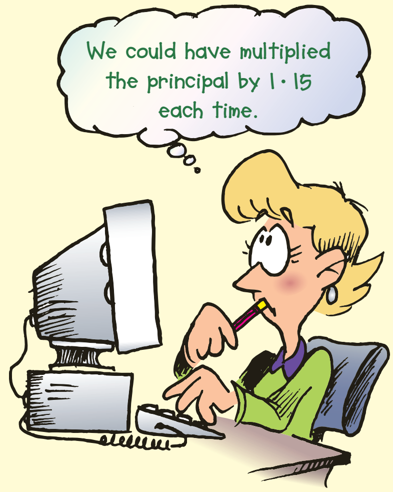
- When *simple interest* is applied, the interest earned in one year (or time period) does not itself earn interest in following years. The interest in any time period is calculated using the original principal and so is the same for each time period.
- When *compound interest* is applied, the interest earned in one year (or time period) itself earns interest in following years. The interest is added to the previous principal and so the amount of interest in each time period increases from one period to the next.

For compound interest, interest earned in previous time periods also earns interest.

Consider what happens when $40 000 is invested for 4 years at 15% pa compound interest. Find the total amount of interest earned.

Amount after 1 year = principal + interest

Amount after 1 year = $40 000 + 15% × $40 000 = $40 000 + $6000 = $46 000
Amount after 2 years = $46 000 + 15% × $46 000 = $46 000 + $6900 = $52 900
Amount after 3 years = $52 900 + 15% × $52 900 = $52 900 + $7935 = $60 835
Amount after 4 years = $60 835 + 15% × $60 835 = $60 835 + $9125.25 = $69 960.25



∴ Total interest = final amount − original principal
= $69 960.25 − $40 000
= $29 960.25

worked examples

1 Find the compound interest earned if $9000 is invested for 3 years at 13% pa if interest is compounded yearly. Answer to the nearest cent.

2 $12 000 is invested at a compound interest rate of 9% pa. Interest, however, is compounded monthly. Calculate the amount to which the investment will grow in 2 months.

Solutions

1 In each year, the principal will increase by 13%.

Amount after 1 year
= $9000 + 13% of $9000
= $10 170

Amount after 2 years
= $10 170 + 13% of $10 170
= $11 492.10

Amount after 3 years
= $11 492 + 13% of $11 492.10
≑ $12 986.07

or

Amount after 1 year
= $9000 × 1·13
= $10 170

Amount after 2 years
= $10 170 × 1·13
= $11 492.10

Amount after 3 years
= $11 492.10 × 1·13
≑ $12 986.07

∴ Interest earned in 3 years = final amount − original principal
= $12 986.07 − $9000
= $3986.07

2 9% pa is the same as $\frac{9}{12}$ % per month.
$\frac{9}{12}$ % = $\frac{3}{4}$ % = 0·75%

Amount after 1 month
= $12 000 + 0·75% of $12 000
= $12 090

Amount after 2 months
= $12 090 + 0·75% of $12 090
≑ $12 180.68

or

A_1 = $12 000 × 1·0075
= $12 090
A_2 = $12 090 × 1·0075
≑ $12 180.68

∴ Interest earned in 2 months = final amount − original principal
= $12 180.68 − $12 000
= $180.68

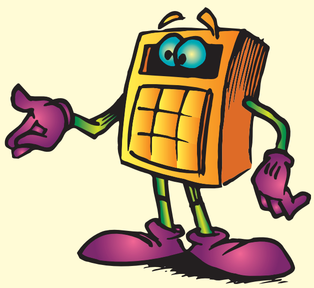
0·75% = 0·75 ÷ 100
= 0·0075

Finding compound interest by repeated multiplication using a calculator

Looking at the second method in the examples above, we can see that to calculate the amount accumulated after *n* time periods, we simply have to multiply the principal by (1 + interest rate) *n* times. The amount $9000 accumulates to at 13% pa after 3 years is given by:

Amount = $9000 × 1·13 × 1·13 × 1·13
= $12 986.07

∴ Interest earned = $12 986.07 − $9000
= $3986.07



Exercise 2:04

Foundation Worksheet 2:04
Compound interest
1 How much will $10 000 grow to at 7% pa compound interest after:
a 1 year? **b** 2 years? **c** 3 years?
2 Find the compound interest for the examples in question **1**.

1 Here, money earned at simple interest and compound interest can be compared.

Rate	*Simple interest on $100 for:*				
	2 years	*4 years*	*6 years*	*8 years*	*10 years*
10% pa	$20	$40	$60	$80	$100

Rate	*Compound interest* on $100 for:*				
	2 years	*4 years*	*6 years*	*8 years*	*10 years*
10% pa	$22.04	$48.94	$81.76	$121.82	$170.70

* compounded monthly

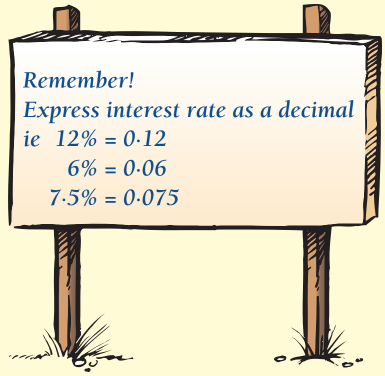
How much more interest is earned on $100 invested at 10% pa compound interest (compounded monthly) than is earned on $100 invested at 10% pa simple interest invested for a period of:

a 2 years? **b** 6 years? **c** 8 years? **d** 10 years?

In these questions, interest is compounded yearly.

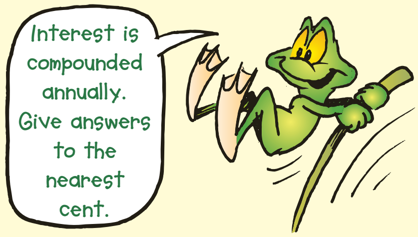
2 By repeated multiplication by (1 + interest rate) find how much each principal accumulates to, for the given number of years (see page 38).

- **a** €2000 at 10% pa for 2 years
- **b** €5000 at 12% pa for 3 years
- **c** €10 000 at 7% pa for 2 years
- **d** €50 000 at $6\frac{1}{2}$% pa for 3 years
- **e** €25 000 at 7·5% pa for 4 years



3 Find the amount received and the compound interest earned if:

- **a** \$400 is invested for 2 years at 12% pa
- **b** \$1900 is invested for 3 years at 13% pa
- **c** \$5350 is invested for 2 years at 9% pa
- **d** \$100 is invested for 3 years at 8% pa
- **e** \$2874 is invested for 2 years at 7·5% pa
- **f** \$650 000 is invested for 2 years at 6% pa
- **g** \$85 700 is invested for 3 years at $12\frac{1}{2}$ pa



4 Naomi invested \$5000 for 4 years at a rate of 10% pa compound interest (compounded yearly). To what amount did the investment grow in this time and what was the total interest earned?

5 Joni was given 8% pa compound interest (compounded yearly) on an investment of \$80 000 over 3 years. What interest did she earn altogether?

6 Find the amount of compound interest earned (to the nearest cent) if:

- **a** £400 is invested for 3 months at 1% per month, compounded monthly
- **b** £53 000 is invested for 2 months at 0·8% per month, compounded monthly
- **c** £8000 is invested at 12% pa for 2 months, compounded monthly
- **d** £2870 is invested at 6% pa for 3 months, compounded monthly

7 Luke had to decide between investing his \$1000 at a simple interest rate of 11% pa for 4 years or investing it at 10% pa compounded interest for the same period of time. Which would be the better investment and by how much?

8 The population of Caramel Creek is 200 and is expected to increase each year by 80% of the previous year's population. What is the expected population in 3 years' time?

9 The value of a block of land has increased by 20% of the previous year's value in each of the last 4 years. If, 4 years ago, the value of the land was \$100 000, what is its value now?

10 If we assume an inflation rate of 8% pa, what would you expect to pay in 3 years' time for:

- **a** a garage that now costs \$30 000?
- **b** a container of ice-cream that now costs \$4.50?
- **c** a toothbrush that now costs \$2.85?

Fun Spot 2:04 | What is the difference between a book and a bore?

Answer each question and put the letter for that question in the box above the correct answer.

Write as a percentage:

H $\frac{1}{20}$ **O** $\frac{3}{4}$

T 0·85 **U** 0·02

Write as a decimal:

O 3% **U** 12·5%

Calculate:

A 1% of \$1 **B** 15% of \$1

C 4% of \$810 **E** $5\frac{1}{2}$% of \$8000

Find the simple interest earned by investing \$100 at 12% pa for:

H 1 year **K** 4 years **N** 6 months

Interest of 10·25% pa was charged on a loan, where interest was calculated daily. What percentage interest (given to three decimal places) would be charged for:

O one day? **P** the month of April? (Take 1 year to be 365 days.)

S Given that $I = PRT$, find the value of R if $I = 180$, $P = 600$ and $T = 3$.

A watch was bought for \$200. What was its value after one year if it depreciated:

T 20%? **U** 11%? **Y** 1·5%?

\$197	0·03	2%	\$32.40	\$0.01	\$6	10%	\$12	0·125	85%	\$178	0·842%	\$160	5%	\$440	\$0.15	0·028%	75%	\$48

• Interest rates change all the time.

2:05 | Depreciation

prep quiz 2:05

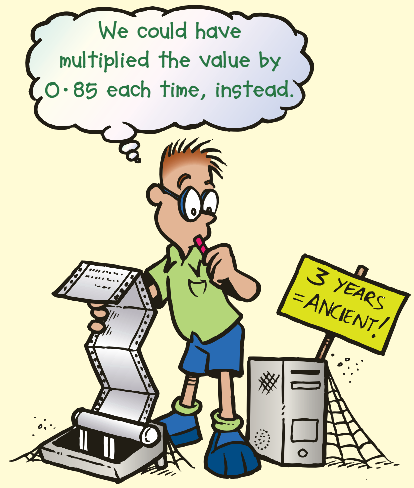
1 100% − 20%
2 100% − 7%
3 20% + 80%
4 7% + 93%
5 Reduce $350 by 20%.
6 Find 80% of $350.
7 Reduce $800 by 7%.
8 Find 93% of $800.
9 Is reducing an amount by 20% the same as finding 80% of the amount?
10 Is reducing an amount by 7% the same as finding 93% of the amount?

- When the value for an item decreases year by year it is said to be **depreciating**. Many of the things we own depreciate at more or less a constant rate. This may be caused by age or by the availability of new models.
- **Depreciation** is the loss in value of an object over a period of time.

Consider what happens when machinery worth $40 000 depreciates for 4 years at a rate of 15% pa. Find the total amount of depreciation.

Value after 1 year
= original value − depreciation for that year

Value after 1 year = $40 000 − 15% × $40 000 = $40 000 − $6000 = $34 000
Value after 2 years = $34 000 − 15% × $34 000 = $34 000 − $5100 = $28 900
Value after 3 years = $28 900 − 15% × $28 900 = $28 900 − $4335 = $24 565
Value after 4 years = $24 565 − 15% × $24 565 = $24 565 − $3684.75 = $20 880.25



∴ Total depreciation = original value − final value
= $40 000 − $20 880.25
= $19 119.75

worked examples

1 A new computer depreciates by 30% per year. If it costs $4000 new, what will it be worth in 3 years?

2 If the population of Sunset Heights is decreasing by 20% of its population every year, what would be the population in two years if it is now 700?

Solutions

1 Each year the value decreases by 30%.

Value after 1 year
= \$4000 − 30% of \$4000
= \$2800

or

Value after 1 year
= 70% of \$4000
= \$2800

Value after 2 years
= \$2800 − 30% of \$2800
= \$1960

Value after 2 years
= 70% of \$2800
= \$1960

Value after 3 years
= \$1960 − 30% of \$1960
= \$1372

Value after 3 years
= 70% of \$1960
= \$1372

∴ The value of the computer after 3 years is \$1372.

2 After a decrease of 20%, 80% of the population remains.

Population after 1 year
= 700 − 20% of 700
= 560

or

Population after 1 year
= 80% of 700
= 560

Population after 2 years
= 560 − 20% of 560
= 448

Population after 2 years
= 80% of 560
= 448

∴ The population of Sunset Heights after two years would be 448.

Exercise 2:05

1 **a** In the first year after purchase, a car costing €14 300 depreciated 18%. What was its value after one year?
b Find the depreciation during the first year if a ring bought for €2300 depreciated at a rate of 9·3% pa.

2 **a** Find the value of a yacht after 2 years if its original value was \$136 000 and the rate of depreciation is 12% pa.
b A machine is purchased for \$8160. If it depreciates at a rate of 8% pa, what will be its value after 2 years?
c Furniture purchased for \$1350 depreciates at a rate of 11% pa. What would be its value after 3 years?
d Find the value of a teacher's library after 3 years if its original value was \$5800 and it depreciated at a rate of 10% pa.
e If the beaver population of our district is 840 and is dropping by 12% pa, how many beavers would you expect to have in the district in 2 years?
f The population of Smallville is dropping at a rate of 4% pa. If its population in 2000 was 56 700, what was its likely population in the year 2003?

3 Find the depreciation (ie loss) for each of the examples in question **2**.

4 **a** Elena bought a video camera for \$1350. What would be the value of the video camera after 3 years if it depreciated at a rate of 16% pa?

b If the \$1350 had been invested for 3 years at 6% pa compound interest, what amount would she have received?

5 **a** Tim bought a second-hand Mercedes for \$15 000. If it depreciates at a rate of 3% pa, what would be its value in 2 years?

b If \$15 000 is invested at 5% pa compound interest (just covering the inflation rate), what would it grow to in 2 years?

6 Julia and Lillian each invested £5000. Julia's investment depreciated at a rate of 3% pa for 3 years, while during that time Lillian's investment grew at a rate of 9% pa (compounded yearly). Find the difference in the value of their investments at the end of the 3 years.

7 Which of the rates 5% pa, 6% pa, 7% pa or 8% pa would cause a car valued at \$6000 to drop in value to \$4826 in 3 years?

8 How many years would it take to cause a machine valued at ¥1 100 000 to drop below ¥700 000 in value if the rate of depreciation is 14% pa?

2:06 | Compound Interest and Depreciation Formulae

prep quiz 2:06

Write each percentage as a decimal.

1 9% **2** $6\frac{1}{2}$% **3** 11·25%

Give the rate per month correct to six decimal places if the rate is:

4 9% pa **5** 8% pa **6** 6·5% pa

Evaluate:

7 $1 + r$, if $r = 0{\cdot}15$ **8** $1 - r$, if $r = 0{\cdot}15$

9 $P(1 + r)^n$, if $P = 40\ 000$, $r = 0{\cdot}15$ and $n = 4$

10 $P(1 - r)^n$, if $P = 40\ 000$, $r = 0{\cdot}15$ and $n = 4$

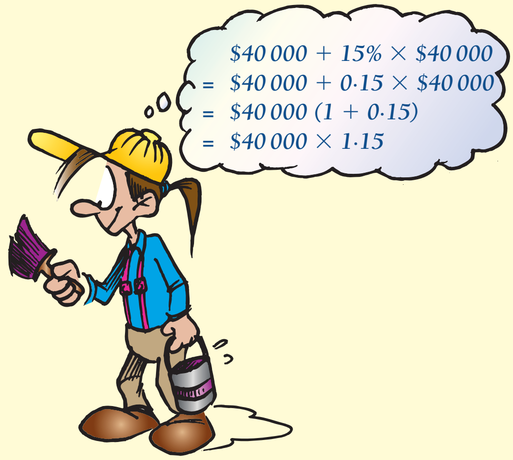
Compound interest formula

On page 38 we saw what happens when \$40 000 is invested for 4 years at 15% pa compound interest.

We can find the amount to which the investment grows in each year by multiplying the amount of the previous year by 1·5 (which is 1 + 15%).

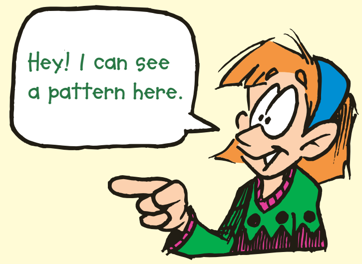
Amount after 1 year = principal + interest

Amount after 1 year = \$40 000 + 15% of \$40 000 = \$40 000 × 1·15 = \$46 000
Amount after 2 years = \$46 000 + 15% of \$46 000 = \$46 000 × 1·15 = \$52 900
Amount after 3 years = \$52 900 + 15% of \$52 900 = \$52 900 × 1·15 = \$60 835
Amount after 4 years = \$60 835 + 15% of \$60 835 = \$60 835 × 1·15 = \$69 960.25



The working above could have been shown as:

Amount after 1 year = \$40 000$(1 + 0{\cdot}15)^1$
Amount after 2 years = \$40 000$(1 + 0{\cdot}15)^2$
Amount after 3 years = \$40 000$(1 + 0{\cdot}15)^3$
Amount after 4 years = \$40 000$(1 + 0{\cdot}15)^4$
= \$69 960.25



Clearly if \$$P$ were to be invested at a rate of r per year, then the amount A to which the investment would grow in n years would be given by $A = P(1 + r)^n$.

Compound interest formula

$$A = P(1 + r)^n$$

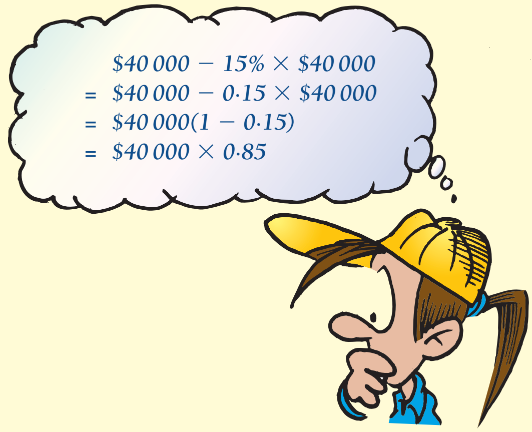
where **P is the principal invested**
n is the number of time periods
r is the rate of compound interest for one time period
A is the amount after n time periods.

Depreciation formula

On page 42 we saw what happens to the value of machinery worth \$40 000 when it depreciates for 4 years at a rate of 15% pa. We can find the value of the machinery each year by multiplying the value of the previous year by 0·85 (which is 1 − 15%).

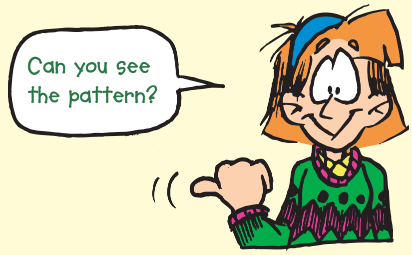
Value after 1 year = original value − depreciation for that year

Value after 1 year = \$40 000 − 15% of \$40 000 = \$40 000 × 0·85 = \$34 000
Value after 2 years = \$34 000 − 15% of \$34 000 = \$34 000 × 0·85 = \$28 900
Value after 3 years = \$28 900 − 15% of \$28 900 = \$28 900 × 0·85 = \$24 565
Value after 4 years = \$240 565 − 15% of \$24 565 = \$24 565 × 0·85 = \$20 880.25



The working above could have been shown as:

Value after 1 year $= \$40\,000(1 - 0{\cdot}15)^1$
Value after 2 years $= \$40\,000(1 - 0{\cdot}15)^2$
Value after 3 years $= \$40\,000(1 - 0{\cdot}15)^3$
Value after 4 years $= \$40\,000(1 - 0{\cdot}15)^4$
$= \$20\,880.25$



Clearly, if an item of value $\$P$ were to depreciate at a rate r per year, then the value A, after n years, would be given by $A = P(1 - r)^n$.

Depreciation formula
$$A = P(1 - r)^n$$

where **P is the original value**
n is the number of time periods
r is the rate of depreciation for one time period
A is the value after n time periods.

worked examples

Compound interest

1. Find the compound interest earned if \$9000 is invested for 5 years at 13% pa. (Answer to the nearest cent.)
2. \$12 500 is invested at a compound interest rate of 9% pa. Interest, however, is compounded monthly. Calculate the amount to which the investment will grow in 4 years.

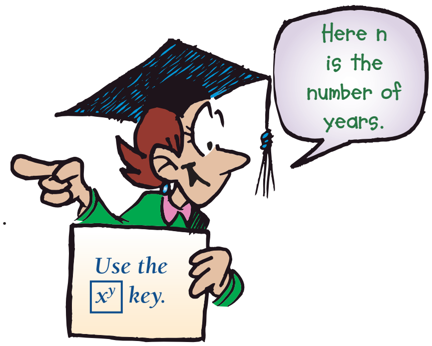
Solutions

1 Here:

$P = \$9000$
$n = 5$
$r = 13\%$
$= 0{\cdot}13$

$A = P(1 + r)^n$
$= \$9000(1 + 0{\cdot}13)^5$
$= \$9000(1{\cdot}13)^5$
$= \$16\,581.92$
$\therefore$ interest $= \$16\,581.92 - \$9000$
$= \$7581.92$

$\therefore$ The compound interest earned in 5 years is \$7581.92.



2 Here:

$P = \$12\,500$
Time = 4 years
= 48 months
$\therefore n = 48$
Rate = 9% pa
= 9 ÷ 12% per month
= 0·75% per month
$\therefore r = 0{\cdot}0075$

$A = P(1 + r)^n$
$= \$12\,500(1 + 0{\cdot}0075)^{48}$
$= \$12\,500(1{\cdot}0075)^{48}$
$= \$17\,892.567$
$\doteqdot \$17\,892.57$

$\therefore$ The amount to which the investment will grow in 4 years is \$17 892.57.

Depreciation

1 Adam buys a second-hand car for \$3400. What will the car be worth in 5 years if each year it depreciates 18%?

2 Carol paid \$2200 for a new video unit. What would be the value of the unit in 6 years if its rate of depreciation is 28% pa?

Solutions

1 Here $P = \$3400$
$n = 5$
$r = 18\%$
$= 0{\cdot}18$

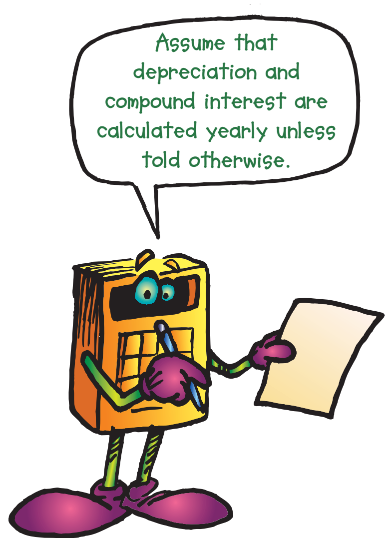
$A = P(1 - r)^n$
$= \$3400(1 - 0{\cdot}18)^5$
$= \$3400(0{\cdot}82)^5$
$\doteqdot \$1260.52$

$\therefore$ The value of the car in 5 years is \$1260.52.

2 Here $P = \$2200$
$n = 6$
$r = 28\%$
$= 0{\cdot}28$

$A = P(1 - r)^n$
$= \$2200(1 - 0{\cdot}28)^6$
$= \$2200(0{\cdot}72)^6$
$\doteqdot \$306.49$

$\therefore$ After 6 years, the value of the unit is \$306.49.



Exercise 2:06

> **Foundation Worksheet 2:06**
> **Compound interest formula**
> 1 Use the formula $A = P(1 + r)^n$ to find the amount received if:
> **a** \$10 000 is invested at 6% pa for 7 years.
> 2 Find the interest for each part in question 1.

1 Use the formula $A = P(1 + r)^n$ to find the amount received if:
- **a** \$4000 is invested for 5 years at 12% pa
- **b** \$860 is invested for 7 years at 7% pa
- **c** \$17 360 is invested for 20 years at 6·25% pa
- **d** \$476.50 is invested for 6 years at $5\frac{1}{2}$% pa

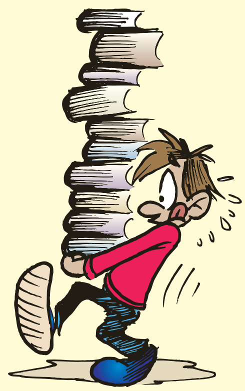
2 Find the interest earned for each of the investments in question **1**.

3 Use the formula $A = P(1 - r)^n$ to find the value of an item worth:
- **a** €3000 after it depreciates 9% pa for 12 years
- **b** €35 000 after it depreciates 25% pa for 4 years
- **c** €465 after it depreciates 30% pa for 7 years

4 Find the amount of compound interest earned (to the nearest cent) if:
- **a** \$400 is invested for 5 years at 12% pa compounded monthly
- **b** \$5350 is invested for 10 years at 9% pa compounded monthly
- **c** \$2874 is invested for 20 years at 7·5% pa compound monthly
- **d** \$500 is invested for 6 years at 8% pa compounded half-yearly

5 Jenny Walsh discovered that she was to inherit the amount to which an investment of her great-grandfather had grown. He had invested 50 pounds at 9% pa compound interest, 100 years ago. Find how much she will receive if one pound is equal to two dollars. (Assume that interest is compounded annually.)

6 Neo Wang had to decide between investing his 1000 Malaysian ringitt at a simple interest rate of 16% pa for 8 years or investing it at 11% pa compound interest for the same period of time. Which would be the better investment and by how much?



7 A library depreciates at a rate of 15% pa. If its value now is \$18 700, what will be its value in:
- **a** 8 years? **b** 15 years? **c** 21 years?

8 What would be the value of a \$20 million ship after 10 years if its rate of depreciation is:
- **a** 9% pa? **b** 18% pa? **c** $17\frac{1}{2}$% pa?

9 After having depreciated at a rate of 12% pa, a printing machine is now worth \$2000. What was the machine's value 7 years ago?

10 A library now worth €17 600 has been depreciating at a rate of 9% pa for the last 8 years. What was its value 8 years ago?

11 \$1000 is invested for 5 years at 12% pa compound interest. To what does this investment grow if interest is compounded:
- **a** yearly? **b** 6 monthly? **c** 3 monthly?
- **d** monthly? **e** fortnightly? **f** daily?

Is there a limit to which the investment can grow as we reduce the time period for compounding the interest? (Consider compounding the interest each hour, minute and second.)

12 Jegan invests \$10 000 at 9% pa simple interest while Su-Lin invests \$10 000 at 9% pa compounded monthly. What is the difference in the value of their investments after:
- **a** 1 year? **b** 5 years? **c** 10 years?

Investigation 2:06 | Compound interest tables

investigation 2:06

If a calculator with a power key or a computer is not available, a table could be used that shows the effects of compound interest. The sample table below shows the amount $1 will grow to for various interest rates and time periods.

Accumulated value of $1 (to four decimal places)									
Number of time periods	*Interest rate*								
	0·5%	*1%*	*5%*	*6%*	*7%*	*8%*	*10%*	*12%*	*15%*
1	1·0050	1·0100	1·0500	1·0600	1·0700	1·0800	1·1000	1·1200	1·1500
2	1·0100	1·0201	1·1025	1·1236	1·1449	1·1664	1·2100	1·2544	1·3225
3	1·0151	1·0303	1·1576	1·1910	1·2250	1·2597	1·3310	1·4049	1·5209
4	1·0202	1·0406	1·2155	1·2625	1·3108	1·3605	1·4641	1·5735	1·7490
5	1·0253	1·0510	1·2763	1·3382	1·4026	1·4693	1·6105	1·7623	2·0114
6	1·0304	1·0615	1·3401	1·4185	1·5007	1·5869	1·7716	1·9738	2·3131
7	1·0355	1·0721	1·4071	1·5036	1·6058	1·7138	1·9487	2·2107	2·6600
8	1·0407	1·0829	1·4775	1·5938	1·7182	1·8509	2·1436	2·4760	3·0590
9	1·0459	1·0937	1·5513	1·6895	1·8385	1·9990	2·3579	2·7731	3·5179
10	1·0511	1·1046	1·6289	1·7908	1·9672	2·1589	2·5937	3·1058	4·0456
11	1·0564	1·1157	1·7103	1·8983	2·1049	2·3316	2·8531	3·4785	4·6524
12	1·0617	1·1268	1·7959	2·0122	2·2522	2·5182	3·1384	3·8960	5·3503
18	1·0939	1·1961	2·4066	2·8543	3·3799	3·9960	5·5599	7·6900	12·3755
24	1·1272	1·2697	3·2251	4·0489	5·0724	6·3412	9·8497	15·1786	28·6252

Examples

Find the accumulated value and hence the compound interest earned when:

1 $10 000 is invested at 8%pa, interest compounded annually for 10 years

2 $50 000 is invested at 12% pa, interest compounded monthly for 18 months

Solutions

1 If interest rate = 8% and time periods = 10, then $1 grows to $2.1589
∴ $10 000 will grow to:
$2.1589 × 10 000 = $21 589
∴ Interest = $21 589 − $10 000
= $11 589

2 If interest rate = 12% pa, then interest rate per month = 1%
So for 18 months, $1 will grow to $1.1961
∴ $50 000 will grow to:
$1.1961 × 50 000 = $59 805
∴ Interest = $59 805 − $50 000
= $9805

Exercises

1 Find the accumulated amount and the interest earned when:
   a \$20 000 is invested at 7% pa, interest compounded annually for 12 years
   b \$150 000 is invested at 10% pa, interest compounded annually for 24 years
   c \$7500 is invested at 6% pa, interest compounded annually for 8 years

2 Find an accumulated amount when \$100 000 is invested at:
   a 12% pa for 2 years, interest compounded annually
   b 1% per month for 24 months, interest compounded monthly

3 Why are the answers different for parts **a** and **b** of question **2** when the interest rate and terms are the same? (2 years = 24 months, 12% pa = 1% per month)

2:07 | Reducible Interest

prep quiz 2:07

Find:

1 15% of \$7000	2 9% of \$12 000	3 $7\frac{1}{2}$% of \$5600

What is the monthly interest rate, if the annual rate is:

4 12%	5 9%	6 8%

Decrease:

7 \$1000 by 10%	8 \$20 000 by 5%	9 \$7000 by 1%	10 \$12 000 by 0·75%

When money is borrowed via a loan it is usually repaid using a method called **reducible interest**. Interest is only calculated on the amount still owing on the loan at any point in time. So, as the loan is paid off, the amount of interest charged reduces. This can be seen in the following example.

worked examples

Example 1

A loan of \$10 000 is charged 10% pa interest and is repaid by annual instalments of \$1500.

∴ The amount of interest after 1 year = \$10 000 × 10%
= \$1000

So, principal plus interest after 1 year = \$10 000 + \$1000
= \$11 000

∴ After repayment is made, amount owing = \$11 000 − \$1500
= \$9500

Now, with reducible interest, the amount of interest for the second year is calculated on a principal, or amount owing, of \$9500.

∴ Amount of interest for second year = \$9500 × 10%
= \$950

So, principal plus interest after 2 years = \$9500 + \$950
= \$10 450

∴ After repayment is made, amount owing = \$10 450 − \$1500
= \$8950

Continuing this process, the interest charged and the amount still owing for the first 5 years of the loan are shown in the following table.

At the end of:	*Interest charged*	*Amount still owing*
1 year	\$1000	\$9500
2 years	\$ 950	\$8950
3 years	\$ 895	\$8345
4 years	\$ 834.50	\$7679.50
5 years	\$ 767.95	\$6947.45

From the table, it can be seen that the amount of interest charged reduces even though the interest rate stays the same. This is because the interest is being calculated only on the amount still owing, which is itself reducing.

Example 2

A home loan of \$300 000 is charged interest at the rate of 9% pa. The interest is calculated monthly and repayments of \$2520 are made at the end of each month so that the loan is paid off after 25 years.

Calculate the amount still owing at the end of 3 months (after the first 3 repayments).
Also determine the total amount of interest paid and the equivalent flat (simple) interest rate.

Solution

The monthly interest rate $= 9\% \div 12$

$= \frac{3}{4}\%$ or $0{\cdot}0075$

- Amount owing after first month = principal + interest − repayment
  $= \$300\,000 + \$300\,000 \times 0{\cdot}0075 - \$2520$
  $= \$300\,000 + \$2250 - \$2520$
  $= \$299\,730$
- Amount owing after second month $= \$299\,730 + \$299\,730 \times 0{\cdot}0075 - \$2520$
  $= \$299\,457.97$
- Amount owing after third month $= \$299\,457.97 + \$299\,457.97 \times 0{\cdot}0075 - \$2520$
  $= \$299\,183.90$

So, the amount still owing after 3 payments of \$2520 (\$7560) is \$299 183.90. It appears that less than \$900 has been paid off the original principal. This is because most of the repayment amount has paid the interest charged.

At the end of 25 years, 300 payments of \$2520 will have been paid, a total of \$756 000. Since the original loan was for \$300 000, the amount of interest paid must have been \$456 000.

continued →→→

To calculate the flat interest rate:

$$\text{Amount of interest for 1 year} = \$456\,000 \div 25$$
$$= \$18\,240$$
$$\therefore \text{Flat interest rate per annum} = \frac{\$18\,240 \times 100\%}{\$300\,000}$$
$$= 6{\cdot}08\%$$

So, for this loan, a flat interest rate of 6·08% pa is equivalent to a reducible interest rate of 9% pa.

Exercise 2:07

1 A loan of $100 000 is charged 10% pa interest. Repayments of $12 000 are paid at the end of each year.

- **a** How much interest is charged at the end of the first year?
- **b** How much is still owed on the loan after the first repayment?
- **c** Using your answer to part **b**, calculate the interest charged for the second year.
- **d** Therefore, how much is still owing after the second repayment?

2 Reducible interest of 8% pa is charged on a €30 000 loan for a car. Repayments of €3000 are paid annually.

- **a** By calculating the interest for one year and subtracting the first repayment, determine how much is still owing at the end of the first year.
- **b** Using your answer to part **a** as the new principal, calculate the amount that will be owing after 2 years.
- **c** How much interest has been paid altogether for the first 2 years?
- **d** Calculate 8% pa simple interest on €30 000 for 2 years.
- **e** How much less interest is paid using reducible interest?

3 A loan of $50 000 is paid off by paying 6% pa reducible interest. Annual repayments of $6000 are paid at the end of each year.

- **a** Determine the amount still owing at the end of the third year.
- **b** How much interest is paid altogether?

4 A home loan of $250 000 is paid off by monthly repayments of $3000. The interest rate is 12% pa (1% per month).

- **a** Determine the amount still owing after the first monthly payment.
- **b** How much would still be owing after the third monthly payment?

5 £80 000 is borrowed at an interest rate of 6% pa. Repayments of £700 are paid monthly. Determine the amount still owing after 3 months.

6 A loan of \$10 000 is borrowed at an interest rate of 7% pa.

a Calculate the interest owed after one year and hence the total amount owing.

b A repayment of \$700 is made at the end of the first year. How much is now owing at the beginning of the second year?

c If the same interest rate and annual repayment continues to be made, how much will be still owing after

i 2 years?

ii 5 years?

iii 20 years?

Loans like those in question 6 are called *interest-only loans*. The principal remains the same and the repayments only pay the interest charged.

7 \$150 000 is borrowed to buy an investment property. The interest rate is 7·5% pa and repayments will be monthly.

a Calculate the interest for the first month.

b How much must be repaid each month if this is to be an interest-only loan. (See question **6**.)

To calculate the amount owing beyond the first few payments for reducible interest loans is tedious. An easy way to see how a loan progresses for a large number of time periods is to use a spreadsheet. (See Investigation 2:07.) You might like to try the following questions to test your accuracy and ability.

8 A loan of \$10 000 is repaid by annual repayments of \$2638. If the interest rate is 10% pa reducible, determine the number of repayments that are needed to repay the loan.

9 Quarterly instalments of \$600 are used to repay a loan of \$20 000. The annual interest rate is 8% (2% per quarter).

a Determine the amount still owing at the end of the first year, that is, after 4 quarterly repayments.

b If interest was calculated annually and repayments of \$2400 were made at the end of each year, how much would still be owing after the first year?

c Noting your answers to parts **a** and **b**, is it better to pay quarterly or to make annual repayments?

investigation 2:07

Investigation 2:07 | Reducible home loan spreadsheet

The table below shows the first few lines of a spreadsheet that tracks the progress of a loan for $180 000 at an interest rate of 6% per annum with monthly repayments of $1700.

The initial principal can be changed by typing a different amount into cell B4 and the repayment can be changed in cell B1. The interest rate can be changed in column C.

The formulae for particular cells are shown.

Formulae: D4 =B4*C4/1200; E4 =B4+D4; F4 =B1; G4 =E4–F4; B5 =G4

	A	B	C	D	E	F	G
1	Repayment	$1,700.00					
2							
3	N	PRINCIPAL(P)	Interest Rate r	INTEREST(I)	P+I	R	P+I–R
4	1	$180,000.00	6.00	$900.00	$180,900.00	$1,700.00	$179,200.00
5	2	$179,200.00	6.00	$896.00	$180,096.00	$1,700.00	$178,396.00
6	3	$178,396.00	6.00	$891.98	$179,287.98	$1,700.00	$177,587.98
7	4	$177,587.98	6.00	$887.94	$178,475.92	$1,700.00	$176,775.92
8	5	$176,775.92	6.00	$883.88	$177,659.80	$1,700.00	$175,959.80
9	6	$175,959.80	6.00	$879.80	$176,839.60	$1,700.00	$175,139.60
10	7	$175,139.60	6.00	$875.70	$176,015.30	$1,700.00	$174,315.30

The formulae shown are applied down each column and the interest rate is also copied down column C.

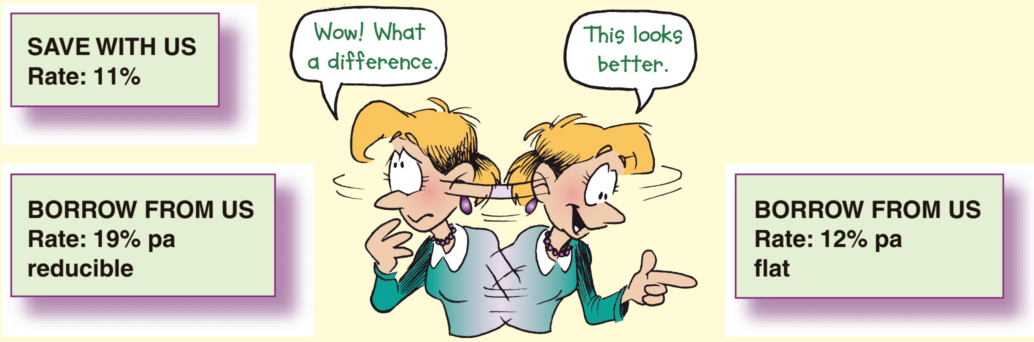
This spreadsheet can be copied from the student CD-ROM.

Experiment with this spreadsheet, changing either the interest rate and/or the repayment to see the effect on the loan.

2:08 | Borrowing Money

Lending institutions such as banks, building societies, credit unions and finance companies borrow money from investors in order to lend at a higher rate. The difference between the rate they offer to investors and the rate they charge for loans must pay the salary of employees, cover advertising, provide facilities, pay taxes and produce a profit as well.

As interest rates for investments rise, so too will the rates for borrowing money.



- *Reducible interest* is calculated on the amount still owing. The interest to be paid is based on the amount owing at the time.
- *Flat rate* is calculated on the original loan even when the loan is almost repaid in full.

Note: A reducible interest rate of 7% pa with monthly rests (ie instalments paid monthly) is equivalent to a flat rate of:

- 3·73% pa paid over 2 years
- 3·76% pa paid over 5 years
- 3·93% pa paid over 10 years
- 4·30% pa paid over 20 years

■ Beware! 5% pa flat rate may be more than 9% pa reducible.

■ A *flat rate* of interest is *simple* interest.

Changing reducible rates to equivalent flat rates

Reducible rate conditions for the following table:

- Interest is calculated and added monthly to the amount owing.
- The monthly reducible rate is taken to be $\frac{1}{12}$ of the yearly rate.
- Equal monthly repayments are made throughout the time period.

Rates are expressed as percentages but they can be changed to decimals.

Annual reducible rate	*Monthly reducible rate*	*Equivalent annual flat rate (correct to 2 dec. pl.) (for monthly rate, divide annual rate by 12)*			
		Over 2 years	*Over 5 years*	*Over 10 years*	*Over 20 years*
4% (0·04)	$0\cdot\dot{3}\%$	2·11%	2·10%	2·15%	2·27%
5% (0·05)	$0\cdot41\dot{6}\%$	2·65%	2·65%	2·73%	2·92%
6% (0·06)	0·5%	3·18%	3·20%	3·32%	3·60%
7% (0·07)	$0\cdot58\dot{3}\%$	3·73%	3·76%	3·93%	4·30%
8% (0·08)	$0\cdot\dot{6}\%$	4·27%	4·33%	4·56%	5·04%
9% (0·09)	0·75%	4·82%	4·91%	5·20%	5·80%
12% (0·12)	1·0%	6·49%	6·69%	7·22%	8·21%
16% (0·16)	$1\cdot\dot{3}\%$	8·76%	9·18%	10·10%	11·70%
20% (0·2)	$1\cdot\dot{6}\%$	11·07%	11·79%	13·19%	15·39%

worked examples

1 A loan of $8000, taken at 8% pa reducible interest (added monthly), is to be repaid in monthly instalments over 5 years.

- **a** Find (using the table on page 155) the equivalent flat rate of interest.
- **b** Use the flat rate in part **a** to calculate the interest charged for the 5 years.
- **c** Find the size (to the nearest dollar) of each monthly instalment.

2 Find the size of each monthly repayment (to the nearest dollar) on a loan of $35 800 taken at a rate of 7% pa reducible, over 20 years.

Solutions

1 **a** From the table, 8% pa reducible over 5 years is approximately 4·33% pa flat.

b Calculating interest at a flat rate is the same as calculating simple interest.

$$\begin{aligned} \therefore\ I &= PRT \\ &= \$8000 \times 4{\cdot}33\% \times 5 \\ &= \$8000 \times 0{\cdot}0433 \times 5 \\ &= \$1732 \end{aligned}$$

■ *P* is the amount borrowed.
R is the flat rate pa.
T is the number of years.

∴ Interest of $1732 was charged for the 5 years.

c

$$\begin{aligned} \text{Amount to be repaid} &= \text{amount borrowed} + \text{interest charged} \\ &= \$8000 + \$1732 \\ &= \$9732 \\ \text{Number of repayments} &= 5 \times 12 \text{ (months in 5 years)} \\ &= 60 \\ \text{Each monthly instalment} &= \frac{\text{amount to be repaid}}{\text{number of repayments}} \\ &= \frac{\$9732}{60} \\ &= \$162 \text{ (to the nearest dollar)} \end{aligned}$$

∴ Each monthly instalment is $162 to the nearest dollar.

2 From the table, 7% pa reducible, over 20 years, is equivalent to a flat rate of 4·30% pa.

$$\begin{aligned} \text{Interest charged} &= PRT \\ &= \$35\,800 \times 4{\cdot}30\% \times 20 \\ &= \$35\,800 \times 0{\cdot}0430 \times 20 \\ &= \$30\,788 \\ \text{Amount to be repaid} &= \text{amount borrowed} + \text{interest charged} \\ &= \$35\,800 + \$30\,788 \\ &= \$66\,588 \\ \text{Number of repayments} &= 20 \times 12 \text{ (months in 20 years)} \\ &= 240 \\ \text{Each monthly instalment} &= \frac{\text{amount to be repaid}}{\text{number of instalments}} \\ &= \frac{\$66\,588}{240} \\ &= \$277 \text{ (to the nearest dollar)} \end{aligned}$$

∴ $277 (to the nearest dollar) is the size of each monthly instalment.

Exercise 2:08

1 Use the table on page 57 to find approximate reducible interest rates equivalent to the following flat rates, if the loan is repaid in monthly instalments over 2 years.

- **a** 2·65% pa flat
- **b** 3·73% pa flat
- **c** 4·27% pa flat
- **d** 6·49% pa flat
- **e** 8·76% pa flat
- **f** 11·07% pa flat

2 As in question 1, find approximate equivalent reducible rates if these instalments are over 5 years.

- **a** 2·65% pa flat
- **b** 3·76% pa flat
- **c** 4·33% pa flat
- **d** 6·69% pa flat
- **e** 9·18% pa flat
- **f** 11·79% pa flat

3 Use the table on page 57 to find the flat rate of interest equal to the reducible rate of:

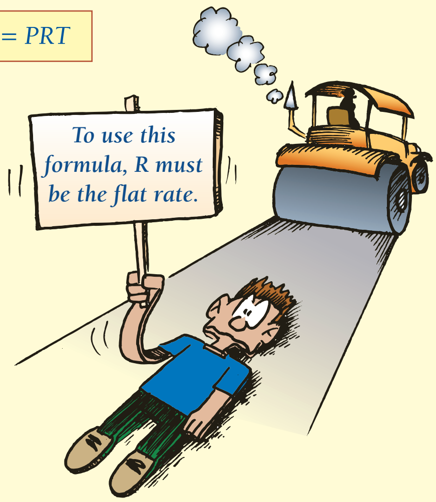
- **a** 4% pa over 10 years
- **b** 6% pa over 2 years
- **c** 9% pa over 5 years
- **d** 12% pa over 20 years
- **e** 16% pa over 2 years
- **f** 7% pa over 10 years
- **g** 20% pa over 5 years
- **h** 8% pa over 20 years

- Assume that reducible interest is charged monthly.
- Assume that repayments are made monthly.

4 Use the formula $I = PRT$ to calculate the interest payable on a loan of:

- **a** \$12 000 at 3·20% pa flat over 5 years
- **b** \$45 000 at 5·80% pa flat over 20 years
- **c** \$18 000 at 2·65% pa flat over 2 years
- **d** \$156 000 at 5·20% pa flat over 10 years
- **e** \$8750 at 6·49% pa flat over 2 years
- **f** \$6700 at 2·10% pa flat over 5 years
- **g** \$16 500 at 13·19% pa flat over 10 years
- **h** \$54 000 at 8·21% pa flat over 20 years

$I = PRT$



5 Use the table on page 57 to determine the lower rate of interest for a loan taken over 5 years.

- **a** 16% pa reducible or 9% pa flat
- **b** 6% pa reducible or 3% pa flat
- **c** 4% pa flat or 7% pa reducible
- **d** 11% pa flat or 20% pa reducible
- **e** 9% pa reducible or 5% pa flat
- **f** 12% pa reducible or 7% pa flat

6 \$5000 was borrowed at an interest rate of 9·51% pa flat. The loan plus interest was repaid in monthly instalments over 5 years.

- **a** Find the interest paid over the 5 years. (Use $I = PRT$.)
- **b** How much was repaid altogether?
- **c** How much was each monthly repayment?

7 Lina borrowed €70 000 at 9% pa reducible interest to be repaid in monthly instalments over 20 years.

a Find (using the table on page 57) the equivalent flat rate of interest.
b Use the flat rate in part **a** to calculate the interest charged.
c Find the size (to the nearest euro) of each monthly instalment.

8 Find the size of each monthly repayment (to the nearest dollar) on a loan of:

a \$45 000, taken at 6% pa reducible over 20 years
b \$9600, taken at 8% pa reducible over 5 years
c \$17 300, taken at 16% pa reducible over 10 years
d \$156 000, taken at 12% pa reducible over 2 years
e \$356, taken at 20% pa reducible over 2 years

9 The Hollier family wished to borrow \$12 000 for a new car. After being told that to receive a loan they must pay for insurance, an establishment fee, a registration and handling fee and government loan duty, they began to ask more questions. They discovered that the interest rate was 11·79% pa flat and that the loan would be repaid in monthly instalments over 5 years.

a What amount would be paid each month?
b Use the table on page 120 to find the equivalent reducible rate to the nearest whole per cent.

10 Qing was told that a loan of \$7000 (after some initial extra charges) would cost her \$155.69 each month for 5 years.

a How much money did she pay altogether (excluding the initial extra charges)?
b How much interest did she pay altogether?
c What would be the yearly interest charged if the same amount is charged each year?
d What percentage is this yearly interest of the \$7000 borrowed? (This is the flat rate of interest charged.)
e What reducible rate is equivalent to this flat rate?

11 Compare the cost of borrowing \$1000 at 12% pa reducible and 12% pa flat over 2 years, if equal monthly repayments are made during the time period. (*Note:* To compare costs, first change 12% pa reducible to a flat rate using the table on page 120). How much more must you pay at 12% pa flat than at 12% pa reducible?

12 Compare the cost of borrowing €28 000 at 16% pa reducible and 16% pa flat over 5 years, if equal monthly repayments are made during that time. How much more must you pay at 16% pa flat than at 16% pa reducible?

Challenge 2:08 | A frightening formula

challenge 2:08

As you may have guessed, the formula that converts a reducible rate to a flat rate is very complicated. The proof is beyond the scope of this course, as is the formula itself. However, this work will really test your ability to use a calculator.

$$\boldsymbol{F = \frac{(1+R)^n(nR-1)+1}{n(1+R)^n - n}}$$

where   ***F*** **is the flat rate per month**
R **is the reducible rate per month**
n **is the number of monthly instalments.**

1 Use the formula to change 12% pa reducible interest over 10 years to an equivalent flat rate expressed as a percentage per annum correct to two decimal places.

Steps:
- **a** Change 12% pa to a monthly rate expressed as a decimal.
- **b** Find the number of instalments in 10 years.
- **c** Substitute these values of R and n into the formula, writing $(1 + R)$ as a single number where it occurs.
- **d** Use your calculator very carefully to find F.
- **e** Multiply F by 12 to find the yearly flat rate expressed as a decimal.

Hint: For this example $F = \dfrac{(0{\cdot}01)^{120}(120 \times 0{\cdot}01 - 1) + 1}{120(1{\cdot}01)^{120} - 120}$

To get F press the following keys slowly to allow time for calculations:

1·01 [x^y] 120 [×] [[(…] 120 [×] 0·01 [−] 1 […)]] [+] 1 [=] [÷]
[[(…] 120 [×] 1·01 [x^y] 120 [−] 120 […)]] [=]

2 Use the formula to change these reducible rates to equivalent flat rates expressed as a percentage per annum correct to two decimal places.

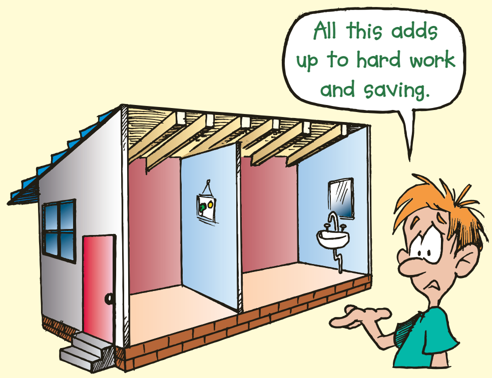
- **a** 24% pa over 2 years
- **b** 18% pa over 5 years
- **c** 22% pa over 10 years
- **d** 15% pa over 20 years

2:09 | Home Loans

Some lending institutions require the borrower to have been a customer for a period of time before they will grant a housing loan.

The **rate of interest** depends on the economic climate and the amount you wish to borrow. The rates are usually about $5\frac{1}{2}$% pa more than the basic rate for savings accounts. If you can get $2\frac{1}{2}$% for your savings on call, you can expect to pay 8% pa reducible (with monthly rests) for a home loan from a bank or building society.

Usually you will be lent only a percentage (perhaps 90%) of the cost of the house and then only if you have an income large enough to ensure that you will be able to repay the loan.



Other costs to the borrower include an establishment fee, a valuation fee (and several inspection fees in the case of a new home), government taxes (stamp duty) and solicitor's costs.

Exercise 2:09

1

Loan amount	*Term of loan (years)* 10	15	20	25
$	$	$	$	$
6·5% pa monthly rests (loans up to $190 000)				
187 000	2123	1629	1394	1263
190 000	2157	1655	1417	1283
7% pa monthly rests (loans from $190 000 to $220 000)				
214 000	2485	1923	1659	1513
217 000	2520	1950	1682	1534
220 000	2554	1977	1706	1555
7·5% pa monthly rests (loans from $220 000 to $250 000)				
229 000	3023	2123	1845	1692
235 000	3102	2178	1893	1737
238 000	3142	2206	1917	1759

Interest rates used here are reducible rates.

Use the table to find:

a the interest rate charged for home loans up to $190 000

b the monthly payment for a loan of $235 000 taken over 20 years

c the largest loan I can afford if I am able to pay up to $1692 per month for 25 years

d the interest rate charged on a loan of:
i $238 000 **ii** $217 000 **iii** $172 000

e the monthly payment for a loan of $217 000 taken over 25 years

f Mario was told that his total monthly financial repayments should not exceed 25% of his monthly gross income. He earns $82 000 pa.
- **i** What is his monthly gross income?
- **ii** What is the largest monthly repayment he should make?
- **iii** What is the most he can borrow if he wishes to repay the loan in 20 years?

2 Pauline and Tarkyn were told that they needed a deposit of \$37 600 before they could borrow enough money to buy their new home.

At the time of their marriage, Pauline had saved \$10 900 while Tarkyn had saved \$14 700. They both worked and although they paid rent of \$1120 each month, they were still able to save \$480 per week.

a When they married, how much less than the deposit did they have?
b How many weeks did it take to save the rest of the deposit?
c If the deposit was 10% of the cost of buying the home (including all extra charges), what was this cost?
d How much money had to be borrowed?
e How much interest is charged for the first month if the interest rate is 7% pa reducible with monthly rests?
f If the contract price for the home was \$360 000, find the stamp duty charges that must be paid to the government if the rate is 2% of the contract price.

3 Olivia borrowed €214 000 which she repaid in monthly instalments over 20 years.

a Use the table in question **1** to determine the size of the monthly payments.
b What is the total amount paid in 20 years?
c How much interest was paid in this time?
d Use the table in question **1** to find the reducible rate of interest paid.
e What is the approximate flat rate equivalent to this reducible rate? (See table, page 57.)

4 The table below shows the monthly repayments per \$1000 of a loan for a variety of interest rates. For a loan of \$80 000 you would have to pay 80 times as much as for a loan of \$1000.

worked example

Find the monthly repayments on a loan of \$280 000 taken over 20 years at 7·5% pa.

Monthly repayment
= 280 × \$8.06
= \$2256.80

Interest rate % pa reducible	*Approximate monthly repayment per \$1000 of loan over:*						
	2 years \$	*5 years* \$	*7 years* \$	*10 years* \$	*15 years* \$	*20 years* \$	*25 years* \$
6·0	44.32	19.33	14.61	11.10	8.44	7.16	6.44
6·5	44.55	19.57	14.85	11.35	8.71	7.46	6.75
7·0	44.77	19.80	15.09	11.61	8.99	7.75	7.07
7·5	45.00	20.04	15.34	11.87	9.27	8.06	7.39
8.0	45.23	20.28	15.59	12.13	9.56	8.36	7.72
8·5	45.46	20.52	15.84	12.40	9.85	8.68	8.05
9·0	45.68	20.76	16.09	12.67	10.14	9.00	8.39

Use the table on page 126 to find the monthly repayments on the following loans:

a \$52 000 taken over 20 years at 7% pa
b \$74 000 taken over 25 years at 8% pa
c \$94 000 taken over 15 years at 7·5% pa
d \$106 000 taken over 20 years at 8·5% pa

5 Use the table in question **4** to calculate the total amount that would be paid if a loan of £1000 is repaid monthly at a rate of 8% pa reducible over:

a 2 years **b** 5 years **c** 10 years
d 15 years **e** 20 years **f** 25 years

6 What would be the cost (interest charged) of a loan of £1000 repaid monthly at a rate of 8% pa reducible over:

a 2 years? **b** 5 years? **c** 10 years?
(*Hint:* Use your answers to question **5**.)

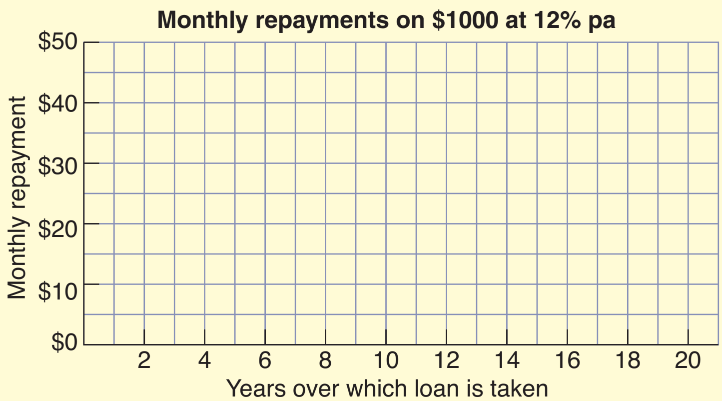
7 Compare the cost (interest charged) of a loan of £1000 taken at a rate of 8% pa flat to one taken at a rate of 8% pa reducible (as in question **6**) over:

a 2 years **b** 5 years **c** 10 years

8 Interest rates for home loans are not fixed. The rate of interest you are paying on your loan may change. This is influenced by changes in the rates banks and building societies have to pay investors.

a Find the increase in monthly repayments for each part in question **4** if the interest rate increased by 0·5% pa.
b Find the decrease in monthly repayments for each part in question **4** if the interest rate decreased by 1% pa.

9 Use the data in question **4** to draw a conversion graph for repayments on a loan of \$1000 at 9% pa over time periods up to 20 years. Use axes like those below.



Use your graph to find the monthly repayment (correct to the nearest dollar) if the loan is taken over:

a 3 years **b** 12 years **c** 18 years

Mathematical Terms 2

compound interest
- Interest earned is added to the principal so that the interest earned for the next time period is calculated on this greater amount.
- Therefore, the interest earned in previous time periods also earns interest.
- The formula for compound interest is:
  $A = P(1 + r)^n$
  where A = amount after n time periods
  P = principal invested
  n = number of time periods
  r = rate of interest

depreciation
- The loss in value of an object over a period of time.
- The formula for calculating the reduced value of an object is:
  $A = P(1 - r)^n$
  where A = value after n time periods
  P = original value
  n = number of time periods
  r = rate of depreciation for one time period

flat rate
- The same as a simple interest rate.

instalment
- A payment made to reduce the amount owing on a loan.

interest
- The payment made for the use of money invested or borrowed.
- Usually expressed as a percentage of the amount borrowed.

principal
- The amount of money invested or borrowed.

reducible interest
- Interest charged only on the amount still owing on a loan.
- Therefore, the amount of interest reduces during the term of the loan.

repayment
- A payment made to reduce the amount owing on a loan.

simple interest
- Interest is paid on the original principal throughout the term of the loan.
- The formula for calculating simple interest is:
  $I = PRT$
  where I = simple interest
  P = principal invested
  R = rate of interest
  T = number of time periods

- If this car depreciates at 15% a year, how long will it take to halve its price if its original price is $150 000?

Mathematical terms 2

diagnostic test

2

Diagnostic Test 2 | Consumer Arithmetic

- Each part of this test has similar items that test a certain question type.
- Errors made will indicate areas of weakness.
- Each weakness should be treated by going back to the section listed.

	Section
1 **a** Find the simple interest charged on €600 at 12% pa for 5 years. **b** What is the simple interest earned by €14 260 invested for 3 years at 4% pa? **c** Find the interest charged on a loan of €85 000 taken over 15 years if a simple interest rate of 8% pa is charged.	2:02
2 **a** Find the simple interest charged on \$860 at $6\frac{1}{2}$% pa for 5 years. **b** What interest would be paid on \$25 000 invested for 6 years at a simple interest rate of 9·3% pa? **c** Find the simple interest earned if \$900 is invested for 3 years at a rate of $6\frac{3}{4}$% pa.	2:02
3 **a** What is the simple interest on £2400 at 8% pa for 5 months? **b** Find the simple interest on £900 for 240 days at a rate of 11% pa. **c** What is the simple interest paid on £1950 invested for 7 months at 0·75% per month?	2:02
4 **a** Jodie borrowed \$78 000 for 4 months at 12·75% pa simple interest to pay for her new house while her old one was being sold. How much interest did she pay? **b** I paid \$8000 to borrow \$50 000 for 2 years. What was the rate of simple interest charged? **c** Troy borrowed \$1200 at 1·5% per month simple interest. Which is the best estimate of the interest charges for 6 months: \$20, \$100, \$200 or \$1800?	2:03
5 **a** Find the compound interest earned if \$9000 is invested for 3 years at 13% pa if interest is compounded yearly. Answer correct to the nearest cent. **b** \$12 000 is invested at a compound interest rate of 9% pa. Interest, however, is compounded monthly. Calculate the amount to which the investment will grow in 2 months. **c** \$12 500 is invested at a compound interest rate of 9% pa. Interest, however, is compounded monthly. Use the formula to calculate the amount to which the investment will grow in 4 years.	2:04 2:06
6 **a** A new computer depreciates by 30% per year. It costs \$4000 new. What will it be worth in 4 years? **b** If the population of Bilby Downs is decreasing by 20% of its population every year, what would be the population in 2 years if it is now 800? **c** Carol paid \$2200 for a new video unit. Use the depreciation formula to find the value of the unit in 6 years if its rate of depreciation is 25% pa.	2:05 2:06

Section

7 **a** Reducible interest at 6% pa is charged on a loan of $50 000. Annual repayments of $5000 are made. Calculate how much is still owing after — 2:07

i 1 year
ii 2 years
iii 3 years

b A loan of $300 000 is charged 8% pa reducible interest. Quarterly payments of $9000 are made. Calculate the amount still owing after 1 year (ie, the first four quarterly payments).

8 **a** Use the table on page 57 to find approximate reducible interest rates equivalent to the following flat rates, if the loan is repaid in monthly instalments over 5 years. — 2:08

i 6·69% pa flat **ii** 11·79% pa flat
iii 3·76% pa flat

b Find the size of each monthly repayment (to the nearest dollar) on a loan of $35 800 taken at a rate of 9% pa reducible, over 20 years. (Use the table on page 57.)

9 Use the table of repayments on page 63, to find the monthly repayments for the following loans. — 2:09

a $30 000 taken over 15 years at 7% pa
b $45 000 taken over 10 years at 6·5% pa
c $75 000 taken over 25 years at 8·5% pa

assignment 2A

Chapter 2 | Revision Assignment

1 Determine the amount of simple interest earned by the following:
   a $1500 at 7% pa over 5 years
   b $950 at 9·5% pa over $3\frac{1}{2}$ years
   c $2200 at 8% pa over 6 months
   d $660 at 1·5% per month over 9 months

2 Bobby bought a video recorder which had a cash price of $850 by paying 10% deposit and the balance over 2 years at a simple interest rate of 15% pa. Find:
   a the deposit
   b the balance owing
   c the amount of interest paid
   d the amount of each monthly repayment

3 Kylie purchased a microwave oven by paying $45 deposit and 18 monthly instalments of $21.30.
   a How much did she pay altogether for the oven?
   b How much interest did she pay if the cash price was $365?

4 Kate's savings account gives interest of 4% pa. Interest is calculated and added to her account every 6 months. If she starts with $2000 in the account and does not use the account for 2 years, what will her new balance be?

5 A loan is to be repaid, with interest, over a 3-year period. Which interest rate would produce the least amount of interest on the loan?
   A 1·2% per month reducible interest
   B 1·2% per month simple (flat) interest
   C 12% per annum reducible interest
   D 12% per annum simple interest

6 $24 000 is invested for 2 years. Interest is paid at 14% pa and is compounded annually. Interest is not withdrawn. How much interest will have been earned after 2 years?

7 Machinery valued at $140 000 depreciates at a rate of 16% pa. Find the value of the machinery after 3 years.

8 A video recorder is purchased under the following terms.

Deposit: $110 Repayments: $41.85 each month for 2 years.

   a Find the total amount paid for the video recorder.
   b If the marked price had been $800, what percentage are the additional charges of the marked price?

9 Use the table on page 120 to convert a reducible rate of 16% pa, with monthly repayments paid over 20 years, to a flat rate. Hence, find the size of each monthly repayment (to the nearest dollar) on a loan of $72 000 taken at a rate of 16% pa reducible over 20 years.

10 Jack decided to invest $5 at 13% pa compound interest for 200 years, with interest to be added annually. His nearest relative living after 200 years would inherit the proceeds.
   a Find the amount the relative should inherit.
   b Find the amount the relative should inherit if the interest rate had been 13% pa simple (flat) interest.

Chapter 2 | Working Mathematically

assignment 2B

1 Find the number of three-digit numbers possible if only the digits 0, 1, 2, 3, 4 and 5 may be used, the number must be a multiple of 5, and no digit may be used more than twice in the same number.

2 Of the 30 students in 10M, 13 love surfing, 15 love hiking and 7 love neither.
   a How many students love both surfing and hiking?
   b How many love surfing but not hiking?
   c How many love hiking but not surfing?

3 A man and his granddaughter share the same birthday. For six consecutive birthdays the man's age is an exact multiple of his granddaughter's age. How old is each of them at the 6th of these birthdays? Can you find another set of numbers that work, but is highly unlikely?

4 Roger started a trip to the country between 8 am and 9 am when the hands of the clock were together. He arrived at his destination between 2 pm and 3 pm, when the hands of the clock were exactly 180° apart. How long did he travel?

5 If $3! = 3 \times 2 \times 1$ (pronounced 3 factorial), $5! = 5 \times 4 \times 3 \times 2 \times 1$ and $10! = 10 \times 9 \times 8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1$, how many zeros are there at the end of the answer to 20!? (A calculator will be of no help.)

6 a For Australia in 1999:
      i How many people were killed on the road?
      ii How many fatalities were there per 100 000 people?
      iii Therefore, what is an estimation of Australia's population according to these figures?
   b Where did Australia rank among these 18 OECD countries in terms of least fatalities per 100 000 population?
   c Compared to 1998, which country in 1999 had:
      i the greatest decrease in fatalities?
      ii the greatest increase in fatalities?

Road safety comparisons: fatalities OECD countries 1999

	Number of fatalities in 1999		
	1999	*% compared to 1998*	*Fatalities per 100 000 persons*
Australia	1 763	0·5%	9·4
Austria	1 079	12·0%	13·4
Czech Republic	1 455	7·0%	14·1
Denmark	500	0·2%	9·4
Finland	427	6·8%	8·3
France	8 487	−4·8%	14·4
Germany	7 749	−0·6%	9·4
Greece	2 131	−4·3%	20·3
Hungary	1 306	−4·7%	12·9
Iceland	21	−22·2%	7·6
Japan	10 372	−4·0%	8·2
Netherlands	1 090	2·3%	7·0
Norway	304	−13·6%	6·9
Poland	6 730	−4·9%	17·4
Spain	5 319	−7·4%	13·5
Sweden	570	7·3%	6·4
Switzerland	583	−2·3%	8·2
United States	41 345	−0·3%	15·3
Total	*91 231*	*−1·9%*	*12·7*

Chapter Review Questions

Technology Applications Activities
1 Compound interest
2 Who wants to be a millionaire?

Technology Applications Drag and Drops
1 Compound interest
2 Depreciation
3 Reducible interest

3

Quadratic Equations

Chapter Contents

Learning Outcomes

Students will be able to:

- Solve quadratic equations by factorising.
- Solve quadratic equations by the use of the formula.
- Solve problems involving quadratic equations.

Areas of Interaction

Approaches to Learning (Knowledge Acquisition, Problem Solving, Logical Thinking), Human Ingenuity

3:01 | Solution Using Factors

prep quiz 3:01

Factorise:	**1** $x^2 + 4x + 3$	**2** $x^2 - 5x + 4$	**3** $x^2 + 5x$
	4 $6x^2 - 3x$	**5** $x^2 - 9$	**6** $4x^2 - 25$
Solve:	**7** $x + 2 = 0$	**8** $3x - 1 = 0$	
	9 $5x = 0$	**10** $2x + 3 = 0$	

A **quadratic equation** is one in which the highest power of the unknown pronumeral is 2. So equations such as:

$x^2 + 4x + 3 = 0$, $\quad x^2 + 5x = 0$,

$x^2 - 25 = 0$ $\quad$ and $\quad 2x^2 - 3x + 7 = 0$

are all quadratic equations.

A quadratic equation is an equation of the 'second degree'.

The solving of a quadratic equation depends on the following observation (called the **Null Factor Law**).

If $ab = 0$, then at least one of a and b must be zero.

worked examples

Solve the quadratic equations:

1 a $(x - 1)(x + 7) = 0$ **b** $2x(x + 3) = 0$ **c** $(2x - 1)(3x + 5) = 0$

2 a $x^2 + 4x + 3 = 0$ **b** $x^2 - 49 = 0$ **c** $2x^2 + 9x - 5 = 0$

3 a $x^2 + x = 12$ **b** $5x^2 = 2x$ **c** $6x^2 = 5x + 6$

Solutions

1 a If $(x - 1)(x + 7) = 0$
then either
$x - 1 = 0$ or $x + 7 = 0$
$\therefore x = 1$ or $x = -7$

b If $2x(x + 3) = 0$
then either
$2x = 0$ or $x + 3 = 0$
$\therefore x = 0$ or $x = -3$

c If $(2x - 1)(3x + 5) = 0$
then either
$2x - 1 = 0$ or $3x + 5 = 0$
$2x = 1$ or $3x = -5$
$\therefore x = \frac{1}{2}$ or $x = \frac{-5}{3}$

A quadratic equation can have two solutions.

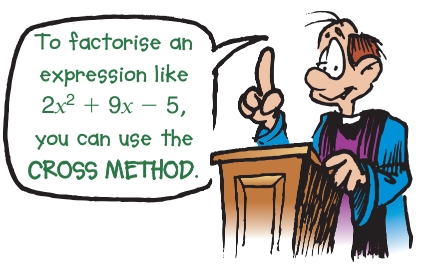
2 To solve these equations, they are factorised first so they look like the equations in example **1**.

a
$x^2 + 4x + 3 = 0$
$(x + 3)(x + 1) = 0$
So $x + 3 = 0$
or $x + 1 = 0$
$\therefore x = -3$ or -1

b
$x^2 - 49 = 0$
$(x - 7)(x + 7) = 0$
So $x - 7 = 0$
or $x + 7 = 0$
$\therefore x = 7$ or -7

or
$x^2 - 49 = 0$
So $x^2 = 49$
$\therefore x = 7$ or -7
ie $x = \pm 7$

c
$2x^2 + 9x - 5 = 0$
$(2x - 1)(x + 5) = 0$
So $2x - 1 = 0$
or $x + 5 = 0$
$\therefore x = \frac{1}{2}$ or -5



continued →→→

3 Before these equations are solved, all the terms are gathered to one side of the equation, letting the other side be zero.

a
$$x^2 + x = 12$$
$$x^2 + x - 12 = 0$$
$$(x + 4)(x - 3) = 0$$
So $x + 4 = 0$
or $x - 3 = 0$
$\therefore\ x = -4$ or 3

b
$$5x^2 = 2x$$
$$5x^2 - 2x = 0$$
$$x(5x - 2) = 0$$
So $x = 0$
or $5x - 2 = 0$
$\therefore\ x = 0$ or $\frac{2}{5}$

c
$$6x^2 = 5x + 6$$
$$6x^2 - 5x - 6 = 0$$
$$(3x + 2)(2x - 3) = 0$$
So $3x + 2 = 0$
or $2x - 3 = 0$
$\therefore\ x = -\frac{2}{3}$ or $\frac{3}{2}$

To solve a quadratic equation:
- **gather all the terms to one side of the equation**
- **factorise**
- **solve the two resulting simple equations.**

Of course, you can always check your solutions by substitution. For example 3a above:

Substituting $x = -4$
$x^2 + x = 12$
L.H.S. $= (-4)^2 + (-4)$
$= 16 - 4$
$= 12$
$=$ R.H.S.

Substituting $x = 3$
$x^2 + x = 12$
L.H.S. $= (3)^2 + (3)$
$= 9 + 3$
$= 12$
$=$ R.H.S.

$\therefore$ Both $x = -4$ and $x = 3$ are solutions.

L.H.S. = left-hand side
R.H.S. = right-hand side

Exercise 3:01

Foundation Worksheet 3:01
Quadratic equations
1 Factorise
a $x^2 - 3x$ b $x^2 + 3x + 2$
2 Solve
a $x(x - 4) = 0$ b $(x - 1)(x + 2) = 0$

1 Find the two solutions for each equation.
Check by substitution to ensure your answers are correct.

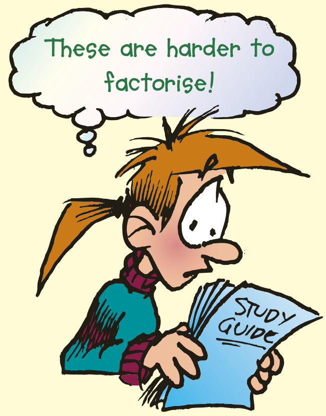
a $x(x - 5) = 0$	**b** $x(x + 7) = 0$	**c** $2x(x + 1) = 0$
d $5a(a - 2) = 0$	**e** $4q(q + 5) = 0$	**f** $6p(p - 7) = 0$
g $(x - 2)(x - 1) = 0$	**h** $(x - 7)(x - 3) = 0$	**i** $(a - 5)(a - 2) = 0$
j $(y + 3)(y + 4) = 0$	**k** $(t + 3)(t + 2) = 0$	**l** $(x + 9)(x + 5) = 0$
m $(a - 6)(a + 6) = 0$	**n** $(y + 8)(y - 7) = 0$	**o** $(n + 1)(n - 1) = 0$
p $(a + 1)(2a - 1) = 0$	**q** $(3x + 2)(x - 5) = 0$	**r** $2x(3x - 1) = 0$
s $(4x - 1)(2x + 1) = 0$	**t** $(3a - 4)(2a - 1) = 0$	**u** $(6y - 5)(4y + 3) = 0$
v $6x(5x - 3) = 0$	**w** $(9y + 1)(7y + 2) = 0$	**x** $(5x - 1)(5x + 1) = 0$

2 After factorising the left-hand side of each equation, solve the following.

a $x^2 + 3x = 0$	**b** $m^2 - 5m = 0$	**c** $y^2 + 2y = 0$
d $6x^2 + 12x = 0$	**e** $9n^2 - 3n = 0$	**f** $4x^2 + 8x = 0$
g $x^2 - 4 = 0$	**h** $a^2 - 49 = 0$	**i** $y^2 - 36 = 0$
j $a^2 - 1 = 0$	**k** $n^2 - 100 = 0$	**l** $m^2 - 64 = 0$
m $x^2 + 3x + 2 = 0$	**n** $a^2 - 5a + 6 = 0$	**o** $y^2 + 12y + 35 = 0$
p $a^2 - 10a + 21 = 0$	**q** $x^2 - 10x + 16 = 0$	**r** $m^2 - 11m + 24 = 0$
s $h^2 + h - 20 = 0$	**t** $x^2 + 2x - 35 = 0$	**u** $a^2 - 4a - 45 = 0$
v $x^2 + x - 56 = 0$	**w** $y^2 - 8y + 7 = 0$	**x** $a^2 + 9a - 10 = 0$

3 Factorise and solve the following.

- **a** $2x^2 + x - 1 = 0$
- **b** $3x^2 + 7x + 2 = 0$
- **c** $3x^2 + 17x + 10 = 0$
- **d** $2x^2 - 11x + 12 = 0$
- **e** $2x^2 - x - 10 = 0$
- **f** $2x^2 - 11x - 21 = 0$
- **g** $4x^2 + 21x + 5 = 0$
- **h** $4x^2 - 19x - 5 = 0$
- **i** $4x^2 - 21x + 5 = 0$
- **j** $5x^2 + 16x + 3 = 0$
- **k** $2x^2 + 13x - 24 = 0$
- **l** $7x^2 + 48x - 7 = 0$
- **m** $4x^2 - 4x - 3 = 0$
- **n** $6x^2 - x - 1 = 0$
- **o** $9x^2 + 9x + 2 = 0$
- **p** $10x^2 + 9x + 2 = 0$
- **q** $12x^2 - 7x + 1 = 0$
- **r** $10x^2 - 13x + 4 = 0$



4 Gather all the terms to one side of each equation and then solve.

- **a** $x^2 = 3x$
- **b** $m^2 = 8m$
- **c** $x^2 = -5x$
- **d** $x^2 = 5x - 4$
- **e** $a^2 = 2a + 15$
- **f** $y^2 = 3y - 2$
- **g** $m^2 = 9m - 18$
- **h** $n^2 = 7n + 18$
- **i** $h^2 = 4h + 32$
- **j** $x^2 + x = 2$
- **k** $y^2 + 2y = 3$
- **l** $x^2 - 7x = -10$
- **m** $y^2 + 3y = 18$
- **n** $t^2 + 3t = 28$
- **o** $y^2 + 2y = 15$
- **p** $2x^2 + x = 1$
- **q** $2x^2 - x = 15$
- **r** $4m^2 - 3m = 6$
- **s** $3x^2 = 13x - 14$
- **t** $5p^2 = 17p - 6$
- **u** $2x^2 = 11x - 5$

Check answers by substitution.

3:02 | Solution by Completing the Square

This method depends upon completing an algebraic expression to form a perfect square, that is, an expression of the form $(x + a)^2$ or $(x - a)^2$.

worked examples

What must be added to the following to make perfect squares?

1 $x^2 + 8x$ **2** $x^2 - 5x$

Solutions

Because $(x + a)^2 = x^2 + 2ax + a^2$, the coefficient of the x term must be halved to give the value of a.

1 $x^2 + 8x + \ldots$

Half of 8 is 4, so the perfect square is:

$x^2 + 8x + 4^2 = (x + 4)^2$

2 $x^2 - 5x + \ldots$

Half of -5 is $-\frac{5}{2}$, so the perfect square is:

$x^2 - 5x + (-\frac{5}{2})^2 = (x - \frac{5}{2})^2$

Now, to solve a quadratic equation using this technique, we follow the steps in the example below.

$$x^2 + 4x - 21 = 0$$
$$x^2 + 4x = 21 \quad \leftarrow \text{Move the constant to the R.H.S.}$$
$$x^2 + 4x + 2^2 = 21 + 2^2 \quad \leftarrow \text{Add (half of } x \text{ coefficient)}^2 \text{ to both sides.}$$
$$\therefore \quad (x + 2)^2 = 25$$
$$x + 2 = \pm\sqrt{25}$$
$$x = -2 \pm 5$$
$$\therefore x = 3 \text{ or } -7$$

Note that the previous example could have been factorised to give $(x - 3)(x + 7) = 0$, which, of course, is an easier and quicker way to find the solution. The method of completing the square, however, can determine the solution of quadratic equations that cannot be factorised. This can be seen in the examples below.

worked examples

Solve:

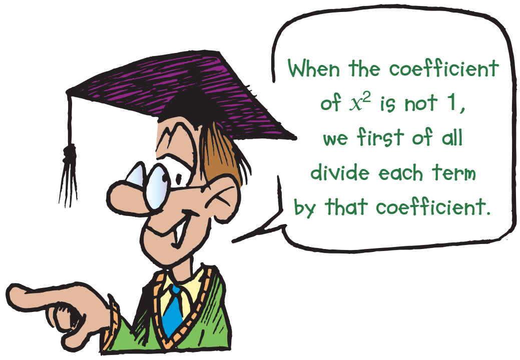
1 $x^2 + 6x + 1 = 0$ **2** $x^2 - 3x - 5 = 0$ **3** $3x^2 - 4x - 1 = 0$

Solutions

1
$$\begin{aligned} x^2 + 6x + 1 &= 0 \\ x^2 + 6x &= -1 \\ x^2 + 6x + 3^2 &= -1 + 3^2 \\ (x + 3)^2 &= 8 \\ x + 3 &= \pm\sqrt{8} \\ \therefore\ x &= -3 \pm \sqrt{8} \end{aligned}$$
ie $x = -3 + 2\sqrt{2}$ or $-3 - 2\sqrt{2}$
($x \doteqdot -0{\cdot}17$ or $-5{\cdot}83$)

Note that the solution involves a square root, ie the solution is irrational. Using your calculator, approximations may be found.

2
$$\begin{aligned} x^2 - 3x - 5 &= 0 \\ x^2 - 3x &= 5 \\ x^2 - 3x + (-\tfrac{3}{2})^2 &= 5 + (-\tfrac{3}{2})^2 \\ (x - \tfrac{3}{2})^2 &= 7\tfrac{1}{4} \\ x - \tfrac{3}{2} &= \pm\sqrt{7\tfrac{1}{4}} \\ \therefore\ x &= \frac{3}{2} \pm \frac{\sqrt{29}}{2} \end{aligned}$$
ie $x = \dfrac{3 + \sqrt{29}}{2}$ or $\dfrac{3 - \sqrt{29}}{2}$
($x \doteqdot 4{\cdot}19$ or $-1{\cdot}19$)



3
$$\begin{aligned} 3x^2 - 4x - 1 &= 0 \\ x^2 - \tfrac{4}{3}x - \tfrac{1}{3} &= 0 \\ x^2 - \tfrac{4}{3}x &= \tfrac{1}{3} \\ x^2 - \tfrac{4}{3}x + (-\tfrac{2}{3})^2 &= \tfrac{1}{3} + (-\tfrac{2}{3})^2 \\ (x - \tfrac{2}{3})^2 &= \tfrac{7}{9} \\ x - \frac{2}{3} &= \pm\frac{\sqrt{7}}{3} \\ \therefore\ x &= \frac{2}{3} \pm \frac{\sqrt{7}}{3} \end{aligned}$$
ie $x = \dfrac{2 + \sqrt{7}}{3}$ or $\dfrac{2 - \sqrt{7}}{3}$
($x \doteqdot 1{\cdot}55$ or $-0{\cdot}22$)

Note:
You can use the following fact to check your answers.
For the equation:
$$ax^2 + bx + c = 0$$
the two solutions must add up to equal $-\dfrac{b}{a}$

In example 1, $(-0{\cdot}17) + (-5{\cdot}83) = -6$ [or $\frac{-6}{1}$]

In example 3, $1{\cdot}55 + (-0{\cdot}22) = 1{\cdot}33$ [$\doteqdot \frac{4}{3}$]

Exercise 3:02

1 What number must be inserted to complete the square?

a $x^2 + 6x + \ldots^2 = (x + \ldots)^2$ **b** $x^2 + 8x + \ldots^2 = (x + \ldots)^2$
c $x^2 - 2x + \ldots^2 = (x - \ldots)^2$ **d** $x^2 - 4x + \ldots^2 = (x - \ldots)^2$
e $x^2 + 3x + \ldots^2 = (x + \ldots)^2$ **f** $x^2 - 7x + \ldots^2 = (x - \ldots)^2$
g $x^2 + 11x + \ldots^2 = (x + \ldots)^2$ **h** $x^2 - x + \ldots^2 = (x - \ldots)^2$
i $x^2 + \frac{5x}{2} + \ldots^2 = (x + \ldots)^2$ **j** $x^2 - \frac{2x}{3} + \ldots^2 = (x - \ldots)^2$

2 Solve the following equations, leaving your answers in surd form.

a $(x - 2)^2 = 3$ **b** $(x + 1)^2 = 2$ **c** $(x + 5)^2 = 5$
d $(x - 1)^2 = 10$ **e** $(x - 3)^2 = 7$ **f** $(x + 2)^2 = 11$
g $(x + 3)^2 = 8$ **h** $(x + 10)^2 = 12$ **i** $(x - 3)^2 = 18$
j $(x + \frac{1}{2})^2 = 5$ **k** $(x - \frac{2}{3})^2 = 3$ **l** $(x + 1\frac{1}{2})^2 = 12$
m $(x - 1)^2 = 2\frac{1}{2}$ **n** $(x + 3)^2 = 4\frac{1}{2}$ **o** $(x - \frac{1}{3})^2 = \frac{5}{9}$

3 Solve the following equations by completing the square. Also find approximations for your answers, correct to two decimal places.

a $x^2 + 2x - 1 = 0$ **b** $x^2 - 2x - 5 = 0$ **c** $x^2 - 4x - 8 = 0$
d $x^2 + 6x - 8 = 0$ **e** $x^2 - 6x + 2 = 0$ **f** $x^2 + 4x + 1 = 0$
g $x^2 + 10x = 5$ **h** $x^2 + 2x = 4$ **i** $x^2 - 12x = 1$
j $x^2 + 5x + 2 = 0$ **k** $x^2 + 7x - 3 = 0$ **l** $x^2 + x - 3 = 0$
m $x^2 + 9x + 3 = 0$ **n** $x^2 + 3x - 5 = 0$ **o** $x^2 - 11x + 5 = 0$
p $x^2 - x = 3$ **q** $x^2 + 3x = 2$ **r** $x^2 - 5x = 1$
s $2x^2 - 4x - 1 = 0$ **t** $2x^2 + 3x - 4 = 0$ **u** $2x^2 - 8x + 1 = 0$
v $3x^2 + 2x - 3 = 0$ **w** $5x^2 - 4x - 3 = 0$ **x** $4x^2 - x - 2 = 0$

3:03 | The Quadratic Formula

As we have seen in the previous section, a quadratic equation is one involving a squared term. In fact, any quadratic equation can be represented by the **general form** of a quadratic equation:

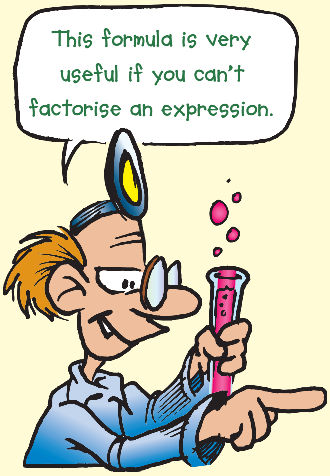
$$ax^2 + bx + c = 0$$

where a, b, c are all integers, and a is not equal to zero.

If any quadratic equation is arranged in this form, a formula using the values of a, b and c can be used to find the solutions.

The *quadratic formula* for $ax^2 + bx^2 + c = 0$ is:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$



PROOF OF THE QUADRATIC FORMULA

$$ax^2 + bx + c = 0$$

$$x^2 + \frac{b}{a}x + \frac{c}{a} = 0$$

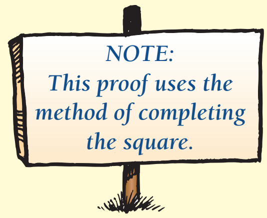
$$x^2 + \frac{b}{a}x = -\frac{c}{a}$$

$$x^2 + \frac{b}{a}x + \left(\frac{b}{2a}\right)^2 = \left(\frac{b}{2a}\right)^2 - \frac{c}{a}$$

$$\left(x + \frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{4a^2}$$

$$x + \frac{b}{2a} = \frac{\pm\sqrt{b^2 - 4ac}}{2a}$$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$



worked examples

Solve the following by using the quadratic formula.

1 $2x^2 + 9x + 4 = 0$ **2** $x^2 + 5x + 1 = 0$ **3** $3x^2 = 2x + 2$ **4** $2x^2 + 2x + 7 = 0$

Solutions

1 For the equation $2x^2 + 9x + 4 = 0$,
$a = 2$, $b = 9$, $c = 4$.

Substituting these values into the formula:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{-9 \pm \sqrt{9^2 - 4 \times 2 \times 4}}{2 \times 2}$$

$$= \frac{-9 \pm \sqrt{81 - 32}}{4}$$

$$= \frac{-9 \pm \sqrt{49}}{4}$$

$$= \frac{-9 \pm 7}{4}$$

$$= -\tfrac{2}{4} \text{ or } -\tfrac{16}{4}$$

$$\therefore x = -\tfrac{1}{2} \text{ or } -4$$

2 For $x^2 + 5x + 1 = 0$,
$a = 1$, $b = 5$, $c = 1$.

Substituting into the formula gives:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{-5 \pm \sqrt{5^2 - 4 \times 1 \times 1}}{2 \times 1}$$

$$= \frac{-5 \pm \sqrt{25 - 4}}{2}$$

$$= \frac{-5 \pm \sqrt{21}}{2}$$

Since there is no rational equivalent to $\sqrt{21}$ the answer may be left as:

$$x = \frac{-5 + \sqrt{21}}{2} \text{ or } \frac{-5 - \sqrt{21}}{2}$$

Approximations for these answers may be found using a calculator. In this case they would be given as:
$x \doteqdot -0{\cdot}21$ or $-4{\cdot}79$ (to 2 dec. pl.)

3 The equation $3x^2 = 2x + 2$ must first be written in the form

$ax^2 + bx + c = 0$,

ie $3x^2 - 2x - 2 = 0$

So a = 3, $b = -2$, $c = -2$.

Substituting these values gives:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{-(-2) \pm \sqrt{(-2)^2 - 4 \times 3 \times (-2)}}{2 \times 3}$$

$$= \frac{2 \pm \sqrt{4 + 24}}{6}$$

$$= \frac{2 \pm \sqrt{28}}{6}$$

So $x = \dfrac{2 + \sqrt{28}}{6}$ or $\dfrac{2 - \sqrt{28}}{6}$

(ie $x \doteqdot 1{\cdot}22$ or $-0{\cdot}55$ to 2 dec. pl.)

4 For $2x^2 + 2x + 7 = 0$,

$a = 2$, $b = 2$, $c = 7$.

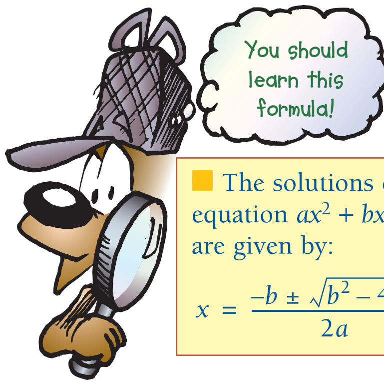
Substituting these values gives:

$$x = \frac{-2 \pm \sqrt{2^2 - 4 \times 2 \times 7}}{2 \times 2}$$

$$= \frac{-2 \pm \sqrt{-52}}{4}$$

But $\sqrt{-52}$ is not real!

So $2x^2 + 2x + 7 = 0$ has no real solutions.



■ The solutions of the equation $ax^2 + bx + c = 0$ are given by:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Exercise 3:03

Foundation Worksheet 3:03

The quadratic formula

1 Evaluate $\dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ if:

a $a = 1$, $b = 3$, $c = 2$ **b** $a = 2$, $b = 5$, $c = -2$

2 Solve:

a $x^2 + 5x + 2 = 0$ **b** $x^2 - 3x - 1 = 0$

1 Use the quadratic formula to solve the following equations. All have rational answers.

a $x^2 + 5x + 6 = 0$ **b** $x^2 + 6x + 8 = 0$
c $x^2 + 10x + 9 = 0$ **d** $x^2 - 3x - 10 = 0$
e $x^2 - 2x - 15 = 0$ **f** $x^2 + 4x - 12 = 0$
g $x^2 - 9x + 14 = 0$ **h** $x^2 - 8x + 12 = 0$ **i** $x^2 - 6x + 5 = 0$
j $3x^2 + 7x + 2 = 0$ **k** $2x^2 + 11x + 5 = 0$ **l** $4x^2 + 11x + 6 = 0$
m $2x^2 - 5x - 3 = 0$ **n** $5x^2 - 9x - 2 = 0$ **o** $3x^2 - 5x + 2 = 0$
p $6x^2 + 7x + 2 = 0$ **q** $6x^2 + 7x - 3 = 0$ **r** $8x^2 - 14x + 3 = 0$

2 Solve the following, leaving your answers in surd form. (Remember: A surd is an expression involving a square root.)

a $x^2 + 4x + 2 = 0$ **b** $x^2 + 3x + 1 = 0$ **c** $x^2 + 5x + 3 = 0$
d $x^2 + x - 1 = 0$ **e** $x^2 + 2x - 2 = 0$ **f** $x^2 + 4x - 1 = 0$
g $x^2 - 2x - 1 = 0$ **h** $x^2 - 7x + 2 = 0$ **i** $x^2 - 6x + 3 = 0$
j $x^2 - 10x - 9 = 0$ **k** $x^2 - 8x + 3 = 0$ **l** $x^2 - 5x + 7 = 0$
m $2x^2 + 6x + 1 = 0$ **n** $2x^2 + 3x - 1 = 0$ **o** $2x^2 - 7x + 4 = 0$
p $3x^2 + 10x + 2 = 0$ **q** $3x^2 - 9x + 2 = 0$ **r** $5x^2 + 4x - 2 = 0$
s $4x^2 - x + 1 = 0$ **t** $3x^2 - 3x - 1 = 0$ **u** $4x^2 - 3x - 2 = 0$
v $2x^2 + 11x - 5 = 0$ **w** $2x^2 - 9x + 8 = 0$ **x** $5x^2 + 2x - 1 = 0$

3 Use the formula to solve the following and give the answers as decimal approximations correct to two decimal places.

a $x^2 - 4x + 1 = 0$	**b** $x^2 - 6x + 3 = 0$	**c** $x^2 + 8x - 5 = 0$
d $x^2 + 9x + 1 = 0$	**e** $x^2 + 2x - 5 = 0$	**f** $x^2 + 3x - 1 = 0$
g $x^2 + 2 = 0$	**h** $x^2 - 7x = 2$	**i** $x^2 = 6x - 11$
j $2x^2 + x - 2 = 0$	**k** $2x^2 - 5x - 2 = 0$	**l** $3x^2 + 9x + 5 = 0$
m $2x^2 = 7x - 2$	**n** $5x^2 - 3x = 4$	**o** $6x^2 = x + 3$

investigation 3:03

Investigation 3:03 | How many solutions?

Consider these three quadratic equations:

A $x^2 + 6x + 5 = 0$ **B** $x^2 + 6x + 9 = 0$ **C** $x^2 + 6x + 12 = 0$

If we use the formula to solve each of these, we get:

A
$$x = \frac{-6 \pm \sqrt{6^2 - 4 \times 1 \times 5}}{2 \times 1}$$
$$= \frac{-6 \pm \sqrt{16}}{2}$$
$$= \frac{-6 \pm 4}{2}$$
$$\therefore x = -1 \text{ or } -5$$

B
$$x = \frac{-6 \pm \sqrt{6^2 - 4 \times 1 \times 9}}{2 \times 1}$$
$$= \frac{-6 \pm \sqrt{0}}{2}$$
$$= -\frac{6}{2}$$
$$\therefore x = -3$$

C
$$x = \frac{-6 \pm \sqrt{6^2 - 4 \times 1 \times 12}}{2 \times 1}$$
$$= \frac{-6 \pm \sqrt{-12}}{2}$$

[$\sqrt{-12}$ has no real solution]

$\therefore$ x has no real solutions

Looking at these equations, it appears that a quadratic equation may have two, one or no solutions. The 'key' is the part of the formula under the square root sign.

The number of solutions is determined by $b^2 - 4ac$.
If $b^2 - 4ac$ is
- ***positive*** **then the equation will have 2 solutions**
- ***zero*** **then the equation will have 1 solution**
- ***negative*** **then the equation will have *no* solution.**

Exercises

By evaluating $b^2 - 4ac$ for each equation, determine how many solutions it will have.

$b^2 - 4ac$ is called the *discriminant*.

1 $x^2 + 4x + 3 = 0$	**2** $x^2 + 4x + 4 = 0$	**3** $x^2 + 4x + 5 = 0$
4 $x^2 - x - 2 = 0$	**5** $x^2 - x = 0$	**6** $x^2 - x + 2 = 0$
7 $4x^2 - 12x + 9 = 0$	**8** $4x^2 - 12x + 7 = 0$	**9** $4x^2 - 12x + 11 = 0$
10 $5x^2 - x + 7 = 0$	**11** $5x^2 - x - 7 = 0$	**12** $9x^2 + 6x + 1 = 0$

3:04 | Choosing the Best Method

prep quiz 3:04

Factorise: **1** $5x^2 - 10x$ **2** $x^2 - 5x - 14$ **3** $x^2 - 81$ **4** $x^2 + 5x + 6$

Solve: **5** $(x - 2)(x + 7) = 0$ **6** $(2x - 3)(3x + 1) = 0$

7 $x^2 - 16 = 0$ **8** $3x^2 - 12x = 0$

9 $x^2 - 3x + 2 = 0$

10 Write down the formula for the solution of the equation: $ax^2 + bx + c = 0$.

Some quadratic equations may appear in a different form from those we have seen so far, but they can always be simplified to the general form $ax^2 + bx + c = 0$. They may then be factorised, or the formula applied, to solve them.

worked examples

Solve the following equations.

1 $x^2 - 2x + 1 = 3x + 6$ **2** $x(x - 5) = 6$ **3** $x = \dfrac{5x - 6}{x}$

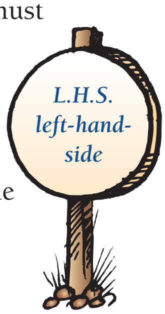
Solutions

1 In this example, all the terms must be gathered to the L.H.S.

$$x^2 - 2x + 1 = 3x + 6$$
$$-3x - 6 \quad -3x - 6$$
$$x^2 - 5x - 5 = 0$$

This cannot be factorised, so the quadratic formula is used.

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$
$$= \frac{5 \pm \sqrt{25 + 20}}{2}$$
$$= \frac{5 \pm \sqrt{45}}{2}$$
$$\therefore x = \frac{5 + \sqrt{45}}{2} \text{ or } \frac{5 - \sqrt{45}}{2}$$



2 Expand and gather the terms to the L.H.S.

$$x(x - 5) = 6$$
$$x^2 - 5x = 6$$
$$x^2 - 5x - 6 = 0$$

Factorising gives:

$$(x - 6)(x + 1) = 0$$
$$\therefore x = 6 \text{ or } -1$$

3 $x = \dfrac{5x - 6}{x}$

Multiplying both sides by x gives:

$$x^2 = 5x - 6$$

ie $x^2 - 5x + 6 = 0$

Factorising gives:

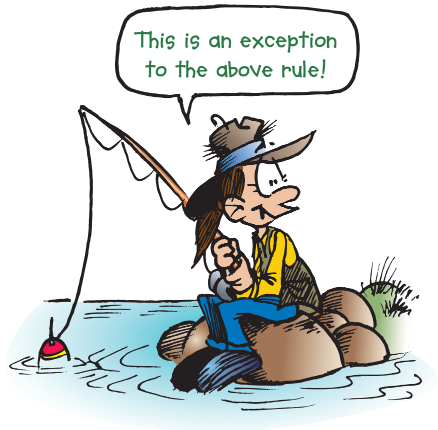
$$(x - 2)(x - 3) = 0$$
$$\therefore x = 2 \text{ or } 3$$

When solving a quadratic equation:

Step 1: Express the equation in general form
$ax^2 + bx + c = 0$

Step 3: Factorise, if you can, and solve it
OR
Use the formula $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

worked example



$(a + 7)^2 = 6$

For equations like this, where one side is a perfect square, it is easier to follow the final steps in the method of completing the square.

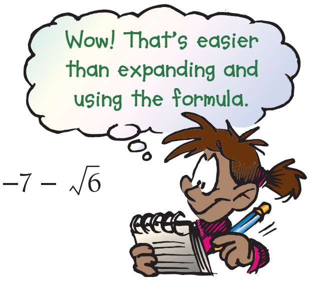
Solution

$(a + 7)^2 = 6$

$\therefore\ a + 7 = \pm\sqrt{6}$

$a = -7 \pm \sqrt{6}$

$a = -7 + \sqrt{6}$ or $-7 - \sqrt{6}$



Exercise 3:04

1 Solve the following quadratic equations. (Give irrational answers to 2 dec. pl. if necessary.)

a $x^2 + 7x + 6 = 0$	**b** $x^2 - 8x + 12 = 0$	**c** $x^2 + 5x - 24 = 0$
d $x^2 - 3x + 1 = 0$	**e** $x^2 + 3x - 3 = 0$	**f** $x^2 + 4x + 2 = 0$
g $x^2 + 8x = 0$	**h** $x^2 - 10x = 0$	**i** $5x^2 - 10x = 0$
j $x^2 - 81 = 0$	**k** $x^2 - 121 = 0$	**l** $4x^2 - 9 = 0$
m $2x^2 + 4x + 1 = 0$	**n** $3x^2 - x - 1 = 0$	**o** $2x^2 - 5x + 1 = 0$
p $2x^2 + 6x + 4 = 0$	**q** $3x^2 + 15x + 18 = 0$	**r** $2x^2 - 6x - 8 = 0$
s $x^2 = 6x + 27$	**t** $x^2 = 13x - 36$	**u** $2x^2 - 5x = 12$
v $25 = 10x - x^2$	**w** $36 = 13x - x^2$	**x** $2 = 9x - 5x^2$

2 Rearrange each equation below to the form $ax^2 + bx + c = 0$, and solve.

a $x^2 + 9x = 2x - 12$	**b** $x^2 + 20 = 8x + 5$	**c** $x^2 - 4x + 10 = 2x + 2$
d $3x^2 + 5x = 2x^2 - 6$	**e** $4x^2 + 5x = 3x^2 - 2x$	**f** $x^2 + 3x - 10 = 3x - 1$
g $x^2 + 5x = 3x + 1$	**h** $x^2 + 7 = 5 - 4x$	**i** $2x + 1 = x^2 + x$
j $x(x + 5) = 6$	**k** $x(x - 7) = 18$	**l** $x^2 = 4(x + 8)$
m $(m - 1)^2 = 4$	**n** $(x + 3)^2 = 9$	**o** $(x + 5)^2 = 11$
p $(2a + 1)^2 = 16$	**q** $(5y - 3)^2 = 7$	**r** $(6n - 7)^2 = 3$
s $x = \dfrac{2x + 15}{x}$	**t** $x = \dfrac{3x + 28}{x}$	**u** $1 = \dfrac{2 - x^2}{x}$
v $x = \dfrac{5x - 3}{x}$	**w** $\dfrac{3(x + 1)}{x} = x$	**x** $2(x + 2) = \dfrac{1}{x}$

Fun Spot 3:04 | What is an Italian referee?

Work out the answer to each question and put the letter for that part in the box that is above the correct answer.

Solve:

A $(x + 3)(x - 5) = 0$

R $(2x - 1)(x - 7) = 0$

M $(x + 1)(x - 1) = 0$

R $(3x + 5)(2x - 3) = 0$

Solve by factorising:

M $x^2 - 5x + 6 = 0$

E $x^2 - 5x - 6 = 0$

A $x^2 - x - 6 = 0$

I $x^2 + x - 6 = 0$

Solve:

N $(x - 2)^2 = 5$

O $(x + 7)^2 = 9$

P $(x + 1)^2 = 2$

U $(x + 3)^2 = 3$

$-2, 3$	$\frac{1}{2}, 7$	$-4, -10$	$2, 3$	$-3, 5$	$2 \pm \sqrt{5}$	$-3 \pm \sqrt{3}$	$-1, 1$	$-1 \pm \sqrt{2}$	$-3, 2$	$-\frac{5}{3}, \frac{3}{2}$	$-1, 6$

fun spot 3:04

3:05 | Problems Involving Quadratic Equations

prep quiz 3:05

From the list of numbers, 1, 2, $3\frac{1}{2}$, 5·2, 9, 10, write down the numbers which are:

1 integers **2** odd **3** square

Write down the next 2 consecutive integers after: **4** 8 **5** n

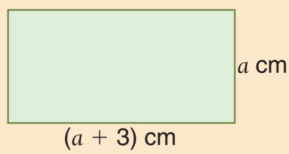
Write down the next 2 consecutive even numbers after: **6** 10 **7** x (x is even)

Write down expressions for:

8 a number that is 3 less than x

9 the area of this rectangle

10 the perimeter of this rectangle



Sometimes, when solving a problem or applying a given formula, a quadratic equation may be involved. Consider the following examples.

worked examples

1 The product of two consecutive positive even numbers is 48. Find the numbers. (*Hint:* If the first number is x, then the next even number is $x + 2$).

2 The length of a rectangle is 5 cm longer than its breadth. If the area of the rectangle is 84 cm^2, find the length of the rectangle.

3 A projectile is fired vertically upwards and its height h in metres after t seconds is given by the formula:

$$h = 40t - 8t^2$$

Find the time taken by the projectile to first reach a height of 48 metres.

continued →→→

Solutions

1 The problem gives the equation:

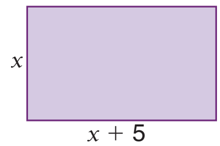
$$x(x + 2) = 48$$

Solving this gives:

$$x^2 + 2x = 48$$
$$x^2 + 2x - 48 = 0$$
$$(x + 8)(x - 6) = 0$$
$$\text{ie} \quad x = -8 \text{ or } 6$$

Since the numbers are positive, x must equal 6.

∴ The two consecutive integers are 6 and 8.

2 

If the breadth is x, then the length is $x + 5$.

Since the area of a rectangle is equal to length times breadth, then:

$$x(x + 5) = 84$$
$$x^2 + 5x = 84$$
$$x^2 + 5x - 84 = 0$$
$$(x + 12)(x - 7) = 0$$
$$\text{So} \quad x = -12 \text{ or } 7$$

Now, since the dimensions must be positive, the breadth must be 7 cm.

∴ The length = 12 cm.

3 $h = 40t - 8t^2$

To find the time t when the height $h = 48$ metres, substitute into the formula:

so $\quad 48 = 40t - 8t^2$

Gathering all the terms on the L.H.S.

$$8t^2 - 40t + 48 = 0$$
$$8(t^2 - 5t + 6) = 0$$
$$8(t - 2)(t - 3) = 0$$
$$t = 2 \text{ or } 3$$

∴ The projectile was at a height of 48 m after 2 seconds and 3 seconds.
So it was first at a height of 48 m after 2 seconds.

Exercise 3:05

1 Find the two positive integers required, if:

a the numbers are consecutive and their product is 20

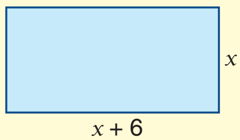
b the numbers are consecutive and their product is 90

c the numbers are consecutive even numbers and their product is 120

d the numbers are consecutive odd numbers and their product is 63

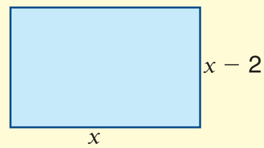
2 **a** The sum of a positive integer and its square is 90. Find the number.
b The sum of a positive integer and its square is 132. Find the number.
c The difference between a positive integer and its square is 56. Find the number.
d The square of a number is equal to 5 times the number. What are the two possible answers?
e When a number is subtracted from its square, the result is 42. Find the two possible solutions.

3 **a**

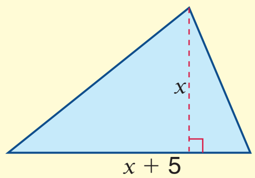


Find the dimensions of this rectangle if the length is 6 cm longer than the breadth and its area is 40 cm^2.

b The width of a rectangular room is 2 metres shorter than its length. If the area of the room is 255 m^2, find the dimensions of the room.

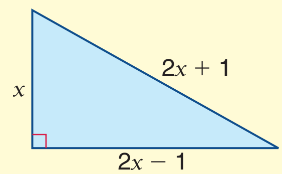


c



The base of a triangle is 5 cm longer than its height. If the area of the triangle is 7 cm^2, find the length of the base.

d A right-angled triangle is drawn so that the hypotenuse is twice the shortest side plus 1 cm, and the other side is twice the shortest side less 1 cm. Find the length of the hypotenuse.



4 **a** Michelle threw a ball vertically upwards, with its height h, in metres, after a time of t seconds, being given by the formula:

$$h = 8t - t^2$$

Find after what time the ball is first at a height of 12 m.

b The sum, S, of the first n positive integers is given by the formula

$$S = \frac{n}{2}(n + 1)$$

Find the number of positive integers needed to give a total of 78.

c For the formula $s = ut + \frac{1}{2}at^2$, find the values of t if:
i $s = 18$, $u = 7$, $a = 2$
ii $s = 6$, $u = 11$, $a = 4$
iii $s = 7$, $u = 1$, $a = 6$

5 An n-sided polygon has $\frac{1}{2}n(n - 3)$ diagonals. How many sides has a figure if it has 90 diagonals?

6 Jenny is y^2 years old and her daughter Allyson is y years old. If Jenny lives to the age of $13y$, Allyson will be y^2 years old. How old is Allyson now? (*Note:* the difference in ages must remain constant.)

7 Kylie bought an item for $\$x$ and sold it for \$10.56. If Kylie incurred a loss of x per cent, find x.

8 A relationship that is used to approximate car stopping distances (d) in ideal road and weather conditions is:

$$d = t_r v + kv^2$$

where t_r is the driver's reaction time, v is the velocity and k is a constant.

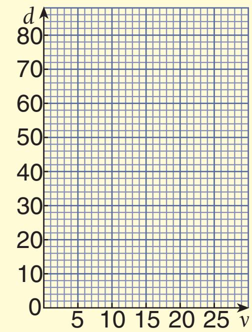
a Stirling's reaction time was measured to be 0·8 seconds. The distance it took him to stop while travelling at 20 m/s (72 km/h) was 51 metres. Substitute this information into the formula to find the value of k.

b If, for these particular conditions, Stirling's breaking distance is given by

$$d = 0{\cdot}8v + 0{\cdot}0875v^2$$

complete the table below, finding d correct to the nearest metre in each case.

v (in m/s)	0	5	10	15	20	25
d (in m)						



c Graph d against v using the number plane shown on the right. What kind of curve is produced?

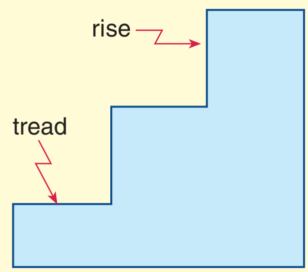
d Use your graph to find the velocity (in m/s) that would produce a stopping distance of 40 metres. Check your accuracy by solving the equation

$40 = 0{\cdot}8v + 0{\cdot}0875v^2$ using the formula $v = \dfrac{-b + \sqrt{b^2 - 4ac}}{2a}$

e What factors would determine the safe car separation distance in traffic?

challenge 3:03

9 The rise and tread of a staircase have been connected using the formula $r = \frac{1}{2}(24 - t)$, or $r = 66/t$ where r and t are measured in inches. (One inch is about 2·54 centimetres.)



a If the tread should not be less than 9 inches, what can be said about the rise?

b Graph both functions on the same set of axes and compare the information they provide.

c What are the points of intersection of the two graphs? Check the accuracy of your graphs by solving the two simultaneous equations,

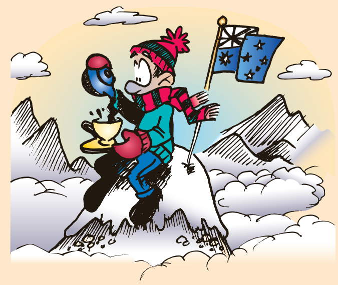
$$\begin{cases} r = \frac{1}{2}(24 - t) & \ldots \quad \text{(1)} \\ r = 66/t & \ldots \quad \text{(2)} \end{cases}$$

(*Hint:* Substitute (2) into (1) and solve the resulting quadratic equation.)

d Convert formulae (1) and (2) to formulae applicable to centimetres rather than inches.

e Do the measurements of staircases you have experienced fit these formulae? Investigate other methods used by builders to determine r and t.

Investigation 3:05 | Temperature and altitude



Please use the Assessment Grid on the following page to help you understand what is required for this Investigation.

The following formula has been used to give the boiling point of water at various heights above sea-level:

$h = 520(212 - T) + (212 - T)^2$

where height, h, is in feet and the temperature, T, is in degrees Fahrenheit (°F).

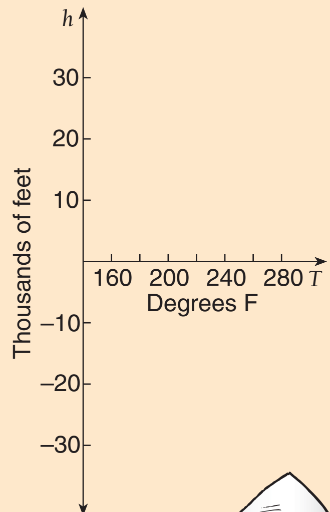
1 Show that $h = (212 - T)(732 - T)$.

2 At what height above sea-level does water boil at:
a 200°F? **b** 250°F?

3 Plot a graph of h against T. (Use values of T from 160°F to 280°F. Use values of h from –30 000 feet to 30 000 feet.)

T in °F	160	180	200	220	240	260	280
h in feet	29 744						–30 736

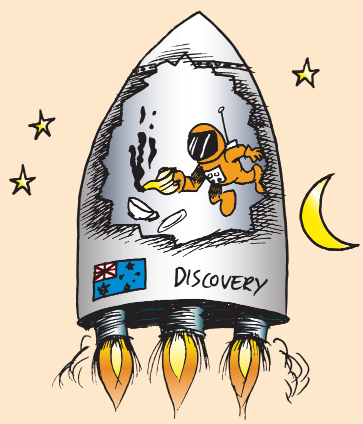
4 Use your graph to find the temperature at which water boils:
- **a** at Flinders Peak (1155 feet above sea-level)
- **b** atop Mt Everest (28 000 feet above sea-level)
- **c** at the bottom of the Mindanao trench (35 000 feet below sea-level).



5 Check your answers to question **4** by substituting each height into the formula and solving the resulting quadratic equation.

6 Change the units on the axes of your graph so that they are in degrees Celsius (°C) and thousands of metres. To do this, use the formula $C = \frac{5}{9}(F - 32)$ and the approximation, 1 foot = 0·305 metres.

7 Discuss:
- Over what temperature range is this formula useful or valid?
- Find the height at which the space shuttle orbits. Can the formula be used there? Is the shuttle pressurised to some equivalent height above sea-level?
- Do all the results you have found make sense and seem logical?
- How accurately have you stated your answers? Is this reasonable?



Assessment Grid for Investigation 3:05 | Temperature and altitude

The following is a sample assessment grid for this investigation. You should carefully read the criteria *before* beginning the investigation so that you know what is required.

		Assessment Criteria (C, D) for this investigation		Achieved ✓
Criterion C Communication in Mathematics	a	None of the following descriptors have been achieved.	0	
	b	There is a basic use of mathematical language and representation. Lines of calculation and reasoning are insufficient.	1	
			2	
	c	There is satisfactory use of mathematical language and representation. Graphs and explanations are clear but not always thorough or complete.	3	
			4	
	d	A good use of mathematical language and representation. Graphs are accurate, to scale and fully labeled. Calculations and explanations are complete and concise.	5	
			6	
Criterion D Reflection in Mathematics	a	None of the following descriptors have been achieved.	0	
	b	An attempt has been made to explain whether the results make sense and are consistent. An attempt has been made to address aspects of parts 4 and 7.	1	
			2	
	c	There is a correct but brief explanation of whether results make sense and how they were found. Some description of the relevant temperature range(s) is given. Some consideration of the accuracy and appropriateness of results is given.	3	
			4	
	d	There is a critical explanation of the results obtained with thorough checking. There is a detailed discussion and analysis of when the formula can be used and of which temperature range(s) are applicable and why. The accuracy and appropriateness of results is fully justified.	5	
			6	

Fun Spot 3:05 | Did you know that 2 = 1?

Now that your algebra skills are more developed, you can be let into the secret that 2 really is equal to 1.

Proof: Assume that $x = y$.

Multiply both sides by x,	$x^2 = xy$
Subtract y^2 from both sides,	$x^2 - y^2 = xy - y^2$
Factorise both sides,	$(x - y)(x + y) = y(x - y)$
Divide by $(x - y)$,	$x + y = y$
Now, if we let $x = y = 1$, then $2 = 1$	Q.E.D.

Mathematical Terms 3

coefficient
- The number that multiplies a pronumeral in an equation or algebraic expression.
  eg $3x^2 - x + 5 = 0$
  coefficient of x^2 is 3
  coefficient of x is -1

completing the square
- Completing an algebraic expression to form a perfect square, ie $(x + a)^2$ or $(x - a)^2$.
  eg To complete the square for $x^2 + 6x$, the number 9 is added,
  thus: $x^2 + 6x + 9 = (x + 3)^2$

factorise
- To write an expression as the product of its factors.
- The reverse of expanding.

quadratic equation
- An equation in which the highest power of the unknown pronumeral is 2.
  eg $x^2 - 16 = 0$, $x^2 + 5x + 6 = 0$
- A quadratic equation may have two solutions.

quadratic formula
- A formula that gives the solutions to equations of the form $ax^2 + bx + c = 0$.
- $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

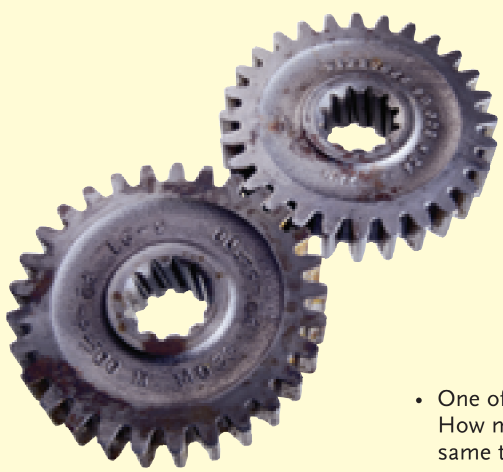
Mathematical terms 3

diagnostic test 3

Diagnostic Test 3 | Quadratic Equations

- Each part of this test has similar items that test a certain question type.
- Errors made will indicate areas of weakness.
- Each weakness should be treated by going back to the section listed.

	Section
1 Solve these equations: **a** $(x + 7)(x - 3) = 0$ **b** $x(x - 5) = 0$ **c** $(2x - 1)(x + 1) = 0$ **d** $(3x + 2)(4x - 5) = 0$	3:01
2 Factorise and solve: **a** $x^2 + 5x = 0$ **b** $x^2 + 9x + 14 = 0$ **c** $x^2 - 49 = 0$ **d** $2x^2 + 5x - 3 = 0$	3:01
3 What number must be inserted to complete the square? **a** $x^2 + 6x + \ldots$ **b** $x^2 - 4x + \ldots$ **c** $x^2 + 3x + \ldots$ **d** $x^2 - x + \ldots$	3:02
4 Solve the following by completing the square. **a** $x^2 + 2x - 2 = 0$ **b** $x^2 - 6x + 1 = 0$ **c** $x^2 - 3x - 5 = 0$ **d** $2x^2 - 10x = 1$	3:02
5 Solve using the quadratic formula. (Leave answers in surd form.) **a** $x^2 + x - 3 = 0$ **b** $x^2 - 5x + 2 = 0$ **c** $2x^2 + 4x + 1 = 0$ **d** $3x^2 + 2x - 2 = 0$	3:03
6 Solve the following: **a** $x^2 - x + 1 = 4x + 7$ **b** $x(x - 5) = x - 9$ **c** $(x + 4)^2 = 6$ **d** $x = \frac{2x + 8}{x}$	3:04



- One of these gear wheels has 28 teeth and the other has 29 teeth. How many revolutions of each wheel must be completed for the same two teeth to be in the same position next to each other?

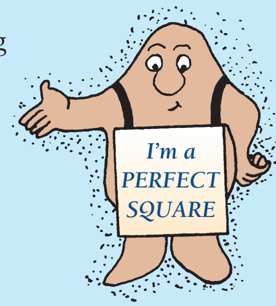
Chapter 3 | Revision Assignment

1 Solve the following quadratic equations using the method you feel is most appropriate.

a $x^2 + x - 30 = 0$
b $x(x - 7) = 0$
c $(x + 1)^2 = 9$
d $2x^2 + 7x - 15 = 0$
e $x^2 + 2x = 24$
f $(x + 2)(3x - 1) = 0$
g $x^2 - 14x + 49 = 0$
h $5x(2x - 3) = 0$
i $x^2 - 100 = 0$
j $x^2 - 5x - 14 = 0$
k $x^2 = 28 + 3x$
l $x^2 + 5x + 1 = 0$
m $(x - 3)^2 = 2$
n $10x^2 - 3x - 1 = 0$
o $x^2 - 20 = 0$
p $5x^2 + 3x = 0$
q $x^2 + 10x + 25 = 0$
r $x^2 + 2x - 4 = 0$
s $3x^2 - x - 2 = 0$
t $2x^2 + 5x + 1 = 0$
u $x(x + 5) = 24$
v $(x + 2)^2 = x + 2$
w $x = \dfrac{5x - 2}{x}$
x $(2x + 1)^2 = (x + 3)^2$

2 Solve by completing the square.

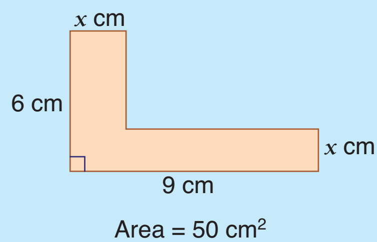
a $x^2 + 4x - 32 = 0$
b $x^2 - 3x - 40 = 0$
c $x^2 - 10x + 4 = 0$
d $2x^2 + 6x - 3 = 0$



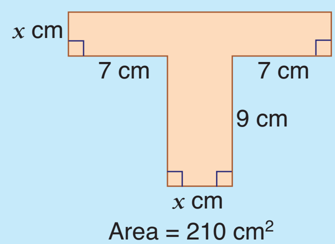
3 Find three consecutive positive integers if the sum of their squares is 50.

4 Find x in the following figures.

a



b



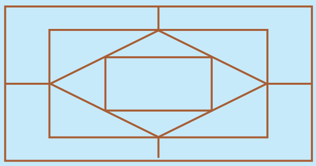
5 If a rectangular field has an area of 0·28 ha and its length is 30 m more than its width, find the width of the field.

assignment 3B

Chapter 3 | Working Mathematically

1 An odd number between 301 and 370 has three different digits. If the sum of its digits is five times the hundreds digit, what is the digit?

2 What is the minimum number of colours needed to shade this diagram if no two adjacent regions may have the same colour?



3 Hai Young's passbook savings account allowed her to deposit or withdraw at any time. Her interest, which was 2·5% pa, was calculated on the minimum monthly balance and was paid twice yearly into her account. She could withdraw up to €500 in cash per day or any amount in the form of a cheque. Cheques for the payment of bills (third party cheques) were provided free of charge. She was able to start her account with as little as €1.

a What is the minimum balance required?

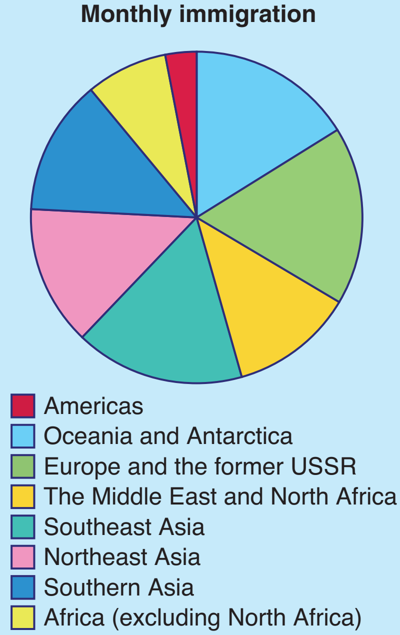
b On what amount is the interest calculated?

c Does the interest earned in one month automatically begin to earn interest during the next month?

4 Decrease $360 by 20% and then increase the result by 20%. What is the difference between $360 and your final answer?

5 50% more than what number is 25% less than 60% more than 10?

6 The monthly immigration to Australia for May 2004 is shown by this pie chart.



a Which region provided:

i the most immigrants?

ii the least immigrants?

b Measure the angle of each sector to determine the percentage of immigrants from:

i Southeast Asia

ii Africa

c If the total number of immigrants was 9690, how many (to the nearest hundred) came from:

i the Americas?

ii Oceania and Antarctica?

Chapter Review Questions

Technology Applications Activities

Completing the square

Technology Applications Drag and Drops

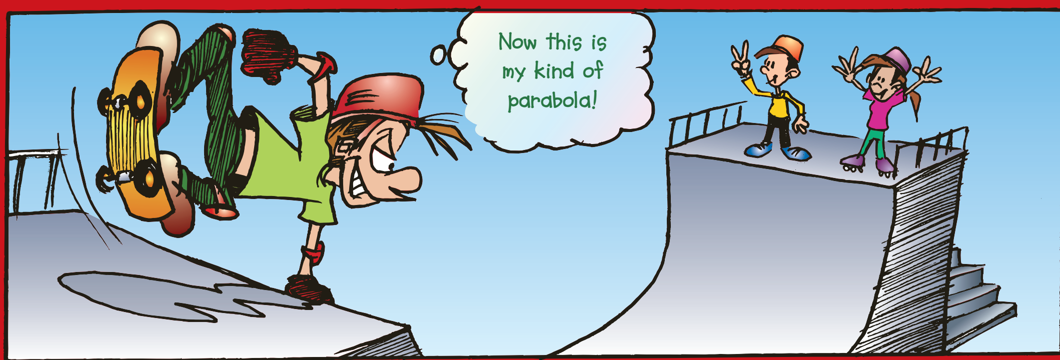
1 Quadratic equations 1

2 Quadratic equations 2

3 Completing the square

4

Number Plane Graphs and Coordinate Geometry



Chapter Contents

Learning Outcomes

Students will be able to:

- Draw and interpret graphs including parabolas, cubics, hyperbolas, exponentials and circles.
- Apply coordinate geometry techniques, including the distance gradient and midpoints of intervals, to solve problems.

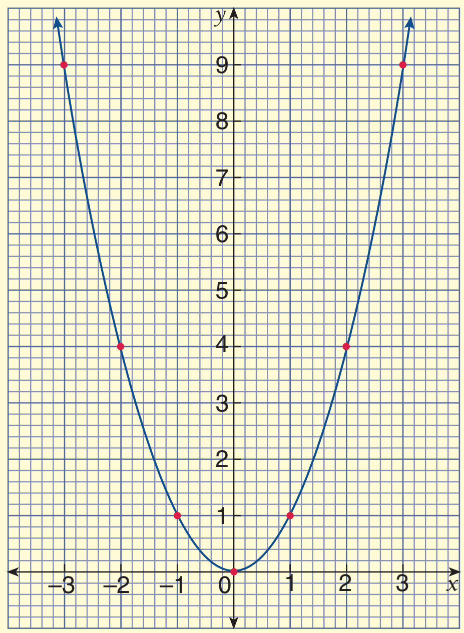
Areas of Interaction

Approaches to Learning (Knowledge Acquisition, Logical Thinking, IT Skills, Reflection), Human Ingenuity

4:01 | The Parabola

• The shape of the parabola is clearly demonstrated by the water arcs of this fountain.

Up until this point, all the graphs have been straight lines. In this section, we will look at a most famous mathematical curve, the parabola.



- The equations of parabolas are called quadratic equations and have x^2 as the highest power of x.
- The simplest equation of a parabola is:

$$y = x^2$$

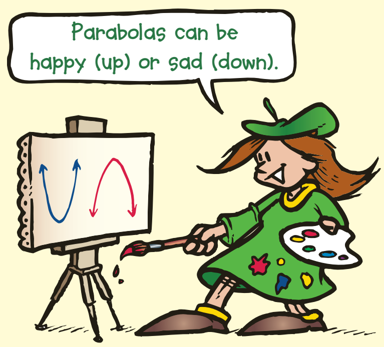
- As with the straight line, the equation is used to find the points on the curve. Some of these are shown in the table.

  $y = x^2$

x	−3	−2	−1	0	1	2	3
y	9	4	1	0	1	4	9

  For an accurate graph, many points would have to be plotted.

- From the graph we can see that the parabola has a turning point, or vertex, which is the minimum value of y on $y = x^2$.
- The y-axis is an axis of symmetry of the curve, so the right side of the curve is a reflection of the left side. This can be seen when points on either side of the axis are compared.
- The parabola is concave up, which means it opens out upwards.



Exercise 4:01

> *Note:* A graphics calculator or computer graphing software could be used in the following exercise as an alternative to plotting points.

1 Complete the following tables and then graph all four curves on one number plane.
Hint: On the y-axis, use values from 0 to 12.

a $y = x^2$

x	−3	−2	−1	0	1	2	3
y							

b $y = 2x^2$

x	−2	−1	−0·5	0	0·5	1	2
y							

c $y = 3x^2$

x	−2	−1	−0·5	0	0·5	1	2
y							

d $y = 0{\cdot}5x^2$

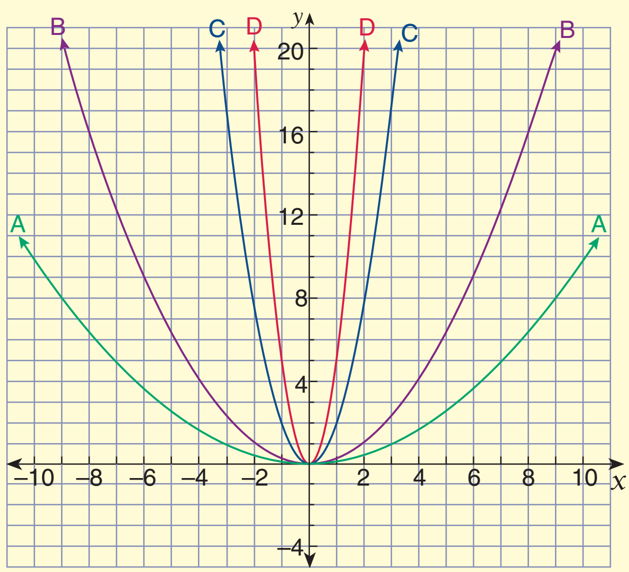
x	−3	−2	−1	0	1	2	3
y							

For the equation $y = ax^2$, what is the effect on the graph of varying the value of a?

2 Match each of the parabolas A to D with the equations below.

a $y = 0{\cdot}25x^2$
b $y = 5x^2$
c $y = 2x^2$
d $y = 0{\cdot}1x^2$

> These parabolas are all concave up.



3 Complete the following tables and then graph all four curves on one number plane.
Hint: On the y-axis, use values from −2 to 13.

a $y = x^2$

x	−3	−2	−1	0	1	2	3
y							

b $y = x^2 + 2$

x	−3	−2	−1	0	1	2	3
y							

c $y = x^2 + 4$

x	−3	−2	−1	0	1	2	3
y							

d $y = x^2 - 2$

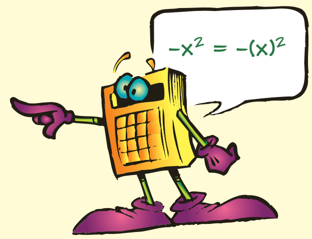
x	−3	−2	−1	0	1	2	3
y							

What is the difference in the curves $y = x^2$ and $y = x^2 + 2$?
Can you see that the shape of the curve is the same in each case?
For the equation $y = x^2 + c$, what is the effect on the graph of varying the value of c?

4 **a** Complete the table of values for $y = -x^2$ and sketch its graph.

$y = -x^2$

x	-3	-2	-1	-0.5	0	0.5	1	2	3
y									



b Sketch the graph of $y = -2x^2$.

c For $y = ax^2$, what does the graph look like if the value of a is negative?

5 On the same number plane, sketch the graphs of $y = -x^2 - 1$ and $y = -x^2 + 2$.

6 Match each of the following equations with the graphs A to E.

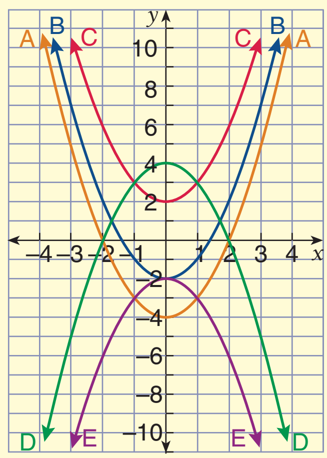
a $y = -x^2 - 2$

b $y = x^2 + 2$

c $y = x^2 - 2$

d $y = x^2 - 4$

e $y = 4 - x^2$



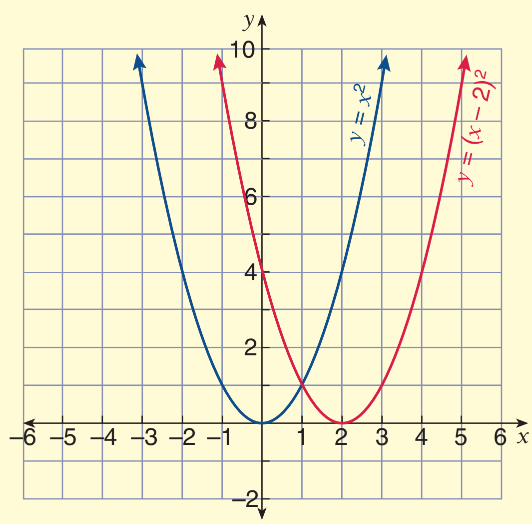
7 The graphs of $y = x^2$ and $y = (x - 2)^2$ are shown on the diagram.

a How are the two graphs related?

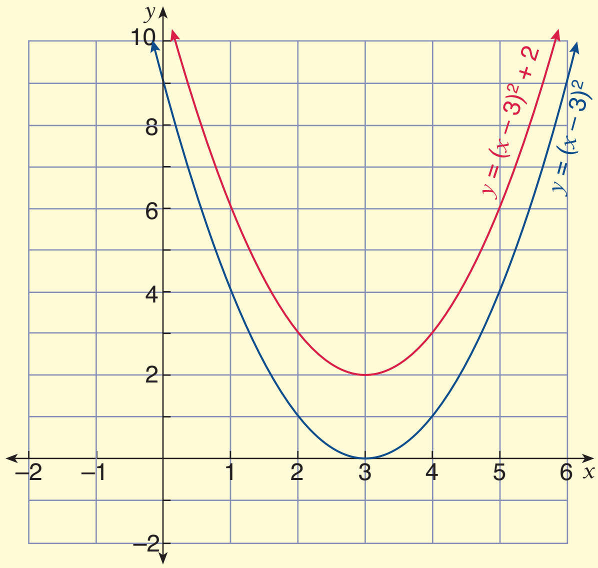
b Graph the parabola $y = (x + 2)^2$. How is it related to the graph of $y = x^2$?

c Sketch the graphs of $y = (x - 4)^2$ and $y = (x + 1)^2$ on different number planes.

d How are the graphs of $y = (x - h)^2$ and $y = (x + h)^2$ related to the graph of $y = x^2$?



8 The graphs of $y = (x - 3)^2$ and $y = (x - 3)^2 + 2$ are shown on the diagram.



a How is the graph of $y = (x - 3)^2 + 2$ related to the graph of $y = (x - 3)^2$?

b How would the graph of $y = (x - 3)^2 - 2$ be obtained from the graph of $y = (x - 3)^2$?

c Sketch the graph of $y = (x + 3)^2 + 2$.

d Use the questions in **a** to **c** to explain the connection between the equation of the parabola and the coordinates of its vertex.

e Sketch the graph of each parabola on a separate number plane.

i $y = (x - 2)^2 + 3$ **ii** $y = (x + 2)^2 + 3$

iii $y = (x - 4)^2 - 2$ **iv** $y = (x + 1)^2 - 2$

9 On separate number planes, sketch the following parabolas:

a $y = (x + 4)^2$ **b** $y = -(x + 4)^2$ **c** $y = -(x + 4)^2 + 3$

10 Find the equation of the parabola that results from performing the following transformations on the parabola $y = x^2$.

a moving it up 2 units **b** moving it down 2 units

c moving it 2 units to the right **d** moving it 2 units to the left

e turning it upside down and then moving it up 4 units

f turning it upside down and then moving it down 2 units

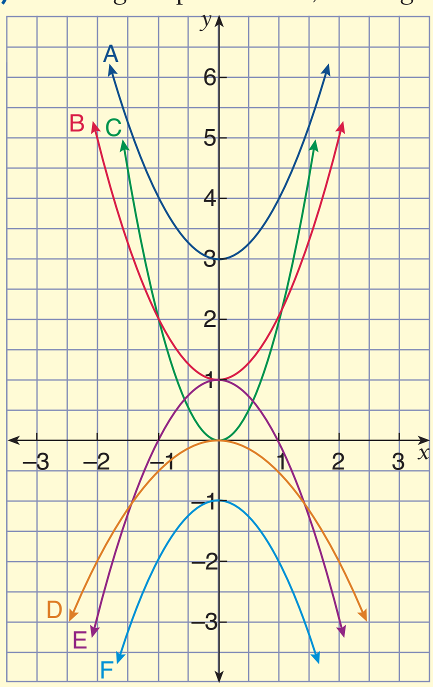
g moving it up 2 units and then reflecting it in the *x*-axis

h moving it 2 units to the right and then turning it upside down

i moving it up 2 units and then moving it 2 units to the left.

j turning it upside down, moving it 3 units to the left and then moving it down 2 units.

11



Match each equation with the corresponding parabola in the diagram.

a $y = 2x^2$

b $y = -\frac{1}{2}x^2$

c $y = x^2 + 1$

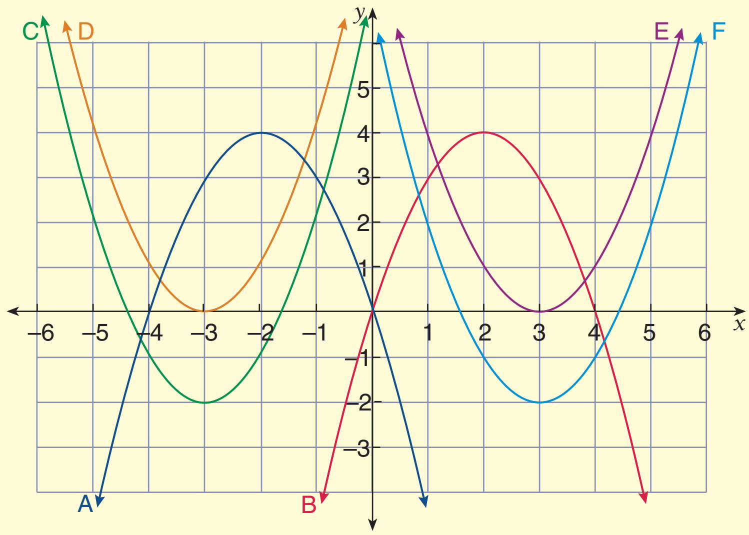
d $y = x^2 + 3$

e $y = -x^2 - 1$

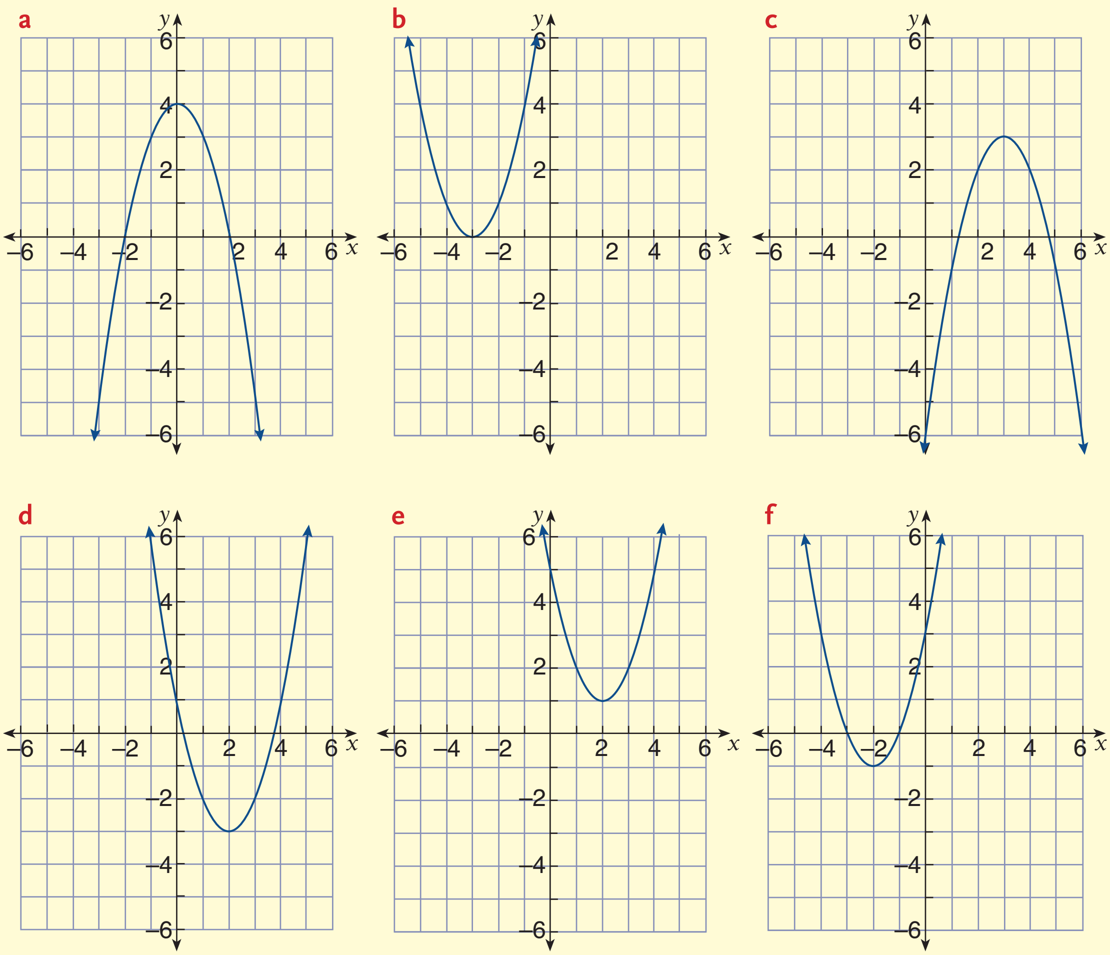
f $y = 1 - x^2$

12 Match each equation with the corresponding parabola A to F in the diagram.

- **a** $y = (x - 3)^2$
- **b** $y = (x + 3)^2$
- **c** $y = -(x + 2)^2 + 4$
- **d** $y = -(x - 2)^2 + 4$
- **e** $y = (x + 3)^2 - 2$
- **f** $y = (x - 3)^2 - 2$



13 The parabolas shown are the result of translating and/or reflecting the parabola $y = x^2$. Find the equation of each parabola.

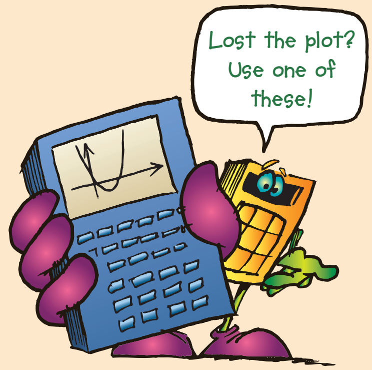


investigation 4:01

Investigation 4:01 | The graphs of parabolas

Please use the Assessment Grid on the following page to help you understand what is required for this Investigation.

A graphics calculator or computer graphing package are excellent tools for investigating the relationship between the equation of a parabola and its graph.



Use either of the above to investigate graphs of the following forms for varying values of *a*, *h* and *k*.

1 $y = ax^2$

2 $y = ax^2 \pm k$

3 $y = (x \pm h)^2$

4 $y = (x \pm h)^2 + k$

Use the patterns you have found above to write a report on each of the 4 forms, explaining how the features of the graph, such as the concavity, the position of the vertex and the number of *x*-intercepts, are related to the values of *a*, *h* and *k*.

Can you now use choose and apply appropriate techniques to find:

i the equation of a parabola with its vertex at (3, 2)?

ii the equation of the parabola with a vertex at (1, −4) and a *y*-intercept at (0, −2)?

Make sure you fully explain and support your answers.

Reflection:
Are your results consistent and do they make sense?
Are the effects of *a*, *h* and *k* the same for non-integer values?
Can you find and discuss some real life uses or applications of parabolic curves?

• The dish of a radio-telescope is parabolic in shape.

Assessment Grid for Investigation 4:01 | The graphs of parabolas

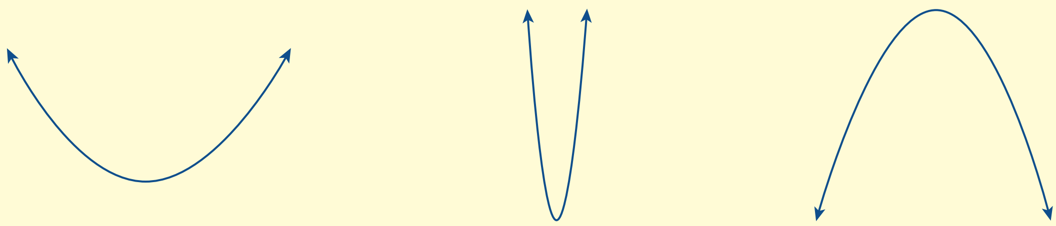
The following is a sample assessment grid for this investigation. You should carefully read the criteria *before* beginning the investigation so that you know what is required.

Assessment Criteria (B, C, D) for this investigation				Achieved ✓
Criterion B Investigating Patterns	a	None of the following descriptors have been achieved.	0	
	b	Some help was needed to describe the patterns evident in the effects of a, h and k.	1	
			2	
	c	Mathematical techniques have been used to describe the effects of a, h and k and explain their relationship to the shape and position of the parabola, with attempts to find the two equations.	3	
			4	
	d	The student has clearly identified the patterns related to a, h and k, explained their effects on the parabola and used these patterns to find the two equations required.	5	
			6	
	e	The student has completed all parts above with thorough explanation and support throughout.	7	
			8	
Criterion C Communication in Mathematics	a	None of the following descriptors have been achieved.	0	
	b	There is a basic use of mathematical language and representation. Lines of reasoning are insufficient.	1	
			2	
	c	There is satisfactory use of mathematical language and representation. Graphs and explanations are clear but not always logical or complete.	3	
			4	
	d	A good use of mathematical language and representation. Graphs are accurate, to scale and fully labeled. Explanations are complete and concise.	5	
			6	
Criterion D Reflection in Mathematics	a	None of the following descriptors have been achieved.	0	
	b	An attempt has been made to explain whether the results make sense and are consistent. An attempt has been made to make connection to real-life applications.	1	
			2	
	c	There is a correct but brief explanation of whether results make sense and how they were found. Some description of the effects of a, h and k is given with relevant links to real life applications. Some consideration of the accuracy of results is given and use of non-integer values.	3	
			4	
	d	There is a critical explanation of the results obtained and their related effects. Consistency and relevance is considered and there is a detailed explanation of the importance of parabolas in real life. The effect of non-integer values is discussed.	5	
			6	

4:02 | Parabolas of the Form $y = ax^2 + bx + c$

In the last section, we saw that all parabolas have the same basic shape.

- They are all concave up or concave down with a single vertex or turning point.
- They are symmetrical about an axis of symmetry.



We looked at the connection between the parabola's shape and its equation and at what numbers in the equation influenced the steepness, the concavity and the position of the graph on the number plane.

In this section, we look at how to sketch the parabola when its equation is given in the form $y = ax^2 + bx + c$. We will also look at how to find features of the parabola, such as the x- and y-intercepts, the axis of symmetry, the vertex and the maximum or minimum value of y.

Finding the *y*-intercept

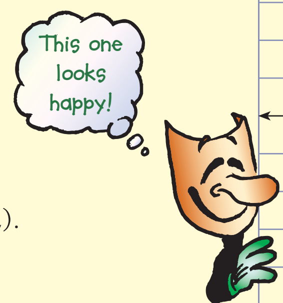
To find the y-intercept of $y = x^2 + x - 12$, we let x be zero.

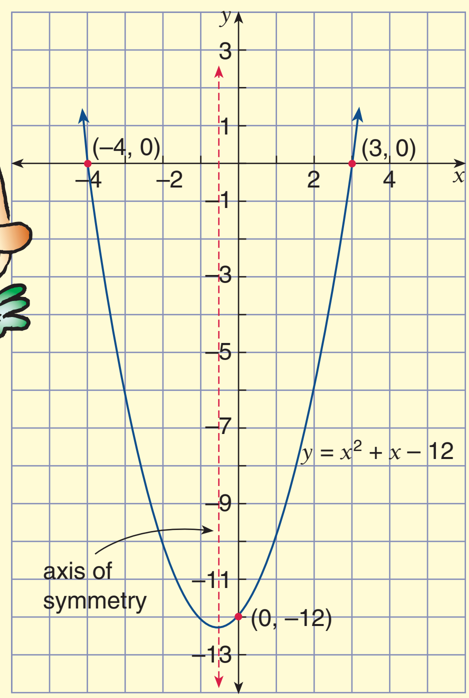
$y = x^2 + x - 12$

When $x = 0$, $y = -12$

$\therefore$ The y-intercept is -12.

$\therefore$ The curve cuts the y-axis at $(0, -12)$.





Finding the *x*-intercepts

To find the x-intercepts of $y = x^2 + x - 12$, we let y be zero.

$y = x^2 + x - 12$

When $y = 0$, $\quad 0 = x^2 + x - 12$

Solving this, $\quad 0 = (x + 4)(x - 3)$

$\therefore x = -4$ or 3

$\therefore$ The x-intercepts are -4 and 3.

$\therefore$ The curve cuts the x-axis at $(-4, 0)$ and $(3, 0)$.

Note: If $x^2 + x - 12 = 0$ was difficult to factorise, then the formula

$x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ could have been used to find the x-intercepts.

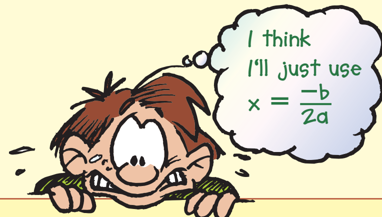
Finding the axis of symmetry

Since the parabola $y = x^2 + x - 12$ has a vertical axis of symmetry, the axis of symmetry will cut the x-axis half-way between (−4, 0) and (3, 0), which are the two x-intercepts. The axis of symmetry will have the equation $x = \frac{-4+3}{2}$,

ie $x = -\frac{1}{2}$ is the axis of symmetry.

The axis of symmetry of the parabola $y = ax^2 + bx + c$ is given by the equation

$$x = \frac{-b}{2a}$$



For the parabola $y = ax^2 + bx + c$, the x-intercepts are found by solving $0 = ax^2 + bx + c$.

$$\text{ie } x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

The axis of symmetry will cut the x-axis half-way between these two values.

$$x = \frac{\left(\frac{-b + \sqrt{b^2-4ac}}{2a} + \frac{-b - \sqrt{b^2-4ac}}{2a}\right)}{2}$$

$$x = \frac{\left(\frac{-2b}{2a}\right)}{2} = \frac{-2b}{2a} \div 2$$

$$x = \frac{-b}{2a}$$

∴ The axis of symmetry is $x = \frac{-b}{2a}$

For the parabola $y = x^2 + x - 12$,
$a = 1$, $b = 1$, $c = -12$.
∴ The axis of symmetry is

$$x = \frac{-1}{2(1)}$$

ie $x = -\frac{1}{2}$

Finding the vertex (or turning point) and the maximum or minimum value

As the vertex lies on the axis of symmetry, its x-coordinate will be the same as that of the axis of symmetry. The y-coordinate can be found by substituting this x value into the equation of the parabola.

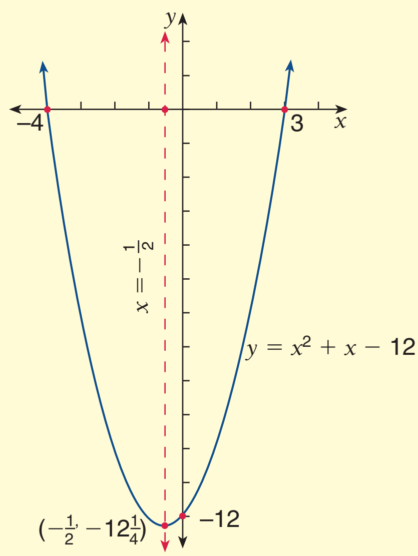
For $y = x^2 + x - 12$, the axis of symmetry is $x = -\frac{1}{2}$.

Now, when $x = -\frac{1}{2}$,

$y = (-\frac{1}{2})^2 + (-\frac{1}{2}) - 12$

$y = -12\frac{1}{4}$

∴ The vertex of the parabola is $(-\frac{1}{2}, -12\frac{1}{4})$.



The minimum or maximum value of y will occur at the vertex. The parabola will have:

- a minimum value of y if the parabola is concave up (when the coefficient of x^2 is positive, eg $y = 2x^2$)
- a maximum value of y if the parabola is concave down (when the coefficient of x^2 is negative, eg $y = -2x^2$)

Hence, on the parabola $y = x^2 + x - 12$, the minimum value of y is $-12\frac{1}{4}$ when $x = -\frac{1}{2}$.

The method of completing the square can also be used to find the minimum or maximum value and the vertex as shown below.

$y = x^2 + x - 12$

$y = (x^2 + x + \frac{1}{4}) - \frac{1}{4} - 12$

$y = (x + \frac{1}{2})^2 - 12\frac{1}{4}$

As $(x + \frac{1}{2})^2$ is always greater than or equal to 0, the minimum value of y will be $-12\frac{1}{4}$ when $x = -\frac{1}{2}$ and the vertex is the point $(-\frac{1}{2}, -12\frac{1}{4})$.

worked examples

For each equation, find:

a the y-intercept

b the x-intercept

c the axis of symmetry

d the vertex (turning point)

Use these results to sketch each graph.

1 $y = x^2 + 2x - 3$ **2** $y = 2x^2 + 4x + 3$ **3** $y = 4x - x^2$

Solutions

1 a For the y-intercept, let $x = 0$

$y = (0)^2 + 2(0) - 3$

The y-intercept is -3.

b For the x-intercepts, let $y = 0$

$0 = x^2 + 2x - 3$

$(x + 3)(x - 1) = 0$

The x-intercepts are -3 and 1.

c Axis of symmetry:

$x = \frac{-3 + 1}{2}$ (midpoint of x-intercepts)

$\therefore x = -1$ is the axis of symmetry.

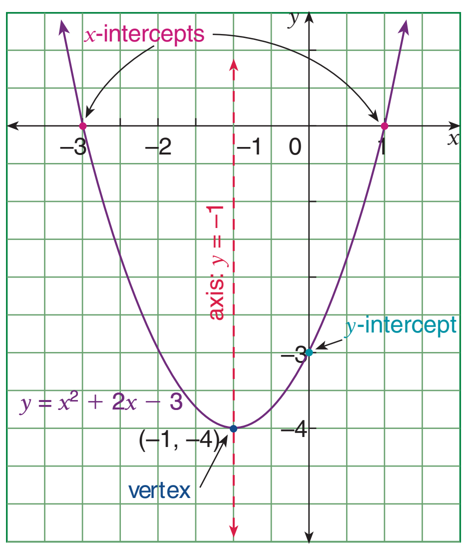
d To find the vertex, substitute $x = -1$ into the equation to find the y-value.

$y = (-1)^2 + 2(-1) - 3$

$y = -4$

$\therefore$ The vertex is $(-1, -4)$.

We now plot the above information on a number plane and fit the parabola to it.

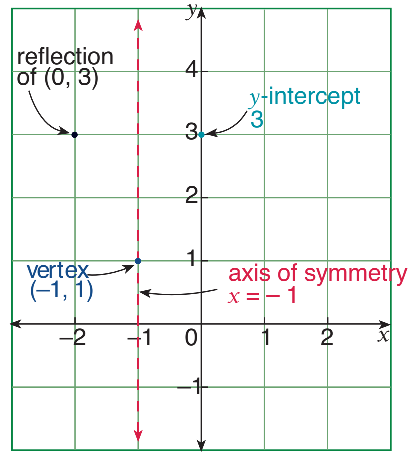


continued →→→

2 **a** To find the y-intercept of $y = 2x^2 + 4x + 3$, let x be zero.

$y = 2(0)^2 + 4(0) + 3$

$\therefore$ The y-intercept is 3.



b For the x-intercepts, solve $2x^2 + 4x + 3 = 0$. However, when we use the formula, we get

$$x = \frac{-4 \pm \sqrt{4^2 - 4 \times 2 \times 3}}{2 \times 2}$$

$$= \frac{-4 \pm \sqrt{-8}}{4}$$

This gives us a negative number under the square root sign.

You can't find the square root of a negative number!

Thus, there are no solutions, so the parabola does not cut the x-axis.

c $y = 2x^2 + 4x + 3$ has $a = 2$, $b = 4$, $c = 3$.

$\therefore$ Axis of symmetry is $x = \frac{-b}{2a}$.

ie $x = \frac{-4}{2(2)}$

$\therefore$ The axis of symmetry is $x = -1$.

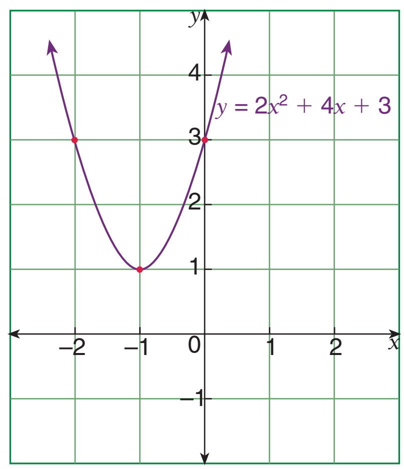
d The vertex is the turning point of the curve, and is on the axis of symmetry $x = -1$.

When $x = -1$,

$y = 2(-1)^2 + 4(-1) + 3$

$= 1$

$\therefore$ The vertex is $(-1, 1)$.



3 **a** For $y = 4x - x^2$, if $x = 0$, then $y = 0$.
$\therefore$ The curve cuts the y-axis at the origin.

b When $y = 0$, $\quad 4x - x^2 = 0$

$$x(4 - x) = 0$$

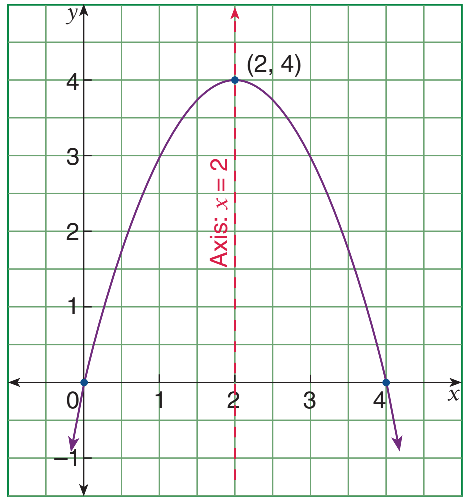
$\therefore$ The x-intercepts are 0 and 4.

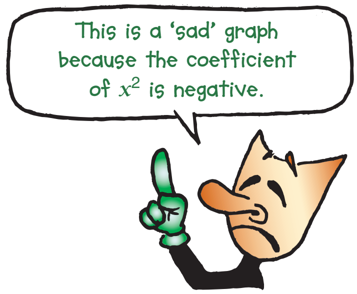
c Axis: $x = 2$ (midpoint of x-intercepts)

d When $x = 2$, $\quad y = 4(2) - (2)^2$

$$= 4$$

$\therefore$ The vertex is $(2, 4)$.

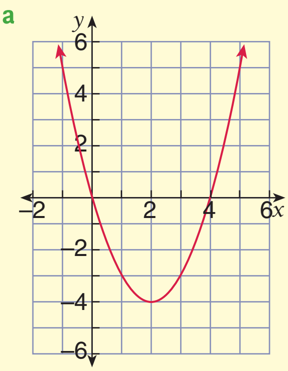




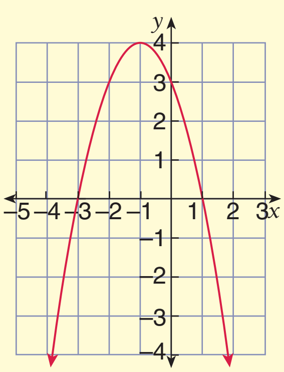
Exercise 4:02

1 For each of the graphs, find:
- **i** the y-intercept
- **ii** the x-intercepts
- **iii** the equation of the axis of symmetry
- **iv** the coordinates of the vertex

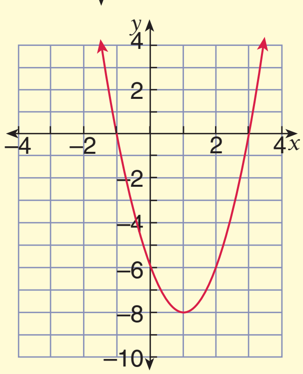
a



b



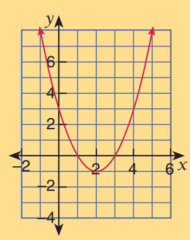
c



Foundation Worksheet 4:02

The parabola $y = ax^2 + bx + c$

1 a



Use the graph to find:
- **i** the y-intercept **ii** the x-intercepts
- **iii** the equation of the axis of symmetry
- **iv** the coordinates of the vertex

2 a For the parabola $y = x^2 + x + 2$, find:
- **i** the y-intercept **ii** the x-intercepts
- **iii** the equation of the axis of symmetry
- **iv** the coordinates of the vertex

2 Find the y-intercepts of the following parabolas.

a $y = x^2 - 6x + 5$ **b** $y = 2x^2 - 8$ **c** $y = (x - 2)(x + 3)$

3 Find the x-intercepts of the following parabolas.

a $y = x^2 - 2x - 8$ **b** $y = 3x^2 + 10x - 8$ **c** $y = (x - 3)(4x + 7)$

4 Find the equation of the axis of symmetry and the coordinates of the vertex of the following parabolas.

a $y = (x - 3)(x - 5)$ **b** $y = 3(x - 2)(x + 6)$ **c** $y = -\frac{1}{2}(x + 4)(2 - x)$

d $y = x^2 - 6x + 7$ **e** $y = 3x^2 - 9x + 14$ **f** $y = 4 - 3x - x^2$

5 Find the minimum value of y on the following parabolas.

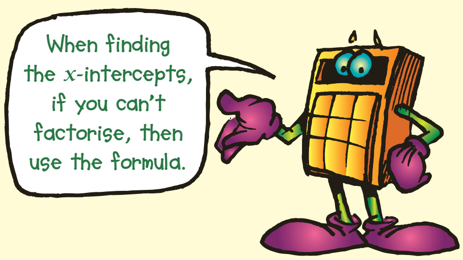
a $y = x^2 - 6x - 2$ **b** $y = 4x^2 - 4x + 6$ **c** $y = 9x^2 - 30x + 18$

6 Find the maximum value of y on the following parabolas.

a $y = 1 - 2x - x^2$ **b** $y = -4x^2 + 20x - 27$ **c** $y = 7 - 12x - 9x^2$

7 For the parabola $y = x^2 + 2x - 8$, find:

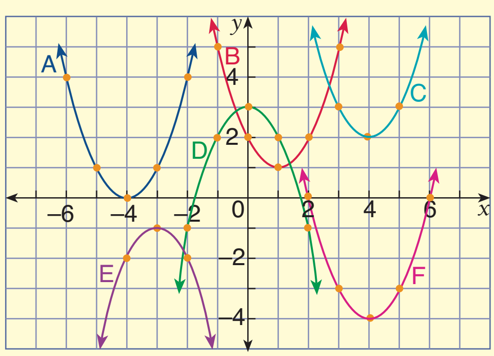
a the y-intercept
b the x-intercepts
c the axis of symmetry
d the vertex
e hence, sketch its graph



8 Repeat the steps in question **7** to graph the following equations, showing all the relevant features.

a $y = x^2 - 6x + 5$ **b** $y = x^2 - 6x$
c $y = 2x^2 - 8x - 10$ **d** $y = -x^2 + 4x - 3$
e $y = -x^2 + 6x - 9$ **f** $y = 2x^2 + 4x + 2$
g $y = x^2 - 3x - 4$ **h** $y = 2x^2 - 3x - 2$
i $y = -2x^2 - 3x - 1$

9 Match each graph with one of the equations written below the diagram. Each graph has an x^2 shape.



a $y = -x^2 + 3$ **b** $y = x^2 - 2x + 2$
c $y = x^2 - 8x + 12$ **d** $y = x^2 + 8x + 16$
e $y = -x^2 - 6x - 10$ **f** $y = x^2 - 8x + 18$

1 Find the turning point.
2 Is it happy ∪ *(a is +ve) or sad* ∩ *(a is −ve)?*
3 Visualise the graph before you sketch.

10 Sketch each set of three parabolas on the same number plane.

	i	ii	iii
a	$y = x^2 - 4$	$y = x^2 - 4x$	$y = x^2 - 4x + 4$
b	$y = 9 - x^2$	$y = 9x - x^2$	$y = 10 + 9x - x^2$
c	$y = (x - 3)(x + 5)$	$y = 2(x - 3)(x + 5)$	$y = (3 - x)(5 + x)$
d	$y = x^2 - 2x - 8$	$y = 2x^2 - 4x - 16$	$y = 8 + 2x - x^2$

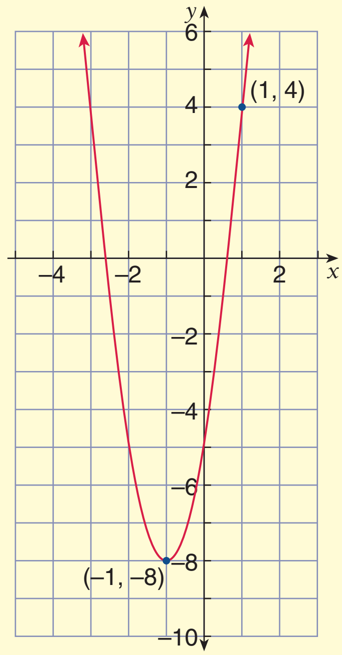
11 Sketch the graph of each quadratic relationship, showing all relevant features.

- **a** $y = 2x^2 - 8$
- **b** $y = 16 - x^2$
- **c** $y = (x + 2)(x - 6)$
- **d** $y = x^2 + 4x + 3$
- **e** $y = x^2 - 8x + 7$
- **f** $y = x^2 - 5x$
- **g** $y = (3 - x)(7 + x)$
- **h** $y = 24 - 2x - x^2$
- **i** $y = 4x^2 + 16x + 7$
- **j** $y = 2x^2 + 9x - 5$
- **k** $y = 4x^2 - 36x + 56$
- **l** $y = 2x^2 - 5x - 7$

12
- **a** Sketch the graphs of $y = x^2 + 3x$ and $y = 2x^2 + 6x$. Compare the two graphs and describe the difference between them.
- **b** Sketch the graphs of $y = (x - 3)(x + 2)$ and $y = 3(x - 3)(x + 2)$. Compare the two graphs and describe the difference between them.

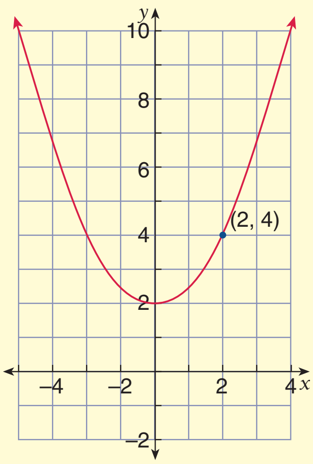
13 The parabola in the diagram has its vertex at $(-1, -8)$ and it passes through the point $(1, 4)$. The equation of the parabola has the form $y = ax^2 + bx + c$.

- **a** Use the y-intercept to show that $c = -5$.
- **b** Use the equation of the axis of symmetry to show that $b = 2a$ and that the equation of the parabola is of the form $y = ax^2 + 2ax - 5$.
- **c** Substitute the coordinates of the vertex or the point $(1, 4)$ to find the value of a.
- **d** What is the equation of the parabola?

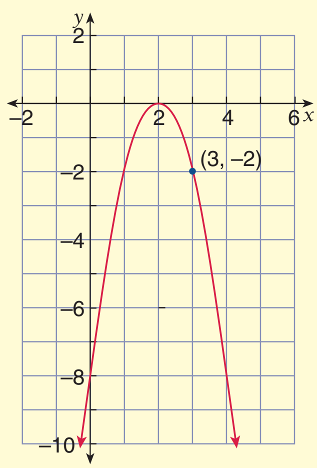


14 Use the method of question **13** to find the equation of each of the following parabolas.

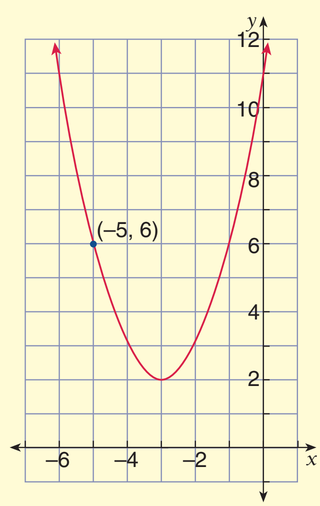
a



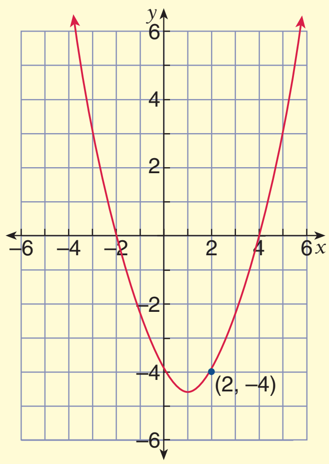
b



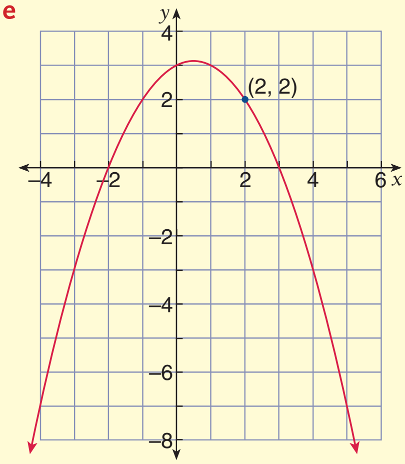
c



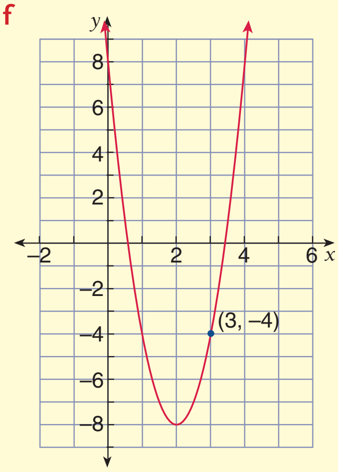
d



e



f

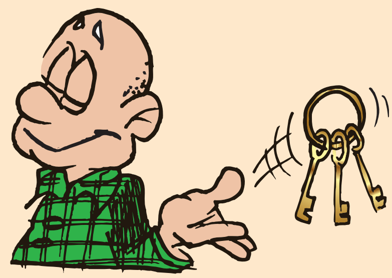


fun spot

4:02

Fun Spot 4:02 | Why didn't the bald man need his keys?

Work out the answer to each question and put the letter for that part in the box that is above the correct answer.



Factorise:

E $x^2 - 3x - 4$ **S** $x^2 - 16$

O $x^2 - 4x$ **S** $x^2 + 3x - 4$

What is the axis of symmetry for:

H $y = x^2 - 4x + 4$? **S** $y = x^2 + 4x$?

K $y = x^2 - 4$? **O** $y = x^2 - 3x - 4$?

Where does each parabola below cut the y-axis?

I $y = x^2 - 4x + 4$ **H** $y = x^2 - 4x$ **T** $y = x^2 - 4$

What is the vertex for each parabola?

L $y = x^2 - 4x + 4$ **C** $y = x^2 + 4x$ **L** $y = x^2 + 2x - 3$

$(0, 0)$	$(x-4)(x+1)$	$(-1, -4)$	$x = \frac{3}{2}$	$(x-4)(x+4)$	$(0, -4)$	$x = 2$	$(0, 4)$	$x = -2$	$(2, 0)$	$x(x-4)$	$(-2, -4)$	$x = 0$	$(x+4)(x-1)$

4:03 | The Hyperbola: $y = \frac{k}{x}$

prep quiz 4:03

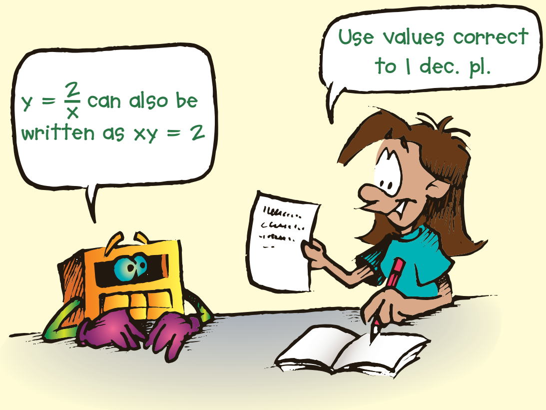
Find the value of $\frac{2}{x}$ when x is: **1** $\frac{1}{2}$ **2** 1 **3** 2 **4** 4 **5** −4

If $y = \frac{8}{x}$, what is the value of y when x is: **6** 2? **7** 4? **8** −8?

9 If $y = \frac{4}{x}$, what happens to y as x increases from 1 to 40?

10 If $y = \frac{4}{x}$, what happens to y as x decreases from −1 to −40?

We need to take many points when graphing a curve like $y = \frac{2}{x}$, as it has two separate parts. The curve of such an equation is called a **hyperbola**.



> $y = \frac{k}{x}$
>
> **is a *hyperbola* if *k* is a constant (eg 1, 2 or 4).**

worked example

$y = \frac{2}{x}$

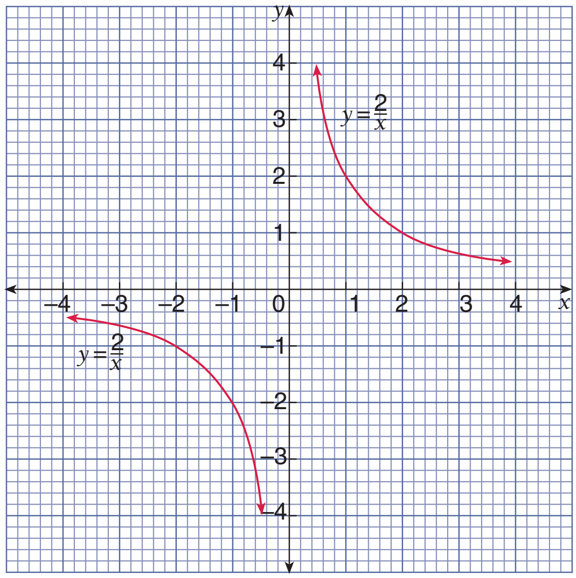
x	−4	−3	−2	−1	−0·5	0	0·5	1	2	3	4
y	−0·5	−0·7	−1	−2	−4	–	4	2	1	0·7	0·5

- Notice that there is no value for y when $x = 0$.

  When $x = 0$, $y = \frac{2}{x}$ becomes $y = \frac{2}{0}$. This value cannot exist as no number can be divided by 0.
- What will happen to the y values as the x values get closer to 0?
- What will happen to the y values as the x values become larger?

continued →→→

Plotting the points in the table gives us the graph of $y = \frac{2}{x}$.



Note:

- The hyperbola has two parts.
- The parts are in opposite quadrants and are the same shape and size.
- The curve is symmetrical.
- The curve approaches the axes but will never touch them.
- The x- and y-axes are called **asymptotes** of the curve.
- No value for y exists when $x = 0$.

Exercise 4:03

1 Use your calculator to complete the tables below, giving values for y correct to two decimal places.

a $y = \frac{2}{x}$

x	–4	–2	–1	2	4	8
y						

b $y = \frac{6}{x}$

x	–6	–2	–1	1	2	6
y						

c $y = \frac{-1}{x}$

x	–4	–2	–1	1	2	4
y						

d $y = \frac{-8}{x}$

x	1	2	3	4	6	8
y						

2 Complete the table below for $y = \frac{4}{x}$.

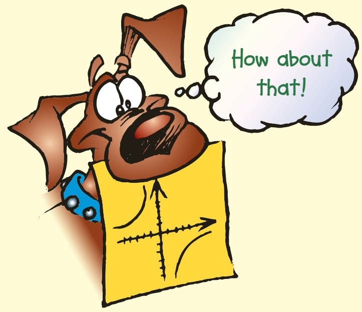
x	–8	–4	–2	–1	–0·5	0·5	1	2	4	8
y										

Use a sheet of graph paper to graph the curve $y = \frac{4}{x}$, using your table. Use values –8 to 8 on both axes.

3 Graph the curve $y = \frac{-1}{x}$ by first completing the table below.

x	−4	−2	−1	−0·5	−0·25	0·25	0·5	1	2	4
y										

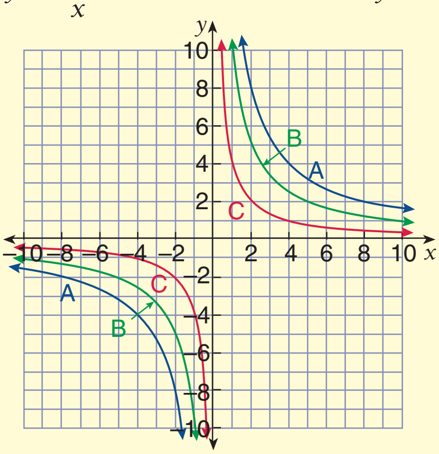
What does a negative value of k do to the graph?

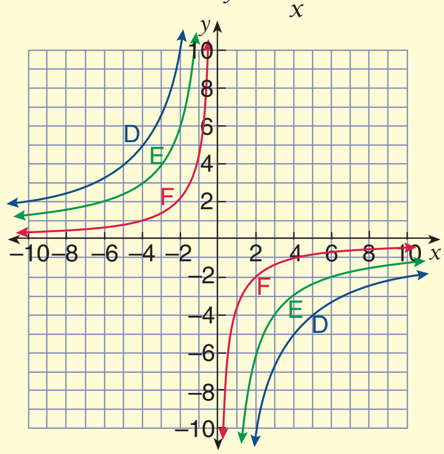


4 Match each of the graphs A to F with the following equations.

a $y = \frac{4}{x}$ **b** $y = \frac{-4}{x}$ **c** $y = \frac{-20}{x}$

d $y = \frac{10}{x}$ **e** $y = \frac{-12}{x}$ **f** $y = \frac{16}{x}$





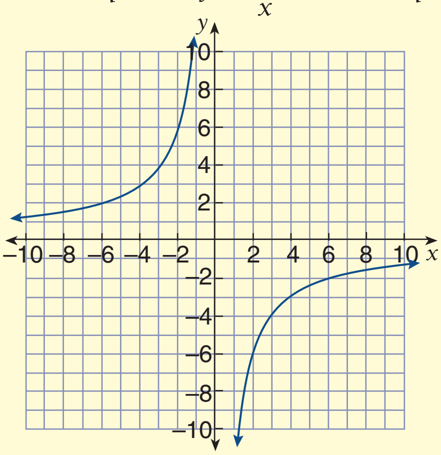
5 **a** Does the point (4, 2) lie on the hyperbola $y = \frac{8}{x}$?

b If the point (3, −6) lies on the hyperbola $y = \frac{k}{x}$, what is the value of k?

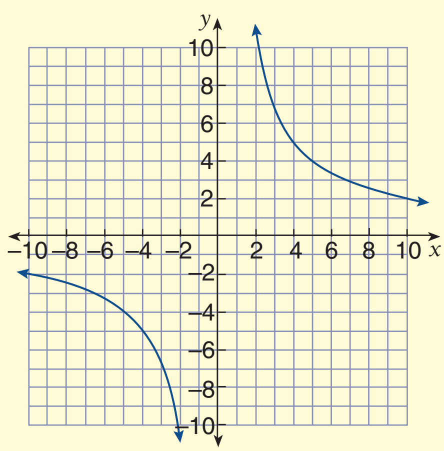
c The hyperbola $y = \frac{k}{x}$ passes through the point (10, 2). What is the value of k?

6 For each of the following, find a point that the hyperbola passes through and, by substituting this in the equation $y = \frac{k}{x}$, find the equation of the hyperbola.

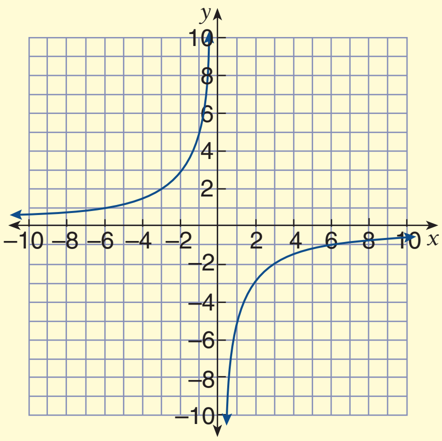
a



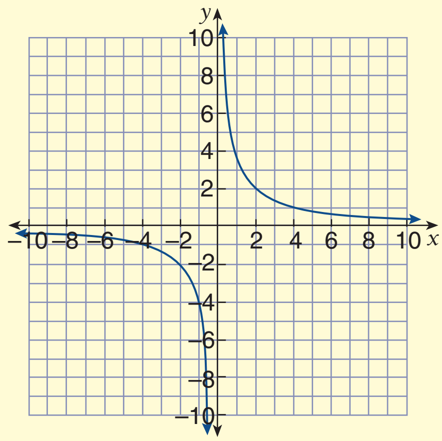
b



c



d



4:04 | Exponential Graphs: $y = a^x$

prep quiz 4:04

Find the value of: **1** 2^3 **2** 2^5 **3** 2^0 **4** 2^{-1} **5** 2^{-5}

If $y = 2^{-x}$, find y when x is: **6** 1 **7** 3 **8** −2

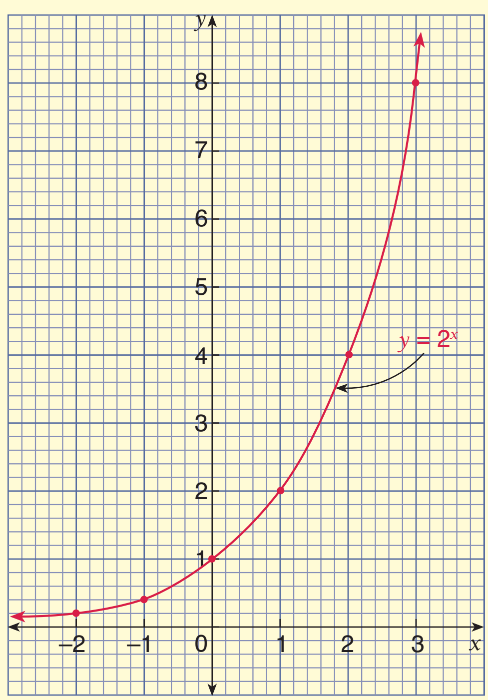
Use your calculator to find, to one decimal place, the value of:

9 $2^{1\cdot5}$ **10** $2^{-2\cdot5}$

A curve whose equation is of the form $y = a^x$ is called an **exponential curve**. On the following number plane, the graph of $y = 2^x$ has been drawn.

$y = 2^x$ is an exponential curve.

- The curve passes through (0, 1) on the y-axis since $2^0 = 1$.
- The curve rises steeply for positive values of x.
- The curve flattens out for negative values of x. The x-axis is an *asymptote* for this part of the curve.
- Because 2^x is always positive, the curve is totally above the x-axis.



Exercise 4:04

1 a Complete the table below for $y = 2^x$ and graph the curve for $-2 \leqslant x \leqslant 3$.

x	−2	−1·6	−1·2	−0·8	−0·4	0	0·4	0·8	1·2	1·6	2	2·5	3
y				0·57									

(In the table, use values of y correct to two significant figures.)

b Complete the table below for $y = 3^x$ and graph the curve for $-2 \leqslant x \leqslant 2$. Use the same diagram you used in part **a**.

x	−2	−1·5	−1	−0·5	0	0·5	1	1·5	2
y									

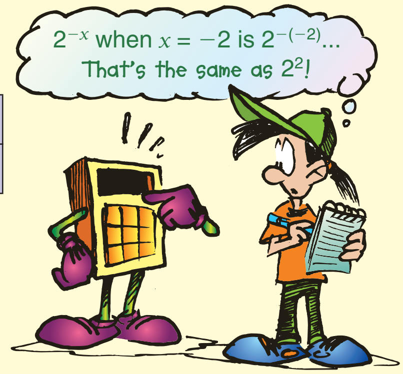
The graph of $y = a^x$ will always pass through (0, 1), since $a^0 = 1$.

c Compare the graphs of $y = 2^x$ and $y = 3^x$. What do you notice?

2 a Complete the table of values for $y = 2^{-x}$ and graph the curve on a number plane.

x	−2	−1·5	−1	−0·5	0	0·5	1	1·5	2
y									

b Compare your graph with $y = 2^x$. What do you notice?



3 a Draw on the same number plane the graphs of $y = 2^x$ and $y = 2^{-x}$.

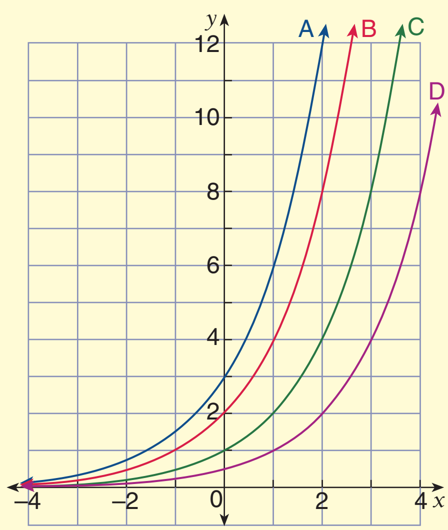
b With reference to the graphs in part **a**, now draw the graphs of $y = -2^x$ and $y = -2^{-x}$ on the same diagram.

c What is the effect of graphing the 'negative' relationships?

4 The graphs of the curves $y = 2^x$, $y = 2 \times 2^x$, $y = 3 \times 2^x$ and $y = 0.5 \times 2^x$ are shown on the number plane.

a Match each of the curves A to D with its equation.

b What is the effect of multiplying an exponential function by a constant, k (ie $y = ka^x$)?



5 **a** For the graph of $y = 6a^x$, where would the curve cut the y-axis? To which end of the x-axis is the curve *asymptotic*? (*Note:* $a > 0$)

b For the graph of $y = 6a^{-x}$, where would the curve cut the y-axis? To which end of the x-axis is the curve *asymptotic*?

c Using your answers to parts **a** and **b**, draw sketches of:

i $y = 4 \times 2^x$ **ii** $y = 2 \times 2^{-x}$ **iii** $y = \frac{1}{2} \times 3^x$ **iv** $y = 5 \times 4^{-x}$

6 The quantity of carbon-14 present after t years is given by the formula:

$$Q = A \times 2^{\frac{-t}{5730}}$$

where Q is the quantity of carbon-14 present,
A was the amount of carbon-14 present at the start,
t is the time in years.

If 10 g of carbon-14 were present at the start, $A = 10$, and the formula becomes:

$$Q = 10 \times 2^{\frac{-t}{5730}}$$

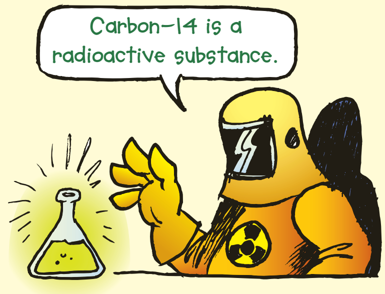
a Find the value of Q when $t = 0$.

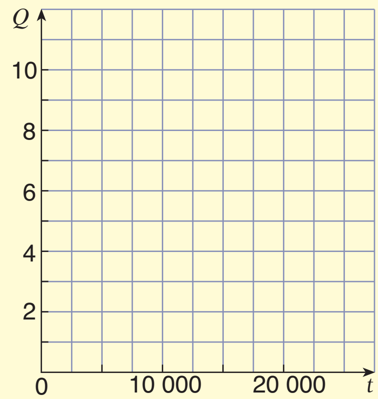
b Find the quantity of carbon-14 remaining after 5730 years. (Carbon-14 has a *half-life* of 5730 years.)

c Find the value of Q when t is:

i 11 460 **ii** 17 190 **iii** 2865

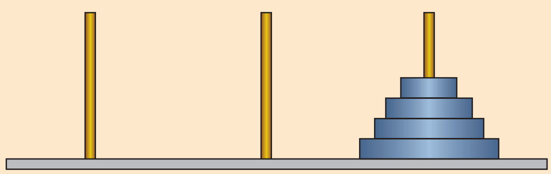
d Use the values found above to sketch the graph of $Q = 10 \times 2^{\frac{-t}{5730}}$ for values of t from 0 to 20 000.





Fun Spot 4:04 | The tower of Hanoi

This famous puzzle consists of three vertical sticks and a series of discs of different radii which are placed on one stick to form a tower, as shown in the diagram. The aim of the puzzle is to move the discs so that the tower is on one of the other sticks. The rules are:



- only one disc can be moved at a time to another stick
- a larger disc can never be placed on top of a smaller one.

The puzzle can be made more difficult by having more discs.

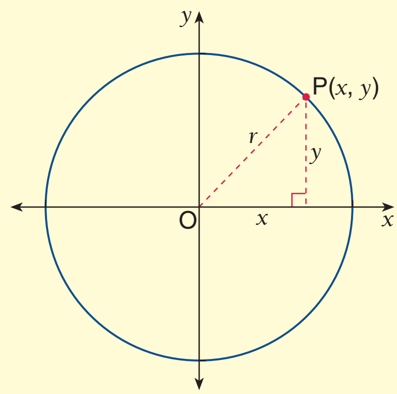
Investigate the minimum number of moves needed if there are 2, 3 or 4 discs. Can you generalise your results?

Can you predict the minimum number of moves needed if there are, say, 8 discs? (*Hint:* An exponential relationship can be found!)

4:05 | The Circle

A circle may be defined as the set of all points that are equidistant (the same distance) from a fixed point called the centre.

- We need to find the equation of a circle of radius r units with the origin O as its centre.
- If P(x, y) is a point on the circle which is always r units from O, then, using Pythagoras' theorem, $x^2 + y^2 = r^2$. This is the equation that describes all the points on the circle.



The equation of a circle with its centre at the origin O and a radius of r units is:

$$x^2 + y^2 = r^2$$

worked examples

1 What is the equation of the circle that has its centre at the origin and a radius of 6 units?

2 What is the radius of the circle $x^2 + y^2 = 5$?

Solutions

1 $r = 6$, so the equation is $x^2 + y^2 = 6^2$
$\therefore\ x^2 + y^2 = 36$ is the equation of the circle.

2 $x^2 + y^2 = 5$ is of the form $x^2 + y^2 = r^2$.
$\therefore\ r^2 = 5$, so $r = \sqrt{5}$
$\therefore$ The radius of the circle is $\sqrt{5}$ units.

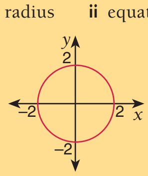
Exercise 4:05

Foundation Worksheet 4:05

The circle

1 For each circle, write down its
i radius ii equation

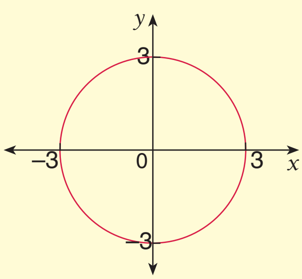
a



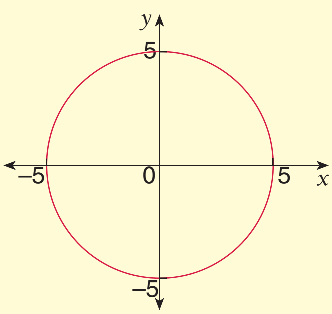
2 Sketch the circle represented by the equations:
a $x^2 + y^2 = 36$ b $x^2 + y^2 = 4$

1 What is the equation of each circle?

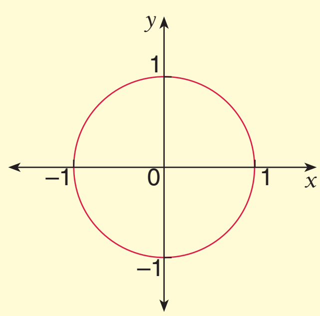
a



b



c



2 What is the equation of a circle with the origin as its centre if the radius is:

a 2 units? **b** 7 units? **c** 10 units?

d $\sqrt{3}$ units? **e** $\sqrt{6}$ units? **f** $2\sqrt{2}$ units?

g $1\frac{1}{2}$ units? **h** $2\frac{1}{4}$ units? **i** 4·2 units?

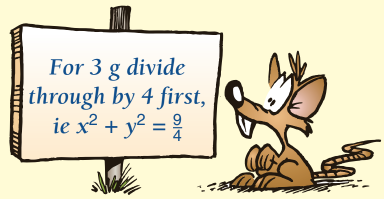
3 What is the radius of these circles?

a $x^2 + y^2 = 64$ **b** $x^2 + y^2 = 81$

c $x^2 + y^2 = 10$ **d** $x^2 + y^2 = 2$

e $x^2 + y^2 = 2{\cdot}25$ **f** $x^2 + y^2 = 6{\cdot}25$

g $4x^2 + 4y^2 = 9$ **h** $9x^2 + 9y^2 = 16$



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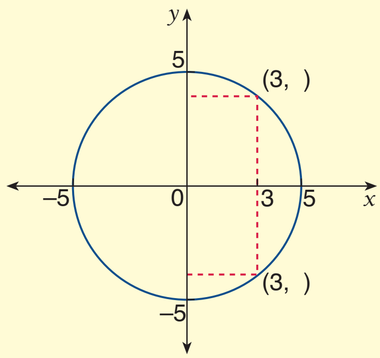


For the circle $x^2 + y^2 = 25$, there are two points that have an x value of 3.

Substituting $x = 3$ into $x^2 + y^2 = 25$

we get $3^2 + y^2 = 25$

$\therefore\ y^2 = 25 - 9$

$= 16$

$y = \pm\sqrt{16}$

$\therefore\ y = \pm 4$ [+4 or −4]

So (3, 4) and (3, −4) are the two points.

a Find the two points on the circle $x^2 + y^2 = 25$ that have an x value of:

i 4 **ii** −3 **iii** 2

b Find the two points on the circle $x^2 + y^2 = 25$ that have a y value of:

i 4 **ii** −3 **iii** 2

5 Graph each of the circles in question **3** on separate number planes.

6 Which equation in each part represents a circle?

a $y = 3x - 1$, $x^2 + 2x = y$, $x^2 + y^2 = 1$

b $xy = 9$, $x^2 + y^2 = 9$, $x + y = 3$

c $x^2 = 4 - y^2$, $x^2 = y^2 - 4$, $x^2 = y + 4$

d $y^2 = 2x + x^2$, $y^2 = 2x^2 + 7$, $y^2 = 2 - x^2$

7 **a** How could it be determined whether a point was inside, outside or lying on a particular circle?

b State whether these points are inside, outside or on the circle $x^2 + y^2 = 20$.

i (2, 4) **ii** (4, 3) **iii** (−3, 3) **iv** (1, −4)

v (−3, 4) **vi** (−4, 2) **vii** $(2\frac{1}{2}, 3\frac{1}{2})$ **viii** (1·5, 4·5)

8 Find the equation of the circle with its centre at the origin that passes through the point:

a (−4, 3) **b** (−2, −3) **c** $(1, \sqrt{3})$

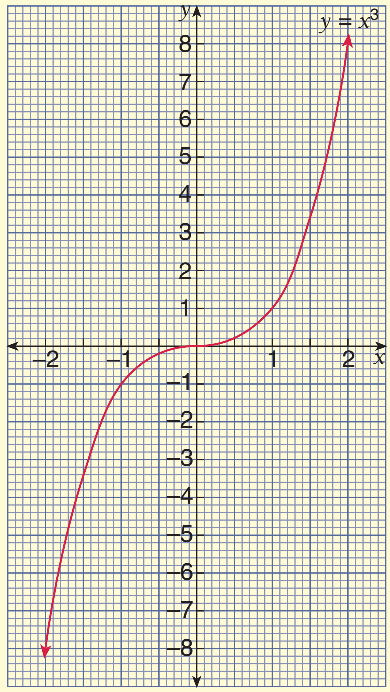
4:06 | Curves of the Form $y = ax^3 + d$

Curves of the form $y = ax^3 + d$ are called *cubics* because of the x^3 term. The simplest cubic graph is $y = x^3$, which occurs when $a = 1$ and $d = 0$.

As with other graphs, a table of values is used to produce the points on the curve.

$y = x^3$

x	−2	−1·5	−1	−0·5	0	0·5	1	1·5	2
y	−8	−3·4	−1	−0·1	0	0·1	1	3·4	8



Features of $y = x^3$:
- It is an increasing curve.
- As x increases, the value of x^3, and hence y, becomes large very quickly. This means it is difficult to fit the points on a graph.
- When x is positive, x^3, and hence y, is positive. When x is negative, x^3, and hence y, is negative. When x is zero, x^3 is zero.

In this section, the relationship between the curve $y = x^3$ and the curve $y = ax^3 + d$ for various values of a and d will be investigated.

Exercise 4:06

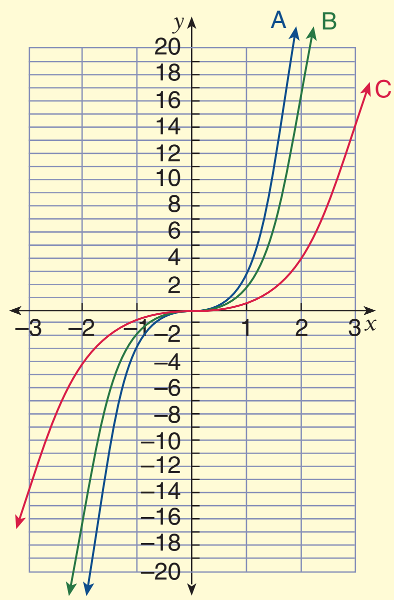
1 **a** Match each of the equations below with the graphs A, B and C.

i $y = 2x^3$ **ii** $y = \frac{1}{2}x^3$ **iii** $y = 3x^3$

b Which graph increases the fastest? (Which is the steepest?)

c Which graph increases the slowest?

d How can you tell which graph is the steepest by looking just at the equations?



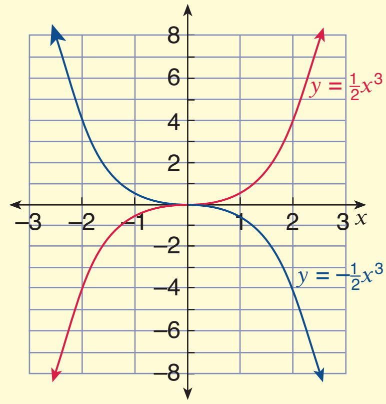
2 Which of the curves is steeper:

a $y = x^3$ or $y = 3x^3$? **b** $y = x^3$ or $y = \frac{1}{2}x^3$? **c** $y = 2x^3$ or $y = 3x^3$?

3 The graphs of $y = \frac{1}{2}x^3$ and $y = -\frac{1}{2}x^3$ are shown.

a How are the graphs related?

b What is the effect on $y = ax^3$ of the sign of a?



4 From your results so far, you should have noticed that all the curves are either *decreasing* or *increasing*. Without sketching, state whether the following are increasing or decreasing.

a $y = 4x^3$ **b** $y = -10x^3$ **c** $y = 0{\cdot}25x^3$

d $y = \frac{1}{5}x^3$ **e** $y = -\frac{1}{3}x^3$ **f** $y = -\dfrac{x^3}{5}$

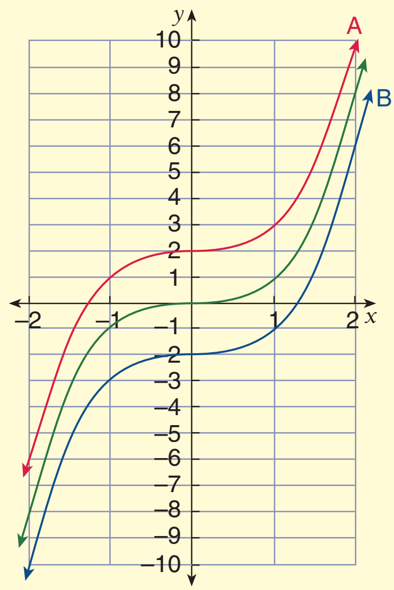
5 **a** Copy and complete the tables of values for the three curves $y = x^3$, $y = x^3 + 2$ and $y = x^3 - 2$.

x	−2	−1	0	1	2
x^3	−8	−1	0	1	8
$x^3 + 2$					
$x^3 - 2$					

b What is the equation of curves A and B?

c How is the graph of $y = x^3 + 2$ related to the graph of $y = x^3$?

d How is the graph of $y = x^3 - 2$ related to the graph of $y = x^3$?



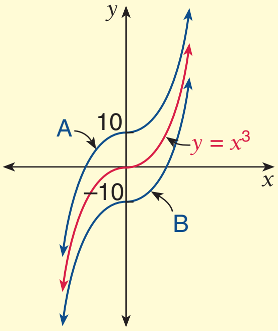
6 Given the graph of $y = \frac{1}{2}x^3$, describe how you would obtain the graphs of:

a $y = \frac{1}{2}x^3 + 1$ **b** $y = \frac{1}{2}x^3 - 1$ **c** $y = \frac{1}{2}x^3 + 2$

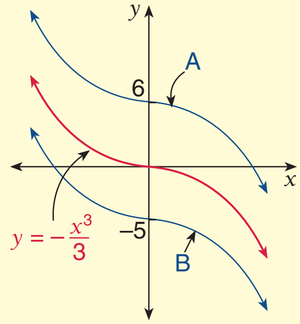
d $y = \frac{1}{2}x^3 - 2$ **e** $y = -\frac{1}{2}x^3$ **f** $y = -\frac{1}{2}x^3 + 1$

7 In each diagram, the two curves A and B were obtained by moving the other curve up or down. Give the equations of the curves A and B.

a



b



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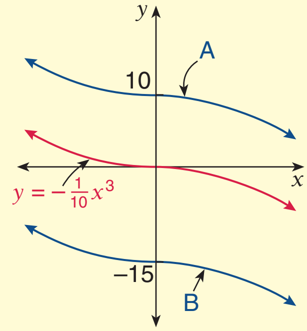


8 Sketch each pair of graphs on the same number plane.

a $y = 2x^3$

$y = 2x^3 - 2$

b $y = -x^3$

$y = -x^3 + 2$

c $y = \frac{1}{4}x^3$

$y = \frac{1}{4}x^3 + 4$

9 Sketch each pair of graphs on the same number plane.

a $y = x^3 + 1$

$y = -x^3 + 1$

b $y = 2x^3 + 1$

$y = x^3 + 1$

10 From the list of equations, write the letter or letters corresponding to the equations of the curves:

a that can be obtained by moving $y = x^3$ up or down

b that are the same shape as A

c that are decreasing

d that pass through (0, 0)

e that can be obtained from the curve $y = -\frac{1}{3}x^3$ by reflection in the y-axis

f that have the largest y-intercepts

A $y = 2x^3$

B $y = x^3 - 1$

C $y = 3 - \frac{1}{3}x^3$

D $y = 2x^3 - 1$

E $y = 3 + 2x^3$

F $y = -\frac{1}{3}x^3$

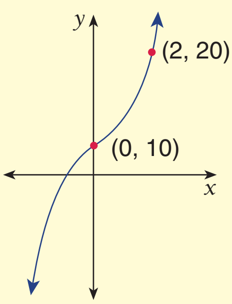
G $y = \frac{1}{3}x^3$

H $y = 2 + x^3$

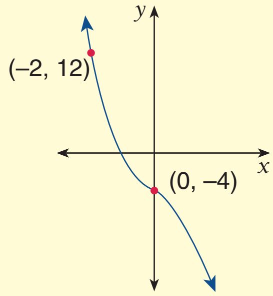
I $y = -\frac{1}{3}x^3 - 2$

11 Give that each of the graphs is of the form $y = ax^3 + d$, find its equation.

a



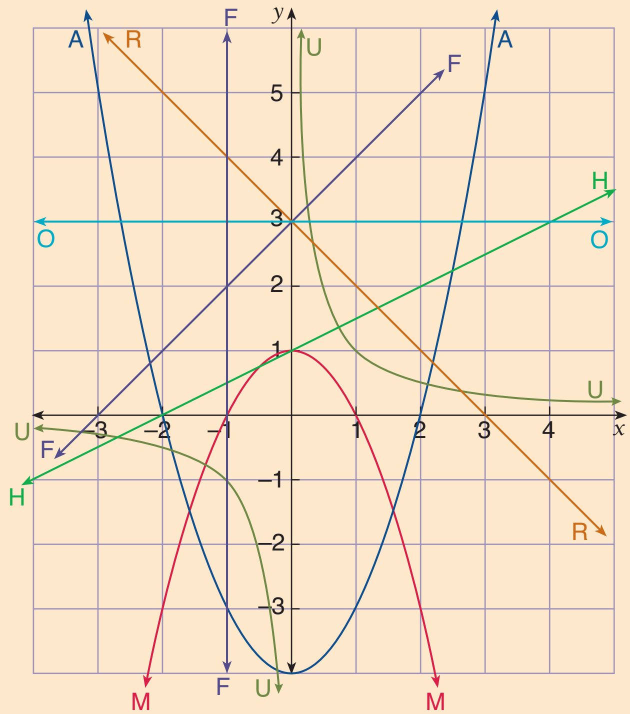
b



12 For equations of the form $y = ax^3 + d$, describe the effect on the graph of different values of a and d.

Fun Spot 4:06 | What is HIJKLMNO?

Work out the answer to each question and put the letter for that part in the box that is above the correct answer.



For the number plane shown, match each graph with its correct equation below.

What is the equation of the parabola that results if the parabola $y = x^2$ is:

O moved up 3 units

A moved down 3 units

E moved 3 units to the right

R moved 3 units to the left

R turned upside down and moved 3 units up

O moved 3 units to the right and turned upside down

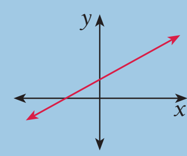
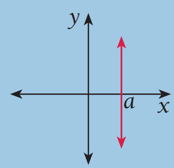
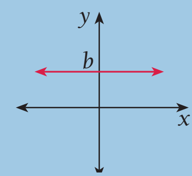
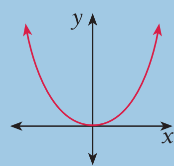
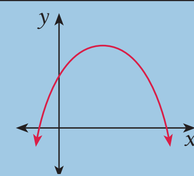
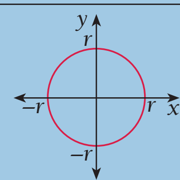
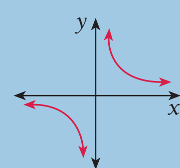
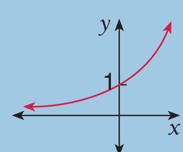
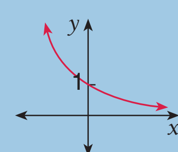
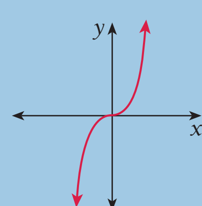
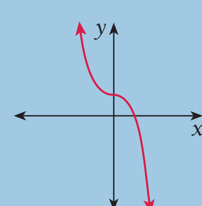
From the equations $y = 2x$, $y = x^2$ and $y = \dfrac{2}{x}$, which one is a:

T parabola? **L** straight line? **W** hyperbola?

															–		2	
$x = -1$	$y = -(x-3)^2$	$y = 3 - x$	$y = 1 - x^2$	$xy = 1$	$y = 2x$	$y = x^2 - 3$	$y = x + 3$	$y = 3$	$y = 3 - x^2$	$y = \frac{2}{x}$	$y = x^2 - 4$	$y = x^2$	$y = (x-3)^2$	$y = (x+3)^2$		$y = \frac{1}{2}x + 1$		$y = x^2 + 3$

4:07 | Miscellaneous Graphs

It is important that you be able to identify the different graphs you have met so far by their equations. Study the review table below and then attempt the following exercise.

Type of graph	*Equation*	*Graph*
Straight line	$y = mx + b$ or $ax + by + c = 0$	
Lines parallel to the axes	$x = a$ or $y = b$	 
Parabola	$y = x^2$ $y = ax^2 + bx + c$	 
Circle	$x^2 + y^2 = r^2$	
Hyperbola	$y = \frac{k}{x}$ or $xy = k$	
Exponential curve	$y = a^x$ or $y = a^{-x}$	 
Cubic curve	$y = x^3$ $y = ax^3 + d$	 

Exercise 4:07

1 From the list of equations given on the right, choose those that represent:

a a straight line
b a circle
c a parabola
d a hyperbola
e an exponential curve
f a cubic curve

A	$x^2 + y^2 = 16$	**B**	$y = 6 - x - x^2$
C	$y = 3x^3$	**D**	$y = 2^{-x}$
E	$y = x^2 - 2$	**F**	$xy = -4$
G	$y = 3^x$	**H**	$x^2 + y^2 = 1$
I	$y = \frac{5}{x}$	**J**	$2x + 4y = 3$
K	$y = 3$	**L**	$y = \frac{1}{3}x^3 - 1$

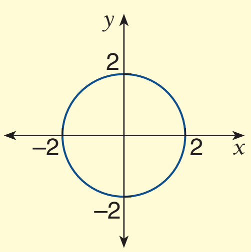
2 Sketch the graphs of the following equations, showing where each one cuts the coordinate axes.

a $y = 2x - 1$ **b** $y = 6 - x$ **c** $x + 3y = 6$
d $x = -1$ **e** $y = 3$ **f** $x = 5$
g $y = x^2 + 2$ **h** $y = x^2 - 4$ **i** $y = (x - 1)^2$
j $y = (x + 1)(x - 3)$ **k** $y = x^2 + 4x - 5$ **l** $y = x^2 + 4x$
m $y = 1 - x^2$ **n** $y = -(x + 1)^2$ **o** $y = 5 - 4x - x^2$
p $x^2 + y^2 = 4$ **q** $x^2 + y^2 = 100$ **r** $x^2 + y^2 = 2$
s $y = \frac{2}{x}$ **t** $xy = 4$ **u** $y = -\frac{3}{x}$
v $y = 4^x$ **w** $y = 2^{-x}$ **x** $y = -3^x$
y $y = 3x^3 - 3$ **z** $y = -3x^3 + 3$ **α** $y = 2x^3 + 2$

3 Match each graph with its equation from the given list.

A	$xy = 9$	**B**	$y = -x^3 + 4$
C	$y = 5^x$	**D**	$x^2 + y^2 = 4$
E	$2x - y + 2 = 0$	**F**	$y = x^3 - 4$
G	$y = 3 + 2x - x^2$	**H**	$y = 5$
I	$y = x^2 - 4$		

a



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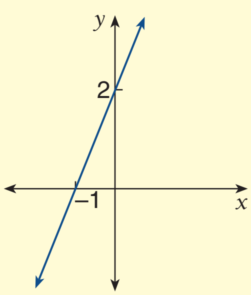

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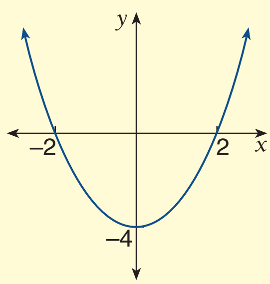

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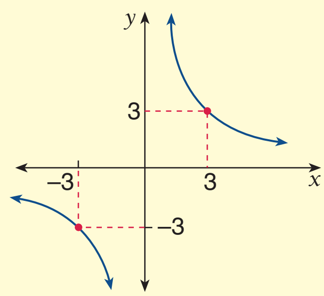

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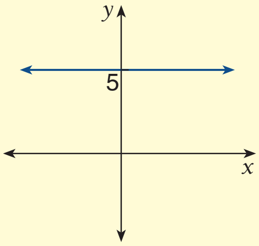

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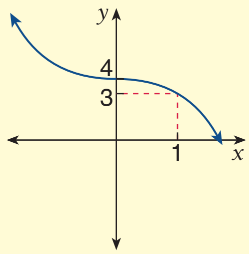

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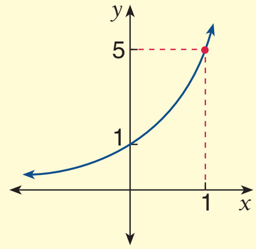

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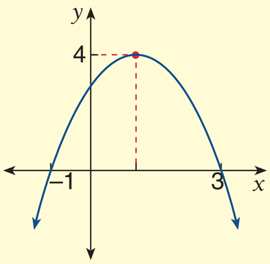

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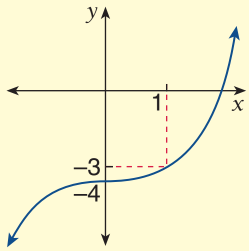
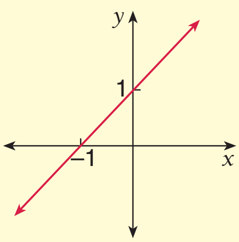


4 Determine the equation of each graph.

a



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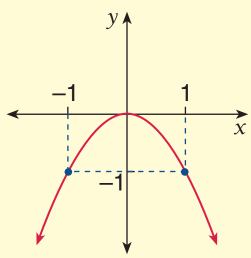

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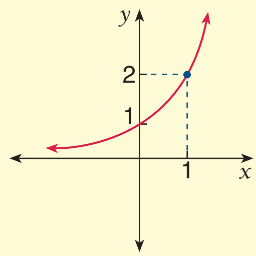

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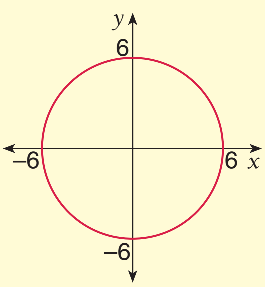

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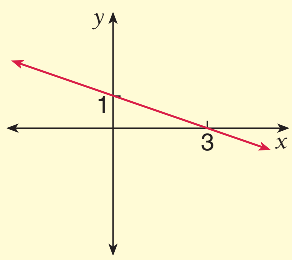

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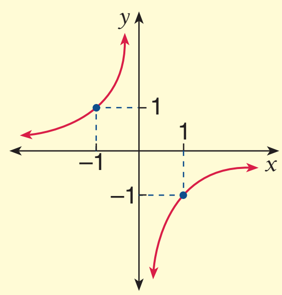

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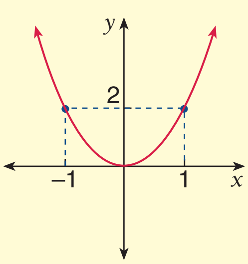

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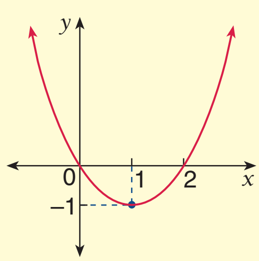

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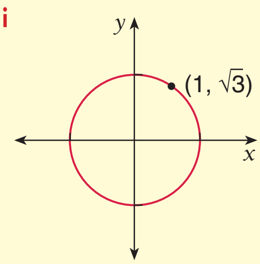

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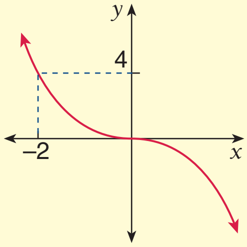

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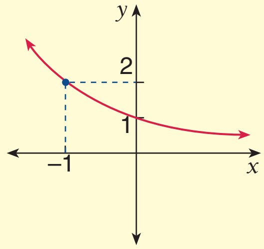

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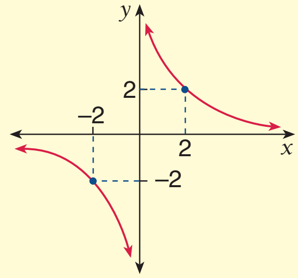

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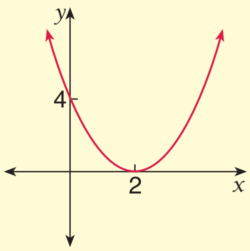

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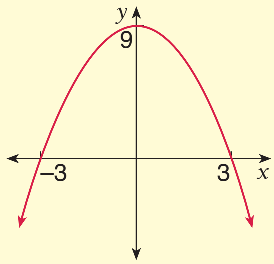

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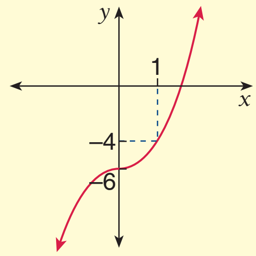

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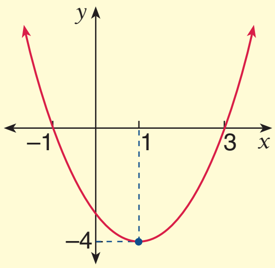

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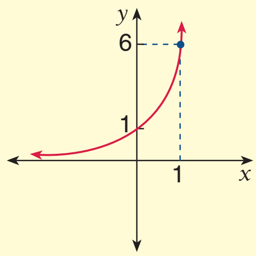

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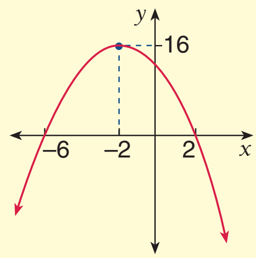


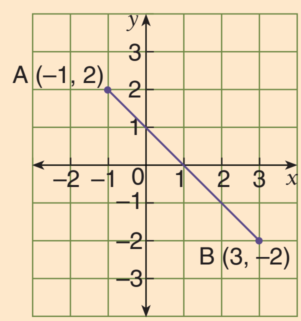
4:08 | Using Coordinate Geometry to Solve Problems

prep quiz 4:08

A(−1, 2) and B(3, −2) are shown on the diagram.

Find:

1. the length of AB
2. the slope of AB
3. the midpoint of AB
4. the *y*-intercept of AB
5. the equation of the line AB



What is the gradient of the line:

6. $y = 1 - 3x$?
7. $2x + y = 4$?
8. What is the equation of the line that passes through (−2, 4) with a slope of 2?

What is the gradient of a line that is:

9. parallel to the line $y = 2x - 3$?
10. perpendicular to the line $y = 2x - 3$?

In Book 3, coordinate geometry was used to investigate:
- the distance between two points
- the midpoint of an interval
- the gradient (or slope) of an interval
- the various equations of a straight line
- parallel and perpendicular lines.

These results can be used to investigate the properties of triangles and quadrilaterals as well as other types of geometrical problems.

The results are reviewed in Chapter 1.

see Chapter 1

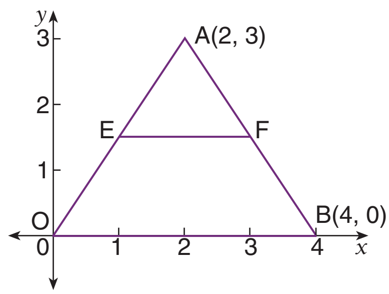
worked examples

Example 1

A triangle is formed by the points O(0, 0), A(2, 3) and B(4, 0). E and F are the midpoints of the sides OB and AB. Show:

a that ΔOAB is isosceles **b** that EF is parallel to OB

Solution 1



a $OA = \sqrt{(2-0)^2 + (3-0)^2}$ $AB = \sqrt{(4-2)^2 + (0-3)^2}$

$= \sqrt{4+9}$ $= \sqrt{4+9}$

$= \sqrt{13}$ $= \sqrt{13}$

∴ ΔOAB is isosceles (two equal sides).

b Now, E is $(1, 1\frac{1}{2})$ and F is $(3, 1\frac{1}{2})$.

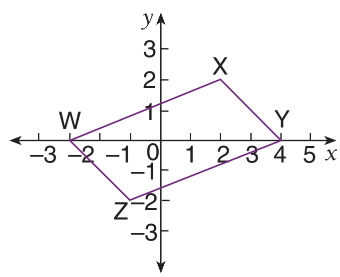
∴ EF is horizontal (E and F have same y-coordinates).
OB is horizontal.
∴ EF is parallel to OB.

Example 2

W(–3, 0), X(2, 2), Y(4, 0) and Z(–1, 2) are the vertices of a quadrilateral.

a Show that WXYZ is a parallelogram

b Show that the diagonals bisect each other.



Solution 2

a Calculating the slopes of the four sides gives the following.

Slope of WX $= \dfrac{2-0}{2-(-3)}$ $= \frac{2}{5}$ Slope of ZY $= \dfrac{0-(-2)}{4-(-1)}$ $= \frac{2}{5}$

Slope of WZ $= \dfrac{0-(-2)}{-3-(-1)}$ $= -1$ Slope of XY $= \dfrac{2-0}{2-4}$ $= -1$

∴ WX || ZY (equal slopes) and WZ || XY (equal slopes)

∴ WXYZ is a parallelogram (opposite sides are parallel).

continued →→→

b Midpoint of XZ = $\left(\frac{2+(-1)}{2}, \frac{2+(-2)}{2}\right)$ Midpoint of WY = $(\frac{1}{2}, 0)$

$= (\frac{1}{2}, 0)$

$\therefore$ $(\frac{1}{2}, 0)$ is the midpoint of both diagonals.

$\therefore$ The diagonals XZ and WY bisect each other.

Exercise 4:08

Foundation Worksheet 4:08

Coordinate geometry

1 a Show that the triangle formed by the points A(−2, 0), B(0, 4) and C(2, 0) is isosceles.

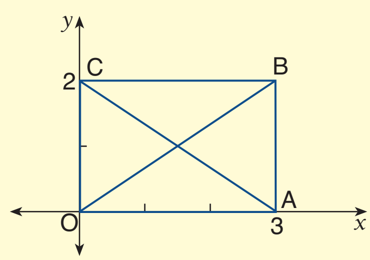
2 a Show that the quadrilateral with its vertices at the points A(−1, 1), B(2, 1), C(3, −1) and D(0, −1) is a parallelogram.

3 A(−2, 0), B (0, 3), C(2, 0) and D(0, −3) are the vertices of a quadrilateral. Show that ABCD is a rhombus.

1 **a** Show that the triangle formed by the points O(0, 0), A(3, 1) and B(1, 3) is isosceles.
b Show that the triangle formed by the points (0, 0), (1, 3) and (7, 1) is right-angled.
c Show that the triangle with vertices at (0, 0), (−2, 2) and (2, 2) is both right-angled and isosceles.

2 **a** Show that the quadrilateral with vertices at A(0, 2), B(3, 0), C(0, −2) and D(−3, 0) is a rhombus.
b A quadrilateral is formed by joining the points O(0, 0), B(1, 2), C(5, 0) and D(4, −2). Show that it is a rectangle.
c The points A(0, 2), B(2, 0), C(0, −2) and D(−2, 0) are joined to form a quadrilateral. Show that it is a square.

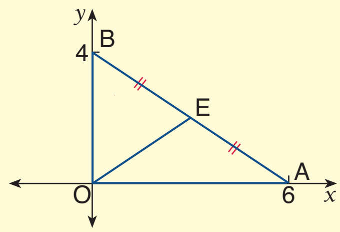
3 **a** If OABC is a rectangle, what are the coordinates of B?
b Find the length of OB and AC. What property of a rectangle have you proved?
c Find the midpoint of OB and AC. What does your answer tell you about the diagonals of a rectangle?



4 The points A(0, 0), B(6, 4) and C(4, −2) form a triangle.
a Find the midpoints of AB and AC.
b Find the slope of the line joining the midpoints in **a**.
c What is the slope of BC?
d What do your answers to parts **b** and **c** tell you?

5 The points A(4, 0), B(4, 4), C(0, 4) and D(0, 0) form a square. Find the slopes of BD and AC. What does your result say about the diagonals BD and AC?

6 A right-angled triangle OAB is shown.

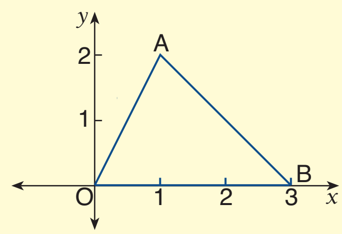


- **a** Find the coordinates of E, the midpoint of AB.
- **b** Find the length of OE.
- **c** Find the length of EA.
- **d** What can you say about the distance of E from O, A and B?

7 A triangle has its vertices at the points A(–1, 3), B(2, 4) and C(1, 1).

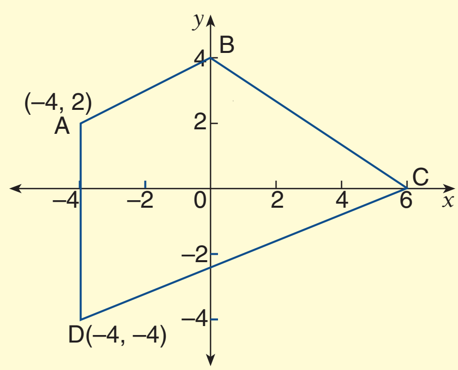
- **a** Show that the triangle is isosceles.
- **b** Find E, the midpoint of AC.
- **c** Find the slope of the line joining E to B.
- **d** Show that EB is perpendicular to AC.
- **e** Describe how you could find the area of ΔABC.

8



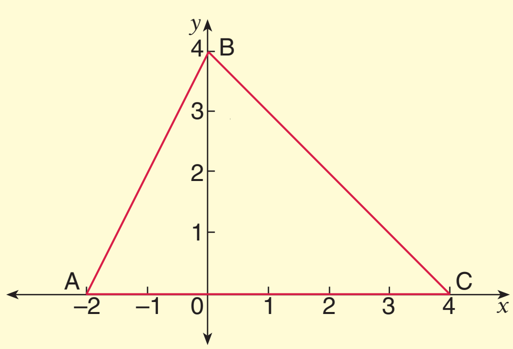
- **a** Find the midpoints of OA and AB.
- **b** Find the length of the line joining the midpoints in **a**.
- **c** Show that your answer in **b** is half the length of OB.

9 ABCD is a quadrilateral.



- **a** Find the coordinates of the midpoints of each side.
- **b** Join the midpoints to form another quadrilateral. What type of quadrilateral do you think it is?
- **c** How could you prove your answer in **b**?

10 The points A(–2, 0), B(0, 4) and C(4, 0) form the vertices of an acute-angled triangle.



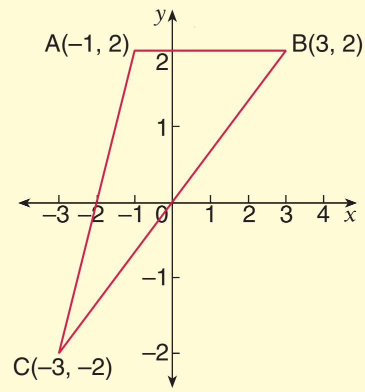
- **a** Find the equation of the perpendicular bisectors of the sides AB, BC and AC.
- **b** Find the point of intersection of the perpendicular bisectors of the sides AB and AC.
- **c** Show that the perpendicular bisector of the side BC passes through the point of intersection found in **b**.

11 A median is a line joining a vertex of a triangle to the midpoint of the opposite side.

a Find the equations of the medians.

b Find the point of intersection of two of the medians.

c Show that the third median passes through the point of intersection of the other two.



• Properties of triangles are fascinating.

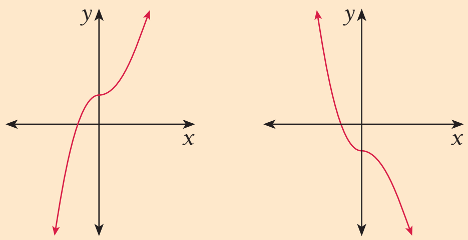
Mathematical Terms 4

circle

- The equation of a circle in the number plane with its centre at the origin is:
  $x^2 + y^2 = r^2$
  where r is the radius.

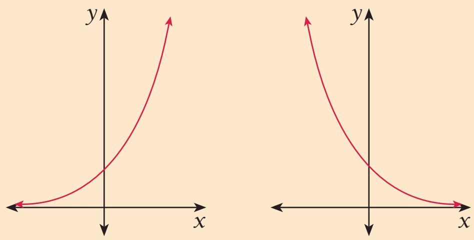
cubic curve

- A curve that contains an x^3 term as its highest power. In this chapter, the curve's equation is of the form:
  $y = ax^3 + d$



exponential curve

- A curve with an equation of the form $y = a^x$, where $a > 0$.



equation

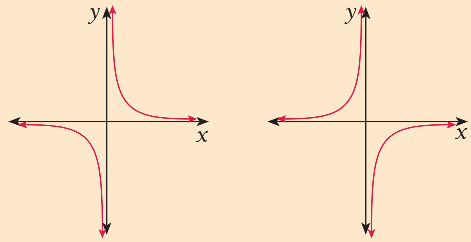
- An algebraic statement that expresses the relationship between the x- and y-coordinates of every point (x, y) on the curve.

graph (of a curve)

- The line that results when the points that satisfy a curve's equation are plotted on a number plane.

hyperbola

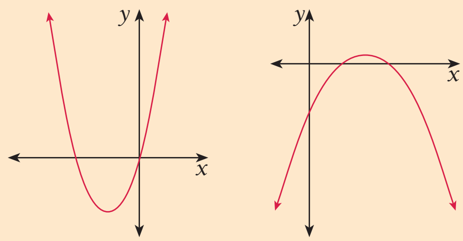
- A curve with the equation $y = \frac{k}{x}$ where k is a constant.



- It has two asymptotes (the x-axis and y-axis), which are lines that the curve approaches but never reaches.

parabola

A curve with the equation $y = ax^2 + bx + c$.



- The line of symmetry of the parabola is its axis of symmetry. The equation of the axis of symmetry is $x = -\frac{b}{2a}$.
- Parabolas can be concave up or concave down.
- The highest (or lowest) value of y on the parabola is the maximum (or minimum) value.
- The point where the parabola turns around is its vertex (or turning point).

x- and y-intercept(s)

- The point(s) where a curve crosses the x- or y-axis.

Mathematical terms 4

Diagnostic Test 4 | Number Plane Graphs and Coordinate Geometry

- These questions reflect the important skills introduced in this chapter.
- Errors made will indicate an area of weakness.
- Each weakness should be treated by going back to the section listed.

Section

1 On the same number plane, sketch the graphs of: — 4:01

a $y = x^2$ **b** $y = x^2 - 4$ **c** $y = x^2 + 2$

2 On the same number plane, sketch the graph of: — 4:01

a $y = 2x^2$ **b** $y = \frac{1}{2}x^2$ **c** $y = -x^2$

3 Sketch the graphs of: — 4:01

a $y = (x - 1)^2$ **b** $y = (x + 3)^2$ **c** $y = (x - 2)^2 + 1$

4 Find the y-intercept of the parabolas: — 4:02

a $y = (x - 1)(x + 3)$ **b** $y = x^2 - 6x$

c $y = 8 - 2x - x^2$ **d** $y = 4x^2 + 8x - 5$

5 Find the x-intercepts for each of the parabolas in question **4**. — 4:02

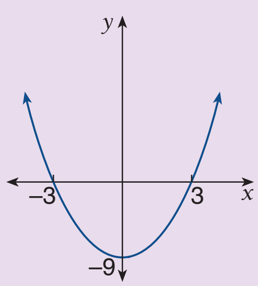
6 Find the equation of the axis of symmetry for each of the parabolas in question **4**. — 4:02

7 Find the vertex of each of the parabolas in question **4**. — 4:02

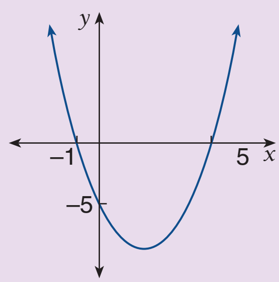
8 Sketch each of the parabolas in question **4**. Also state the maximum or minimum value for each quadratic expression. — 4:02

9 Determine the equation of each parabola. — 4:02

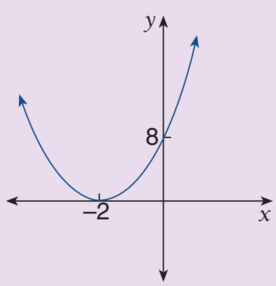
a



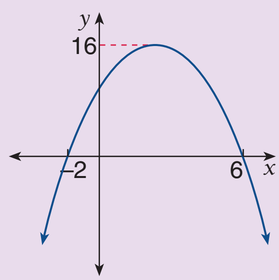
b



c



d



10 Sketch the graphs of: — 4:03

a $y = \frac{3}{x}$ **b** $y = \frac{-2}{x}$ **c** $xy = 6$

Section

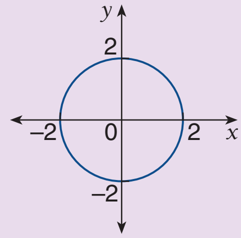
11 Sketch, on the same number plane, the graphs of: 4:04

a $y = 2^x$ b $y = 3^x$ c $y = 2^{-x}$

12 a What is the equation of this circle? 4:05

b What is the equation of a circle that has its centre at the origin and a radius of 7 units?

c What is the radius of the circle $x^2 + y^2 = 3$?



13 Sketch the graphs of: 4:06

a $y = x^3 + 1$ b $y = 1 - x^3$ c $y = 2x^3 - 1$

14 From the list of equations on the right, which one represents: 4:07

a parabolas?

b straight lines?

c circles?

d hyperbolas?

e exponential graphs?

f cubic graphs?

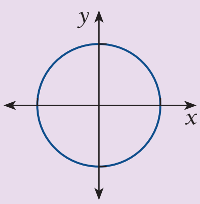
A $y = 9^x$	**B** $y = 3$
C $xy = 9$	**D** $y = 3^{-x}$
E $y = 9 - x^2$	**F** $y = \frac{1}{2}x^3 - 2$
G $x^2 + y^2 = 9$	**H** $y = x^2 + 3x$
I $y = 9 - x$	**J** $x^2 + y^2 = 3$
K $y = 9 - x^3$	**L** $y = \frac{-3}{x}$

15 Match each graph with its equation from the list. 4:07

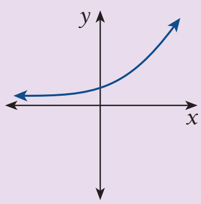
A $y = x^2 - 2$ **B** $y = 2^x$ **C** $xy = 1$

D $y = -2x^2$ **E** $x^2 + y^2 = 2$ **F** $y = 2x^3$

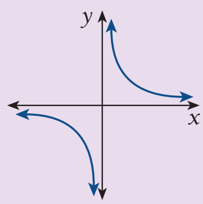
a



b



c



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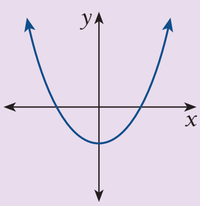
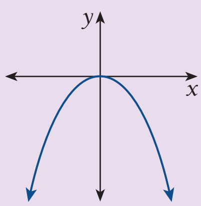
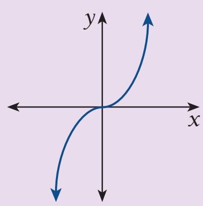


e



f



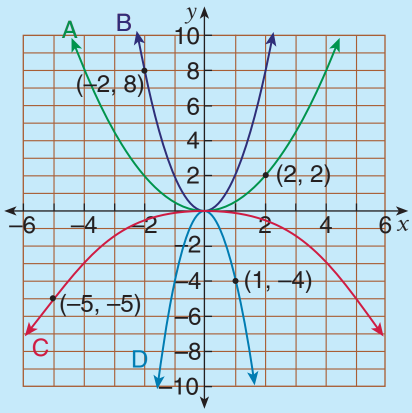
Chapter 4 | Revision Assignment

4A

1 Describe the relationship between the graph of $y = x^2$ and the graph of:

a $y = x^2 + 5$
b $y = x^2 - 5$
c $y = (x - 5)^2$
d $y = (x + 5)^2$
e $y = (x - 5)^2 + 5$

2 The equations of the graphs A to D are of the form $y = ax^2$. Use the given point to find the equation of each curve.

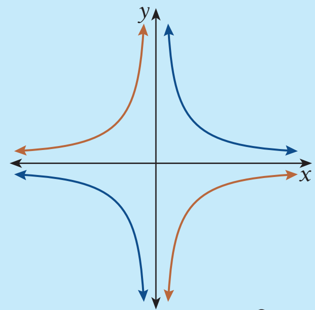


3 A parabola has an equation of $y = 4x^2 - 4x - 3$.

a Find:
- **i** the y-intercept
- **ii** the x-intercepts
- **iii** the equation of the axis of symmetry
- **iv** the coordinates of the vertex

b Sketch the graph of the parabola.

4



The hyperbolas $y = \frac{8}{x}$ and $y = \frac{-8}{x}$ are shown on the diagram. Copy the diagram and sketch and label the hyperbolas $y = \frac{4}{x}$ and $y = \frac{-12}{x}$ onto the diagram.

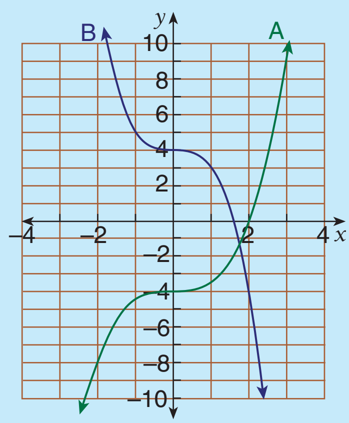
5 On the same number plane, sketch and label the graphs of:

a $y = 3^{-x}$ and $y = -3^{-x}$
b $y = 3x^3$ and $y = -3x^3$

6 On the same number plane, sketch and label the graphs of:

a $y = (x + 3)(x - 2)$ and $y = 2(x + 3)(x - 2)$
b $y = (x + 3)(x - 2)$ and $y = -(x - 3)(x + 2)$

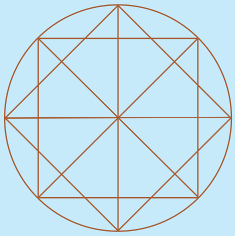
7 The curves A and B have equations of the form $y = ax^3 + d$. Find the equation of each curve.



Chapter 4 | Working Mathematically

assignment 4B

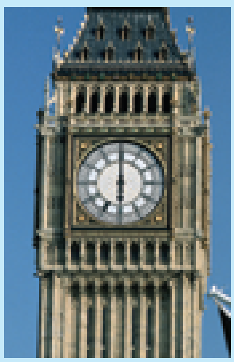
1 Write a set of instructions that would enable a person to redraw the figure shown on the right.



2 Every possible pairing of 5 children was made and their combined weights recorded in the list shown here:
85, 90, 92, 95, 97, 100, 102, 105, 110, 112
(weights measured in kg)
If the weight of every child is a whole number and no two are the same weight, find the individual weight of each child if one of them weighs 50 kg and another weighs 60 kg.

3 Use a calculator to find the answers to 1^2, 11^2, 111^2 and 1111^2. Study the pattern formed by the answers and then write down the answer to $11\ 111\ 111^2$ without using your calculator.

4 If a clock takes 5 seconds to strike 6 o'clock, how long would it take to strike 12 o'clock?



5 On 9 August 2003 at 17:24:31 (Jakarta time), the resident population of Australia was estimated to be 219 883 000. If the population is estimated to be increasing by 1 person every 2 minutes and 35 seconds, on what day will the population reach 20 000 000?

6 The concentration of an acid is measured by its pH. An acid with a concentration of:
0·1 mole/L has a pH of 1
0·01 mole/L has a pH of 2
0·001 mole/L has a pH of 3.

a Write each of 0·1, 0·01, and 0·001 in index notation.
b Explain how the pH is related to the concentration of the acid.
c If acid had a concentration of 0·000 01 mole/L, what is its pH?
d What is the concentration of an acid with a pH of 4·5?
e One acid has a pH of 2 while the other has a pH of 4. Which is the more concentrated acid? How many times is it more concentrated?

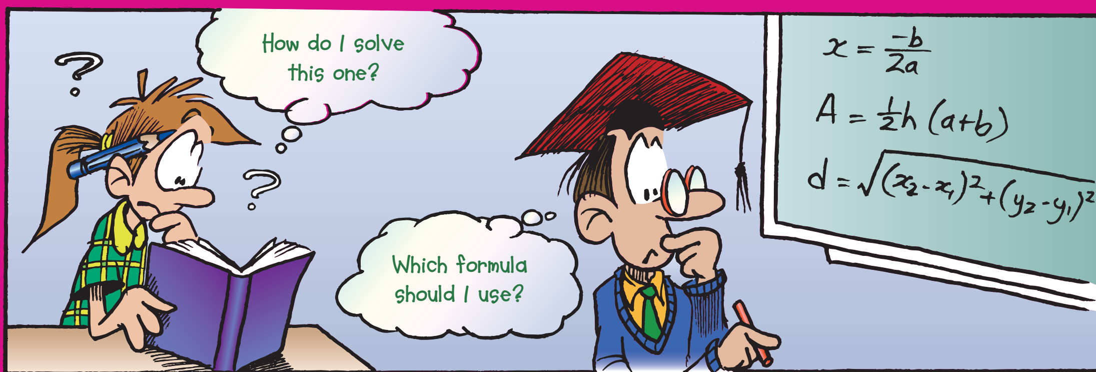
Chapter Review Questions

Technology Applications Drag and Drops
1 Parabolas
2 Identifying graphs

Technology Applications Activities
1 Investigating parabolas
2 Curve stitching

5

Further Algebra



Chapter Contents

Learning Outcomes

Students will be able to:

- Solve simultaneous equations that include quadratics.
- Change the subject of a formula.
- Work with equations that have restrictions on variables.

Areas of Interaction

Approaches to Learning (Knowledge Acquisition, Problem Solving, Communication, Logical Thinking, Reflection), Human Ingenuity

5:01 | Simultaneous Equations Involving a Quadratic Equation

prep quiz 5:01

Solve these quadratic equations:

1 $(x + 3)(x - 2) = 0$ **2** $(2x - 1)(x + 7) = 0$ **3** $5x(x + 4) = 0$

4 $x^2 - 4 = 0$ **5** $x^2 - 3x + 2 = 0$ **6** $2x^2 + 7x - 4 = 0$

Solve these equations:

7 $x^2 - x + 5 = 3x + 2$ **8** $2x^2 - x = 3 - 2x$

Solve these simultaneous equations by substitution:

9 $x + y = 5,\ y = 2x - 1$ **10** $3x - y - 13 = 0,\ y = 7 - 2x$

In Year 9 you saw how to solve two simultaneous equations when both equations were linear, that is, they were of the form $y = 3x + 2$ or $3x - 2y = 6$. Now we shall see how to find the common or simultaneous solutions when one equation is linear but the other is a quadratic equation.

worked example

1 Solve the simultaneous equations $y = x^2$ and $y = x + 2$ using the substitution method.

Solution

$y = x^2$ .... (1)

$y = x + 2$ .... (2)

From (1) we see that x^2 is equal to y.

If we substitute x^2 for y in (2) we have:

$$x^2 = x + 2$$
$$x^2 - x - 2 = 0$$
$$(x + 1)(x - 2) = 0$$
$$\therefore x = -1 \text{ or } 2$$

These values for x can now be substituted into either equation (1) or (2) to find corresponding values for y.

Substitute $x = -1$ in (1)

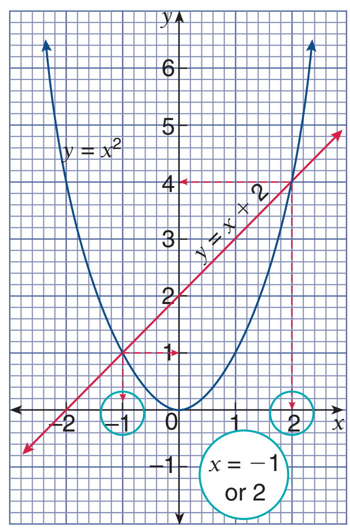
$$y = (-1)^2$$
$$= 1$$
$$\therefore \begin{cases} x = -1 \\ y = 1 \end{cases} \text{ is a solution.}$$

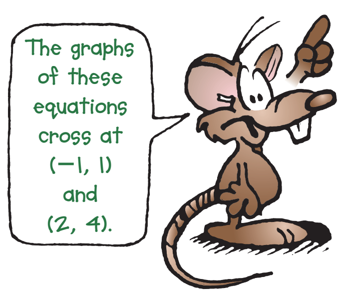
Substitute $x = 2$ in (1)

$$y = (2)^2$$
$$= 4$$
$$\therefore \begin{cases} x = 2 \\ y = 4 \end{cases} \text{ is a solution.}$$

(Check answers by substituting in (1) and (2).)

The solutions are $\begin{cases} x = -1 \\ y = 1 \end{cases}$ and $\begin{cases} x = 2 \\ y = 4 \end{cases}$





continued →→→

2 Use the substitution method to find the common solutions to the equations $y = x^2 - 4$ and $3x + y = 6$.

Solution

$y = x^2 - 4$ .... (1)

$3x + y = 6$ .... (2)

From (1) we see that $x^2 - 4$ is equal to y.

If we substitute $x^2 - 4$ for y in (2), we have:

$3x + (x^2 - 4) = 6$

ie $x^2 + 3x - 10 = 0$

$(x + 5)(x - 2) = 0$

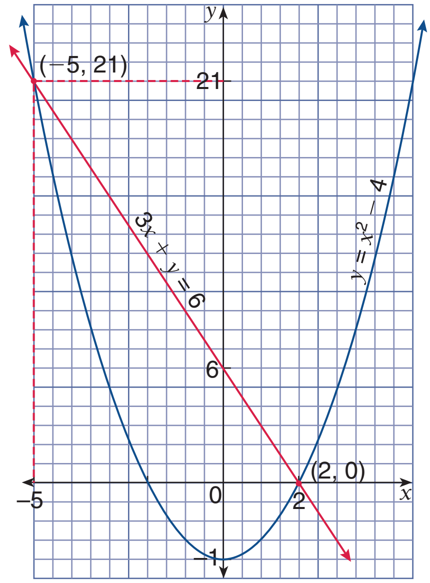
$\therefore x = -5$ or 2

Substituting these values into (1) or (2) we find that:

If $x = -5$, $y = 21$

$x = 2$, $y = 0$

$\therefore$ The solutions are $\begin{cases} x = -5 \\ y = 21 \end{cases}$ and $\begin{cases} x = 2 \\ y = 0 \end{cases}$



Graphical solutions to $y = x^2 - 4$ and $3x + y = 6$.

Exercise 5:01

1 Use this diagram to solve these simultaneous equations.

a $y = x + 6$
$x = 2$

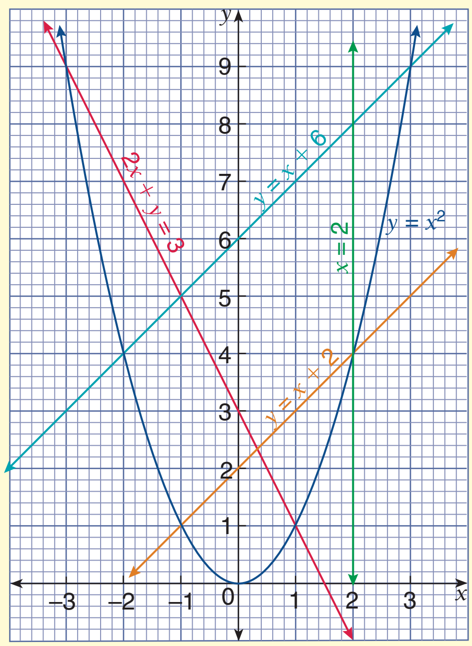
b $y = x + 6$
$2x + y = 3$

c $y = x^2$
$x = 2$

d $y = x^2$
$y = x + 6$

e $y = x^2$
$y = x + 2$

f $y = x^2$
$2x + y = 3$



2 Solve each pair of equations in question **1** using the substitution method and check your solutions with those for question **1**.

3 Use the substitution method to solve these simultaneous equations.

a $\begin{cases} y = x^2 \\ y = -4x - 3 \end{cases}$ **b** $\begin{cases} y = x^2 \\ y = 49 \end{cases}$ **c** $\begin{cases} y = x^2 \\ y = 3x \end{cases}$

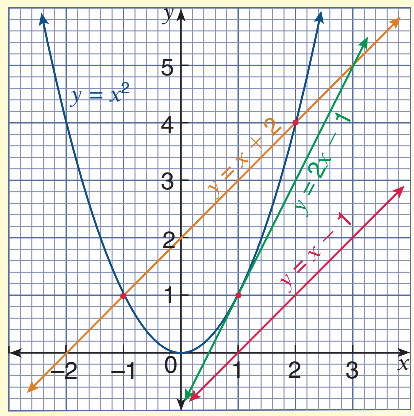
d $\begin{cases} y = x^2 \\ y = x + 56 \end{cases}$ **e** $\begin{cases} y = x^2 \\ y = 10x - 21 \end{cases}$ **f** $\begin{cases} y = x^2 \\ y = -3x - 2 \end{cases}$

g $\begin{cases} y = x^2 + 5 \\ y = 4x + 50 \end{cases}$ **h** $\begin{cases} y = x^2 + 7 \\ y = 8x \end{cases}$ **i** $\begin{cases} y = x^2 - 10 \\ y = -9x \end{cases}$

4 Solve each pair of simultaneous equations using the substitution method.

a $y = x^2$ and $y = x + 2$
b $y = x^2$ and $y = 2x - 1$
c $y = x^2$ and $y = x - 1$

Looking at the graphs of these equations, explain why part **a** has two solutions, part **b** has one solution and part **c** has no solutions.



5 Solve these pairs of simultaneous equations.

a $y = 5x + 6$, $y = x^2$
b $y = x + 3$, $y = x^2 + 1$
c $y = 4x + 9$, $y = x^2 - 3$
d $y = 2x + 4$, $y = x^2 - x$
e $y = 2x + 14$, $y = x^2 - 3x$
f $y = 7 - 3x$, $y = x^2 + 3x$
g $y = x^2 + x - 10$, $y = 2x + 10$
h $y = x^2 + 2x - 8$, $3x + y = 6$
i $y = 4x + 1$, $y = 2x^2 - x - 2$

6 By solving these equations, find the point(s) of intersection of their graphs.

a $y = x^2 - 2$, $y = x$
b $y = x^2 - 10$, $y = 10 - x$
c $y = x^2 + 2$, $y = 4x - 2$
d $y = x^2 + 2x - 20$, $x + y = 8$
e $y = x^2 + 4x + 7$, $2x + y + 2 = 0$
f $y = x^2 + x - 2$, $y = x - 3$

5:02 | Literal Equations: Pronumeral Restrictions

prep quiz 5:02

Complete the following:

1 $a + 15 = 27$
$\therefore a = 27 - \ldots$

2 $m + n = p$
$\therefore m = p - \ldots$

3 $5x = 35$
$\therefore x = \dfrac{35}{\ldots}$

4 $ab = c$
$\therefore b = \dfrac{c}{\ldots}$

Solve:

5 $3x + 13 = 22$

6 $6 - 5n = 21$

7 $\dfrac{2m}{5} = 3$

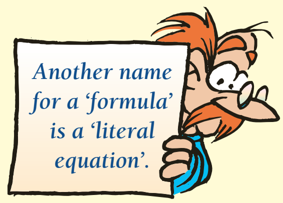
8 $\dfrac{x-3}{2} = 5$

Find x if:

9 $\sqrt{x} = 9$

10 $x^2 = 9$

In Year 9, you were shown how to change the 'subject' of a formula to a specified pronumeral. This was the same as 'solving' the literal equation for that pronumeral.



For example, for $v = u + at$ we could say:

'Change the subject of this formula to a.'

or 'Solve this literal equation for a.'

In each case, the answer is $a = \dfrac{v-u}{t}$.

Steps to follow in solving a literal equation

1. Remove fractions.
2. Expand grouping symbols.
3. Use inverse operations to isolate the pronumeral required.
4. If the required pronumeral appears in more than one term in the equation, gather the terms together and factorise.

worked examples

Solve the equation for a.

1
$$\begin{aligned} v^2 &= u^2 + 2aS \\ v^2 - u^2 &= 2aS \\ \frac{v^2 - u^2}{2S} &= a \\ \therefore a &= \frac{v^2 - u^2}{2S} \end{aligned}$$

2
$$\begin{aligned} R &= \sqrt{\frac{ax}{b}} \\ R^2 &= \frac{ax}{b} \\ bR^2 &= ax \\ \frac{bR^2}{x} &= a \\ \therefore a &= \frac{bR^2}{x} \end{aligned}$$

3
$$\begin{aligned} y &= \frac{a}{a+2} \\ y(a+2) &= a \\ ay + 2y &= a \\ 2y &= a - ay \\ 2y &= a(1-y) \\ \frac{2y}{1-y} &= a \\ \therefore a &= \frac{2y}{1-y} \end{aligned}$$

Inverse operations

$+ \leftrightarrow -$

$\times \leftrightarrow \div$

$(\ )^2 \leftrightarrow \sqrt{\ }$

4 $\frac{1}{x} = \frac{1}{a} + \frac{1}{b}$

Sometimes a different set of steps might be followed to arrive at the same correct result.

Method 1

$\frac{1}{x} - \frac{1}{b} = \frac{1}{a}$ — Subtract $\frac{1}{b}$ from both sides.

$\frac{b-x}{bx} = \frac{1}{a}$ — Rewrite L.H.S. with a common denominator

$\frac{bx}{b-x} = \frac{a}{1}$ — Invert both sides.

$\therefore a = \frac{bx}{b-x}$

Method 2

$\frac{1}{x} = \frac{b+a}{ab}$ — Rewrite R.H.S. with a common denominator

$ab = x(b+a)$ — Cross-multiply.

$ab = bx + ax$ — Expand grouping symbols.

$ab - ax = bx$ — Subtract ax from both sides.

$a(b-x) = bx$ — Factorise L.H.S.

$\therefore a = \frac{bx}{b-x}$ — Divide both sides by '$b - x$'.

When using formulae or literal equations, we should be aware that there may be restrictions on the values a pronumeral may take. Examine the following formulae and their restrictions.

Formula	*Restriction*
$y = \sqrt{x-a}$	$x \geqslant a$, since expressions under square root sign cannot be negative. $y \geqslant 0$, since $\sqrt{\ }$ by definition is positive.
$y = 5(x-a)^2$	$y \geqslant 0$, since $(x-a)^2$ must be greater than or equal to zero, ie $(x-a)^2 \geqslant 0$.
$y = \frac{a}{1-r}$	$r \neq 1$, since the denominator can't equal zero. (The fraction would be undefined.)

Also, there may be assumptions implicit in the formula that will restrict the values a pronumeral can take.

For instance, in a formula such as $v = u + at$ where t represents time spent, it would be assumed that $t \geqslant 0$. (Time spent cannot be negative.)

Exercise 5:02

Foundation Worksheet 5:02

Literal equations

1 Solve for x.
a $y = 3x + 2$ **b** $y = \frac{1}{x+2}$

2 Can x be the value given in each formula?
a $x = 2$ if $y = \sqrt{x-3}$
b $x = 3$ if $y = \frac{1}{x+3}$

1 Make S the subject of each formula.

a $2S + k = 1$ **b** $a = b + cS$ **c** $n = aS - m$
d $Sx - y = z$ **e** $x = y - aS$ **f** $g = Sp - q$
g $a = \frac{b}{S}$ **h** $x = \frac{aS}{b}$ **i** $\frac{St}{u} = v$

2 Expand the grouping symbols, then solve each equation for x.

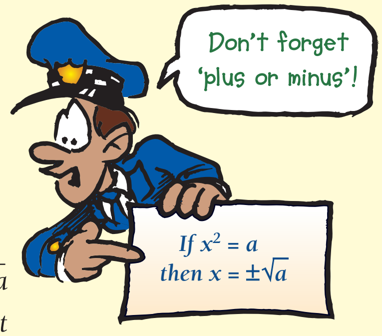
a $3(x - a) = b$ **b** $5(m + x) = n$ **c** $a(x + p) = q$
d $4(2 - x) = y$ **e** $t(v - x) = w$ **f** $m = 3(x + 4)$
g $t = a(2 + x)$ **h** $v = u(1 + x)$ **i** $h = k(x + m)$

3 Solve each literal equation for x.

a $mx^2 = n$ **b** $a = bx^2$ **c** $5x^2 = p$

d $x^2 - a = b$ **e** $t = x^2 + u$ **f** $w = v - x^2$

g $\dfrac{x^2}{a} = y$ **h** $z = \dfrac{x^2}{k}$ **i** $m = \dfrac{nx^2}{3}$



4 Make a the subject of each formula.

a $\sqrt{ab} = c$ **b** $t = \sqrt{5a}$ **c** $c = \sqrt{a - b}$

d $y = \sqrt{a + 5}$ **e** $r = \sqrt{b - a}$ **f** $m = n + \sqrt{a}$

g $p = \sqrt{a} - q$ **h** $x = y\sqrt{a}$ **i** $u = r\sqrt{a} + t$

5 Solve each for x. (Rearrange to make x the subject.)

a $x + a = b - x$ **b** $ax = px + q$

c $x + a = ax + b$ **d** $bx - c = e - dx$

e $\dfrac{x}{3} + \dfrac{x}{5} = a$ **f** $\dfrac{ax}{b} - \dfrac{bx}{4} = 1$

g $\dfrac{x}{x + 2} = a$ **h** $a = \dfrac{x + 2}{x - 3}$

Gather the x terms together and factorise, taking x out as a common factor.

6 Change the subject of each formula to the letter shown in brackets.

	A		B		C	
a	$P = xy$	[x]	$p = q + r$	[q]	$P = Q - R$	[R]
b	$b = \dfrac{xy}{a}$	[y]	$A = \dfrac{3r}{s}$	[r]	$n = \dfrac{2a}{m}$	[m]
c	$a = 3x + y$	[x]	$m = n + at$	[a]	$v^2 = u^2 + 2as$	[a]
d	$x = 5(a + b)$	[b]	$m = 3(n - m)$	[n]	$P = Q(r + t)$	[t]
e	$m = a + n^2$	[n]	$t = v^2 - u^2$	[u]	$A = x^2 + y^2$	[x]
f	$a = \sqrt{bc}$	[c]	$Y = a\sqrt{x}$	[x]	$x = \sqrt{a + y}$	[a]
g	$x = \sqrt{\dfrac{Y}{a}}$	[Y]	$R = \sqrt{\dfrac{ax}{b}}$	[x]	$V = \dfrac{at^2}{2}$	[a]
h	$y = \dfrac{a}{3} + \dfrac{b}{2}$	[a]	$x = \dfrac{m}{5} - \dfrac{n}{3}$	[m]	$\dfrac{A + x}{3} = \dfrac{A + y}{2}$	[A]
i	$m = \sqrt{\dfrac{a + b}{n}}$	[b]	$h = \dfrac{k}{2k + 1}$	[k]	$T = \dfrac{1}{a}\sqrt{\dfrac{m}{n}}$	[m]

7 In these formulae, what values can x possibly take?

a $y = \sqrt{x - 4}$ **b** $A = \sqrt{x + 3}$ **c** $M = \sqrt{x - N}$

d $y = \sqrt{4 - x}$ **e** $K = \sqrt{10 - x}$ **f** $Z = \sqrt{Y - x}$

g $y = \sqrt{3x + 1}$ **h** $A = \sqrt{5x - 2}$ **i** $P = \sqrt{Q - 2x}$

j What is the smallest value that each of the subjects of these formulae can have?

8 What value (or values) can r *not* take in these equations?

a $A = \dfrac{m}{r - 3}$ **b** $M = \dfrac{3}{10 + r}$ **c** $S = \dfrac{1 + r}{1 - r}$

d $P = \dfrac{a}{r - q}$ **e** $K = \dfrac{5}{r + s}$ **f** $Y = \dfrac{a + r}{a - r}$

g $M = \dfrac{1}{(r - 2)(r + 2)}$ **h** $Z = \dfrac{Y}{r^2 - 16}$ **i** $R = \dfrac{3}{r^2 - 3r + 2}$

9 The equation $H = 10 + 9t - t^2$ gives the height H of a ball above the ground at a time t.

a At what height above the ground did the ball start? (ie at $t = 0$)

b At what time will the ball hit the ground? (ie when $H = 0$)

10 **a** Mr Hines earns \$350 plus \$20 commission on each item I that he sells. His wage W could be given by the formula $W = 350 + 20I$
What is the smallest possible value for I and hence the minimum wage, W?

b The surface area of a sphere is given by $S = 4\pi r^2$, where r is the radius of the sphere. What can be said about the value of r? Hence, change the subject of the formula to r.

c The area of a rectangle is given by $A = LB$.

- **i** Change the subject of the formula to B.
- **ii** What can be said about the value of L?
- **iii** For a given area A, if the value of L increases, what will happen to B?

d The time T taken by a pendulum for one swing is given by $T = 2\pi\sqrt{\frac{l}{g}}$ where $g = 9{\cdot}8$ m/s^2 and l is the length of the string in metres.

- **i** What can be said about the value of l?
- **ii** What will happen to T as the value of l increases?
- **iii** Solve this literal equation for l.

• This giant pendulum swings in an office building in Vancouver, Canada.

fun spot

5:02

Fun Spot 5:02 | What small rivers flow into the Nile?

Work out the answer to each question and put the letter for that part in the box that is above the correct answer.

Solve each literal equation for y.

I $2x + y = 5$ **E** $4x - 2y = 7$ **S** $2x = 5 + y$

L $\dfrac{x + 3y}{2} = 4$ **J** $\dfrac{x}{3} + \dfrac{y}{2} = 1$ **E** $2(x + 1) = 5(y - 2)$

V $x = \dfrac{y}{y + 1}$ **N** $2x = \dfrac{y + 2}{y - 2}$ **U** $\dfrac{y + 3}{x + 1} = \dfrac{5 + y}{x - 1}$

$y = \frac{6-2x}{3}$	$y = -x - 4$	$y = \frac{x}{1-x}$	$y = \frac{4x-7}{2}$	$y = \frac{4x+2}{2x-1}$	$y = 5 - 2x$	$y = \frac{8-x}{3}$	$y = \frac{2x+12}{5}$	$y = 2x - 5$

challenge

5:02

Challenge 5:02 | Fibonacci formula

The interesting set of numbers below is known as the Fibonacci sequence.
1, 1, 2, 3, 5, 8, 13, 21, 34, . . .

Library research: Who was Fibonacci? What was his interest in these numbers?

a The first nine terms are given; what are the next three terms?

b This sequence has many curious properties. One curiosity is that the ratio of successive terms gets closer and closer to a particular value. To find this value, evaluate the ratios below correct to four decimal places for the first twelve Fibonacci numbers.

$\frac{3}{2}, \frac{5}{3}, \frac{8}{5}, \frac{13}{8}, \frac{21}{13}, \frac{34}{21}, \ldots$

You should find you are getting closer and closer to the decimal value of the number $\dfrac{1 + \sqrt{5}}{2}$.

(This number is usually denoted by the Greek letter φ (phi). $\varphi : 1$ is called the Golden Ratio.)

c It is easy to see that each term is being generated by adding the previous two together. We could represent this by the statement:

$F_n = F_{n-1} + F_{n-2}$

However, a formula that will calculate the nth Fibonacci number is reasonably complex. Curiously, a fairly simple formula contains the expression for φ given above.

The value of F_n is given by the nearest integer to the expression:

$$\frac{1}{\sqrt{5}}\left(\frac{1 + \sqrt{5}}{2}\right)^n$$

Use this expression to verify the values of F_8 and F_{12} above. Then use the expression to find **i** F_{15}, **ii** F_{20}, **iii** F_{30}.

5:03 | Understanding Variables

prep quiz 5:03

Find the value of $x^2 - 4x$ if x is replaced by:

1 3 **2** 1 **3** −1 **4** −3

Expand:

5 $(3a + 1)(3a - 1)$ **6** $(x + 3)(x - 5)$ **7** $(2y - 3)^2$

Factorise:

8 $100y^2 - 9$ **9** $m^2 - m - 72$ **10** $2x^2 - 8x + 8$

In an algebraic expression, a variable can be given a specific value in order to evaluate the expression. Sometimes it is useful to replace a variable by another variable or expression. Study the following examples carefully.

Variable substitution

worked examples

1 Find an expression for $x^2 + 2x - 3$ if x is replaced by $(x + 1)$

2 Replace a by $\frac{1}{a}$ in the expression $a^2 - \frac{1}{a^2} + 3$.

Solutions

1 $x^2 + 2x - 3$

$(x + 1)^2 + 2(x + 1) - 3$

Replacing x by $(x + 1)$

$= x^2 + 2x + 1 + 2x + 2 - 3$

$= x^2 + 4x$

2 $a^2 - \frac{1}{a^2} + 3$

$\left(\frac{1}{a}\right)^2 - \frac{1}{\left(\frac{1}{a}\right)^2} + 3$ Replacing a by $\frac{1}{a}$

$= \frac{1}{a^2} - a^2 + 3$

$\frac{1}{\left(\frac{1}{a^2}\right)} = 1 \div \frac{1}{a^2}$

$= 1 \times \frac{a^2}{1}$

Factorising using a change of variable

worked examples

3 Factorise $x^2 - 13x + 36$ and hence solve $x^4 - 13x^2 + 36 = 0$.

4 Factorise $x^2 - 4x + 4$ and hence factorise $x^4 - x^2 + 4x - 4$.

continued →→→

Solutions

3 Let $X = x^2$, then the expression becomes $X^2 - 13X + 36$

$= (X - 4)(X - 9)$

Now, substitute x^2 for X

ie $(x^2 - 4)(x^2 - 9)$

So, the equation becomes

$(x^2 - 4)(x^2 - 9) = 0$

ie $x^2 = 4$ or $x^2 = 9$

$\therefore$ $x = \pm 2$ or ± 3

4 $x^2 - 4x + 4 = (x - 2)^2$

Now, $x^4 - x^2 + 4x - 4 = x^4 - (x^2 - 4x + 4)$

$= x^4 - (x - 2)^2$

Let $X = x^2$, and $Y = x - 2$, then the expression becomes

$X^2 - Y^2$

$= (X - Y)(X + Y)$

Now, substituting for X and Y gives:

$[x^2 - (x - 2)][(x^2 + (x - 2)]$

$= (x^2 - x + 2)(x^2 + x - 2)$

Expanding using a change of variable

worked examples

5 Simplify $(a + 4)^2 - (a - 3)^2$.

6 Simplify $[y + 2)^2 - 5][(y + 2)^2 + 5]$.

Solutions

5 Let $A = (a + 4)$, $B = (a - 3)$, then the expression becomes

$A^2 - B^2$

$= (A - B)(A + B)$

Substituting for A and B:

$= [(a + 4) - (a - 3)][(a + 4) + (a - 3)]$

$= 7(2a + 1)$

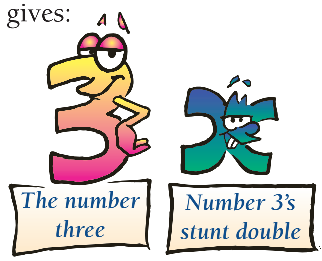
$= 14a + 7$

6 Let $Y = (y + 2)$, then the expression becomes: $(Y^2 - 5)(Y^2 + 5)$

$= Y^4 - 25$

Substituting for Y gives:

$(y + 2)^4 - 25$



Conditions on variables

Because variables represent numbers, there may be limitations or conditions, either stated or implied, for an algebraic expression or equation. Consider the examples below.

worked examples

7 If n is a positive integer, what can be said about the expression $2n + 1$?

8 If $x + y = 20$ and $x > y$, what can be said about x and y?

9 If $a^3 + 3a = 140$ when $a = 5$, solve $(2a - 1)^3 + 3(2a - 1) = 140$, for a.

Solutions

7 If n is a positive integer then $2n$ must be even; thus $2n + 1$ must be an odd number.

8 For $x + y = 20$, if $x = y$ then both would be equal to 10. Since $x > y$, then $x > 10$ and $y < 10$.

9 The a in the first expression has been replaced by $2a - 1$ in the second. Since $a = 5$ gives 140 in the first expression, then $2a - 1 = 5$ will give 140 in the second. Now, we know that $5^3 + 3(5) = 140$. So, if $2a - 1 = 5$, then $a = 3$.

Exercise 5:03

Foundation Worksheet 5:03

Understanding variables

1 Find the new expression for $2a - 5$ if a is replaced by:
a $3x$ **b** $x + 1$ **c** $x^2 + 2$

2 Solve these equations using the given change of variable.
a $x^4 - 10x^2 + 9 = 0$ if $X = x^2$
b $x - 8\sqrt{x} + 15 = 0$ if $X = \sqrt{x}$

1 Find an expression for $3x + 5$ if x is replaced by:

a $2x$ **b** $x + 1$ **c** $4x - 3$
d $5a$ **e** $a - 2$ **f** $a^2 - 2a + 1$

2 Find the new expression for the following, if $(n + 3)$ is substituted for the variable x.

a $3x - 9$ **b** $10 - x$ **c** $x^2 + x$
d $(x + 3)(x - 3)$ **e** $2x^2 - x + 3$ **f** $(2x - 5)^2$

3 Replace the variable a by $-a$ in the following expressions, noting which expressions do not change.

a $5 + 2a$ **b** $a^2 + 4$ **c** $2a^2 + 3a - 5$
d $a^2 + a^3$ **e** $a^4 - 2a^2 + 1$ **f** $a - \frac{1}{a}$

4 Substitute $\frac{1}{x}$ for x in these expressions.

a $x + \frac{1}{x}$ **b** $x^2 + 2 + \frac{1}{x^2}$ **c** $3x - 5 + \frac{2}{x}$

5 By letting $X = x^2$, factorise the following and hence solve each equation for x.

a $x^4 - 5x^2 + 4 = 0$ **b** $x^4 - 20x^2 + 64 = 0$ **c** $x^4 - 5x^2 - 36 = 0$

6 Substitute X for 2^x in the following and hence solve each equation.

a $(2^x)^2 - 3(2^x) + 2 = 0$ **b** $(2^x)^2 - 12(2^x) + 32 = 0$
c $2(2^x)^2 - 33(2^x) + 16 = 0$

7 By changing the variable from a to A^2, solve the following equations. (*Note*: $A = \sqrt{a}$.)

a $a - 7\sqrt{a} + 10 = 0$ **b** $a - 5\sqrt{a} + 4 = 0$ **c** $5\sqrt{a} = a + 6$

8 Follow the steps in example **4**.

a Factorise $x^2 + 2x + 1$ and hence factorise $x^4 - x^2 - 2x - 1$.
b Factorise $x^2 - 6x + 9$ and hence factorise $x^4 - x^2 + 6x - 9$.
c Factorise $a^2 + 4a + 4$ and hence factorise $a^6 - a^2 - 4a - 4$.

9 Replace each binomial in the following by single variables to help simplify each expression.

a $(x + 2)^2 - (x + 1)^2$ **b** $(m + 5)^2 - (m - 5)^2$
c $(a + 7)^2 - (a - 6)^2$ **d** $(n - 10)^2 - (n - 9)^2$

10 By using a suitable substitution, factorise the following.

a $(a + 3)^2 + 2(a + 3) - 8$ **b** $(x - 4)^2 - (x - 4) - 20$
c $(2m + 1)^2 + 3(2m + 1) - 10$ **d** $(4y - 1)^2 - 12(4y - 1) + 20$

11 If n is a positive integer greater than 1, state whether the following statements are true or false.

a $n^2 > n$

b $(n + 2)$ is even

c $\dfrac{1}{n} > 1$

d $1 - n < 0$

e $3n$ is odd

f If n is odd, then n^2 will be odd.

g $\dfrac{1}{n} < \dfrac{1}{2n}$

h $n^3 - n > 2n$

12 For what values of p are the following statements true?

a $2p > p + 2$

b $\dfrac{1}{p} > 2$

c $p^2 > p$

d $2^{p+1} > p^3$

13 Consider the following.

a If $x + y = 10$ and $x > y$, what can be said about x and y?

b If $x + y = 99$ and x is even, what can be said about y?

c If $x > y$, but $x^2 < y^2$, what can be said about x and y?

d If $x^2 + y^2 = 25$, what values can x or y take?

14 **a** If $a^3 + a = 68$ when $a = 4$, what value of a makes $(a - 5)^3 + (a - 5) = 68$ true?

b The equation $\dfrac{5}{y^2} - \dfrac{1}{2y} = 1$ is true when $y = 2$.

Use this fact to solve $\dfrac{5}{(2y+5)^2} - \dfrac{1}{2(2y+5)} = 1$.

c $\sqrt{a+4} - \sqrt{a-1} = 1$ when $a = 5$. Use this fact to solve the equation $\sqrt{2y+4} - \sqrt{2y-1} = 1$.

d If $x^5 - x^3 + x = 219$ when $x = 3$, find the value of x that makes $(4 - x)^5 - (4 - x)^3 + (4 - x) = 219$ true.

- Well done! You have reached the end of this challenging algebra topic.

investigation 5:03

Investigation 5:03 | Number patterns and algebra

Please use the Assessment Grid on the following page to help you understand what is required for this Investigation.

A Football scores

In Australian Rules Football, 6 points are awarded for a *goal*, and 1 point for a *behind*.

- Investigate scores of the form 2.12.24, where the product of the number of goals and the number of behinds equals the number of points. How many such scores are there?
- Note that $xy = 6x + y$, where x is the number of goals and y is the number of behinds. The solutions for x and y must, of course, be positive integers.

B Number patterns

1 Find three consecutive integers such that the first number plus the product of the other two is equal to the first number times the sum of the other two.

Hint: Let the smallest integer be n. The other two integers would be $n + 1$ and $n + 2$. Form an equation and solve for n. (*Note:* Only integers can be considered to be consecutive numbers.)

2 Show that it is not possible to find three integers consecutively differing by 2 that would satisfy the condition given in question **1**. (Such integers would be either consecutive even numbers or consecutive odd numbers.)

3 Would it be possible to find integers consecutively differing by 3, 4, 5 or 6 that would satisfy the condition? If so, find them.

Note: The condition can be expressed as:

$n + (n + a)(n + b) = n(n + a + n + b)$

where $a = 3$ and $b = 6$ if the numbers differ by 3

$a = 4$ and $b = 8$ if the numbers differ by 4

$a = 5$ and $b = 10$ if the numbers differ by 5

$a = 6$ and $b = 12$ if the numbers differ by 6.

Assessment Grid for Investigation 5:03 | Number patterns and algebra

The following is a sample assessment grid for this investigation. You should carefully read the criteria *before* beginning the investigation so that you know what is required.

		Assessment Criteria (B, C) for this investigation		Achieved ✓
Criterion B Investigating Patterns	a	None of the following descriptors have been achieved.	0	
	b	Some help was needed to complete some of the requirements.	1	
			2	
	c	The student independently completes most of the problems and attempts to explain their answers using algebraic patterns.	3	
			4	
	d	The student has correctly identified and used the patterns evident to solve all problems with logical answers and conclusions.	5	
			6	
	e	All exercises are completed correctly with full explanations and justification throughout.	7	
			8	
Criterion C Communication in Mathematics	a	None of the following descriptors have been achieved.	0	
	b	There is a basic use of mathematical language and notation, with some errors or inconsistencies evident. Lines of reasoning are insufficient.	1	
			2	
	c	There is sufficient use of mathematical language and notation. Explanations are clear with algebra skills applied well.	3	
			4	
	d	Correct use of mathematical language and algebraic notation has been shown with complete working. Explanations of all results are logical, complete and concise.	5	
			6	

Mathematical Terms 5

formula (plural: formulae)
- A formula represents a relationship between physical quantities.
- It will always have more than one pronumeral.

  eg $A = L \times B$ represents the relationship between the area (A) of a rectangle and its length (L) and breadth (B).

literal equation
- Another name for a formula.
- It will always have more than one pronumeral.

  eg $v = u + at$

quadratic equation
- An equation in which the highest power of the unknown pronumeral is 2.

  eg $x^2 - 16 = 0$, $x^2 + 5x + 6 = 0$
- A quadratic equation may have two solutions.

simultaneous equations
- When two (or more) pieces of information about a problem can be represented by two (or more) equations.
- The equations are then solved to find the common or simultaneous solution.

  eg the equations $x + y = 10$ and $x - y = 6$ have many solutions but the only simultaneous solution is $x = 8$ and $y = 2$

subject
- The subject of a formula is the pronumeral by itself, on the left-hand side.

  eg in the formula $v = u + at$, the subject is v.

substitution
- The replacing of a pronumeral with a numeral or another variable in a formula or expression.

  eg To substitute 3 for a in the expression $4a - 2$ would give:

  $4(3) - 2$
  $= 12 - 2$
  $= 10$

  To substitute $(a + 3)$ for x in the expression $2x + 1$ would give:

  $2(a + 3) + 1$
  $= 2a + 6 + 1$
  $= 2a + 7$

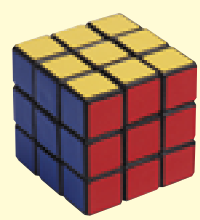
substitution method
- Solving simultaneous equations by substituting an equivalent expression for one pronumeral in terms of another, obtained from another equation.

  eg If $y = x + 3$ and $x + y = 7$, then the second equation could be written as $x + (x + 3) = 7$ by substituting for y using the first equation.

variable
- Another name for a pronumeral in an algebraic expression or equation.
- Variables can be given specific values in order to evaluate an expression.

Mathematical terms 5

- Rubik's cube is a famous mathematical puzzle. The squares on each face can be rotated so that each face is composed of nine squares of the same colour.



diagnostic test 5

Diagnostic Test 5 | Further Algebra

- Each part of this test has similar items that test a certain question type.
- Errors made will indicate areas of weakness.
- Each weakness should be treated by going back to the section listed.

Question	Section
1 Solve each pair of simultaneous equations. **a** $y = 7x + 8$, $y = x^2$ **b** $y = x + 10$, $y = x^2 - 10$ **c** $y = x^2 + 4x - 5$, $x - y = 5$	5:01
2 Solve these literal equations for *a*. **a** $S = m - an$ **b** $P = \dfrac{a^2 + b^2}{2}$ **c** $K = n\sqrt{\dfrac{a}{p}}$ **d** $L = \dfrac{3a}{a - m}$	5:02
3 In these formulae, what values can x *not* take? **a** $y = \sqrt{x - 1}$ **b** $M = \sqrt{5 + 2x}$ **c** $A = \dfrac{3x}{x - 4}$ **d** $P = \dfrac{2}{x^2 - 9}$	5:02
4 Find the new expression, in its simplest form, for: **a** $2x + 7$, if x is replaced by $(a - 4)$ **b** $x^2 - x - 1$, if x is replaced by $(x + 1)$ **c** $a^3 - 2a$, if a is replaced by $(-2a)$ **d** $3m - \dfrac{3}{m}$, if m is replaced by $\left(\dfrac{2}{m}\right)$	5:03
5 If n is a non-zero integer, which statements below are *always* true? (*Remember*: An integer may be negative.) **a** $2n \geqslant n + 2$ **b** $n^2 \geqslant n$ **c** $\dfrac{n}{2} < n$ **d** 2^n will be an integer	5:03

- Does this car go further on a litre of petrol?

Chapter 5 | Revision Assignment

assignment 5A

1 Solve these pairs of simultaneous equations.

a $y = x + 6$, $y = x^2 + 6x$

b $y = 2x(x + 4)$, $3x - y + 3 = 0$

c $y = x^2 - 4x + 3$, $y = 3x - x^2$

2 a Find the point where the two parabolas, $y = x^2 - 4x$ and $y = x^2 - 8x + 12$, intersect.

b Sketch these parabolas on the same number plane, labelling this point of intersection.

3 What values can a take in these expressions?

a $\sqrt{a-4}$ b $\sqrt{4-a}$

c $\sqrt{a+4}$ d $\sqrt{4a}$

4 What values can n *not* take in these expressions?

a $\frac{1}{n-3}$ b $\frac{1}{n+3}$

c $\frac{1}{3n}$ d $\frac{1}{n^2-9}$

5 The equation $h = 60t - 10t^2$ gives the height, in metres, of a ball above the ground after t seconds.

a Find when $h = 0$, and therefore the time taken for the ball to return to the ground.

b Determine the maximum height of the ball. (*Hint:* Find the vertex of the parabola.)

c Hence, what are the possible values for h and t in this formula?

6 Find the equivalent expression for $x^2 + x + 1$ if x is replaced by:

a $2a$ b $a + 2$

c a^2 d $\frac{2}{a}$

7 Substitute X for x^2 in the following and then solve.

a $x^4 - 20x^2 + 64$

b $x^4 - 8x^2 - 9$

c $x^4 + 5x^2 + 4$

8 Noting that each expression is a 'difference of two squares', simplify:

a $(x + 4)^2 - (x - 4)^2$

b $(a + 5)^2 - (a - 2)^2$

c $(3 - 2y)^2 - (3 + 2y)^2$

9 If n is an odd number, what can be said about the following?

a $n + 1$ b $n + 2$

c $2n$ d n^2

e $n!$ if $n! = n(n - 1)(n - 2) \ldots \times 3 \times 2 \times 1$ and $n \geqslant 3$

- How many flights of stairs are in this spiral staircase?

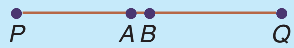
Chapter 5 | Working Mathematically

assignment 5B

1 Natasha had a large bag of sweets. After counting them, she realised that when the total was divided by 2 or 3 or 4 or 5 or 6 there was a remainder of one each time, but when the total was divided by 7 there was no remainder. What is the smallest number of sweets that could have been in the bag?

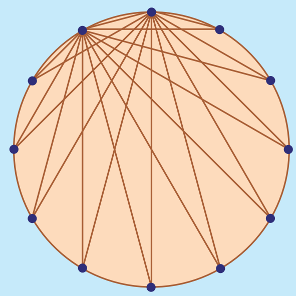
2 A ball is known to rebound to half the height that it drops. If the ball is initially dropped from a height of 12 m, how far will it have travelled by the time it hits the ground for the 6th time?

3 The distance from *P* to *Q* is 36 cm. How far is the point *A* from *P* if it is four times as far from *Q* as it is from *B*, the midpoint of *PQ*?

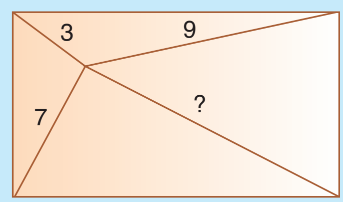


4 A snail begins to climb a wall. Every hour it manages to climb up 30 cm but it gets tired and slides back 10 cm. How long will it take for the snail to climb up the wall if the wall is 2·98 metres high?

5 The partly completed diagram shows how two of the points on the circle have been connected to ten other points around the circle by straight line segments. If every point has to be connected to every other point, how many straight line segments will there be?



6 A point inside a rectangle is 3 units from one corner, 7 units from another and 9 units from another. How far is the point from the fourth corner?



Chapter Review Questions

Technology Applications Activities

Literal equations

Technology Applications Drag and Drops

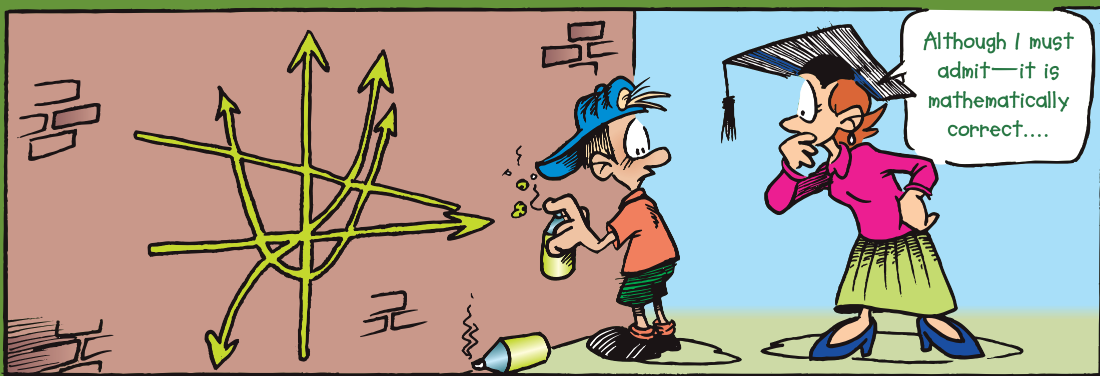
1 Literal equations

2 Further simultaneous equations

• Backgammon is a famous board game involving both chance and strategy.

6

Curve Sketching



Chapter Contents

Learning Outcomes

Students will be able to:

- Sketch a range of curves including parabolas, cubics and circles.
- Find the point of intersection of a range of graphs.
- Describe the main features of a curve from its equation.

Areas of Interaction

Approaches to Learning (Knowledge Acquisition, Problem Solving, Logical Thinking, Reflection), Human Ingenuity

6:01 | Curves of the Form $y = ax^n$ and $y = ax^n + d$

So far the method used to graph a curve has relied on using the equation of the curve to produce a table of values. This gave a set of points on the curve, which could then be plotted. This procedure, although slow, is basic to curve sketching. It is the only way of producing an accurate graph of a curve.

When it is not possible to draw an accurate graph, a sketch is produced. This is an approximation of the graph and is intended to show the main features of the graph.

In this chapter, the approach to curve sketching does not rely on plotting points. It is more general in nature and uses basic number properties relating to the signs of numbers, the powers of numbers and the relative size of numbers. These properties are examined in the following Prep Quiz.

prep quiz 6:01

For **1** to **5**, state whether the statements are true (T) or false (F). Note that n is a positive integer.

1 If x is positive, then x^n is positive.

2 If x is negative, then x^n is negative.

3 If x is a small number (close to zero where $-1 < x < 1$), then x^n is a smaller number (closer to zero).

4 If x is a large number (further than 1 unit from zero), then x^n is a larger number (further from zero).

If $y = x^2$, what happens to y as:

5 x becomes smaller (ie moves closer to zero)?

6 x becomes larger (ie moves further from zero)?

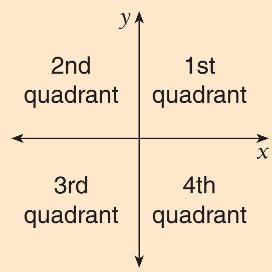
In which quadrant of the number plane would the point (x, y) be found if:

7 x is positive and y is positive?

8 x is positive and y is negative?

9 x is negative and y is positive?

10 x is negative and y is negative?



Curves of the form $y = ax^n$

The reasoning used in the Prep Quiz can be used to sketch the curve $y = ax^n$.

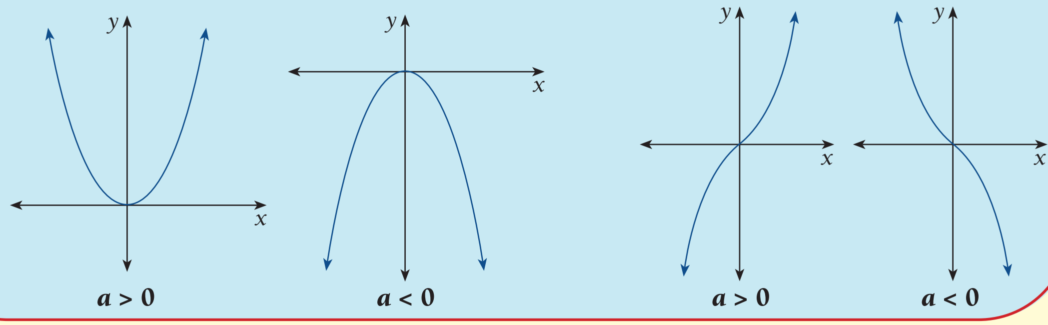
- To find the sign of y when $y = ax^n$ we need to realise that:
  If n is even: x^n is always positive (except for $x = 0$).
  If n is odd: x^n is positive when x is positive
  x^n is negative when x is negative.
  The signs of a and x^n will then determine the sign of y.
- As x becomes smaller, ax^n (and therefore y) becomes smaller.
  As x becomes larger, ax^n (and therefore y) becomes larger.
- If $x = 0$, $y = 0$; and if $y = 0$, $x = 0$. Hence, the curve only crosses the x-axis once, at the point $(0, 0)$.

When considering whether a number is large or small, we are concerned only with its size and not its sign.
Numbers close to zero are small in size.
Numbers far from zero are large in size.

- From this information, we can deduce the following.
  Curves with an even value of n will have shapes like parabolas ($y = ax^2$).
  Curves with an odd value of n will have shapes like cubics ($y = ax^3$).

Curves of the form $y = ax^n$

1 n even **2 n odd**



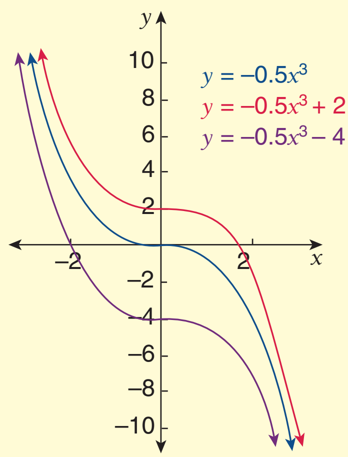
- As n increases, the steepness of the curve increases, eg $y = x^4$ will be steeper than $y = x^2$. The curve with the higher value of n will be above the other curve for all values of x except from −1 to 1.
- As we have already seen in Chapter 5, for the curve $y = ax^3$, changing a changes the 'steepness' of the curve. Hence, for the same value of n, a larger value of a will result in a steeper curve. eg $y = 2x^5$ will be steeper than $y = x^5$.

Curves of the form $y = ax^n + d$

Curves of the form $y = ax^n + d$ can be obtained by translating the curve $y = ax^n$ up or down.

- If d is positive, translate it up d units.
- If d is negative, translate it down d units.

This is illustrated in the diagram on the right for the curves $y = -0{\cdot}5x^3$, $y = -0{\cdot}5x^3 + 2$ and $y = -0{\cdot}5x^3 - 4$



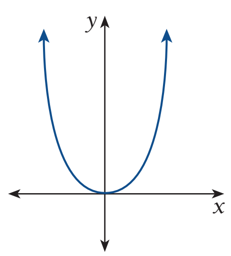
worked examples

Make sketches of the following curves.

1 $y = \frac{1}{2}x^4$ **2** $y = -2x^5$ **3** $y = -1 - 2x^6$

Solutions

1



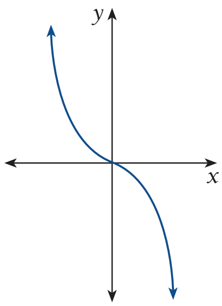
As $\frac{1}{2}x^4$ is positive for all x (except $x = 0$), then y is positive (except when $x = 0$).

When x is small (close to 0), y is small.

As x becomes larger (moves away from 0), $\frac{1}{2}x^4$, and hence y, becomes larger.

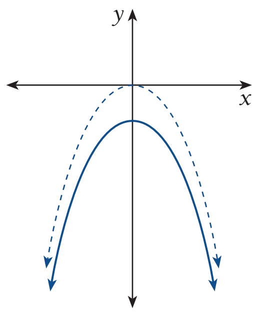
continued →→→

2



When x is positive, $-2x^5$ is negative and y is negative.
When x is negative, $-2x^5$ is positive and y is positive.
When x is small (close to 0), y is small.
As x becomes larger, y becomes larger.

3



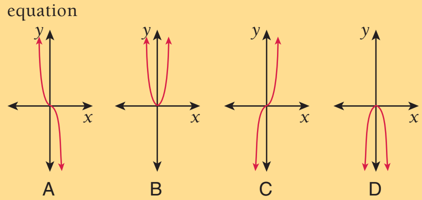
$y = -1 - 2x^6$
$\therefore\ y = -2x^6 - 1$
This is the curve $y = -2x^6$ (shown as a dotted line) moved down 1 unit to get $y = -2x^6 - 1$.

Exercise 6:01

Foundation Worksheet 6:01

Curves of the form $y = ax^n$ and $y = ax^n + d$

1 By examining the signs of x and y, choose which of the curves A to D is the best sketch for the curves with the equation

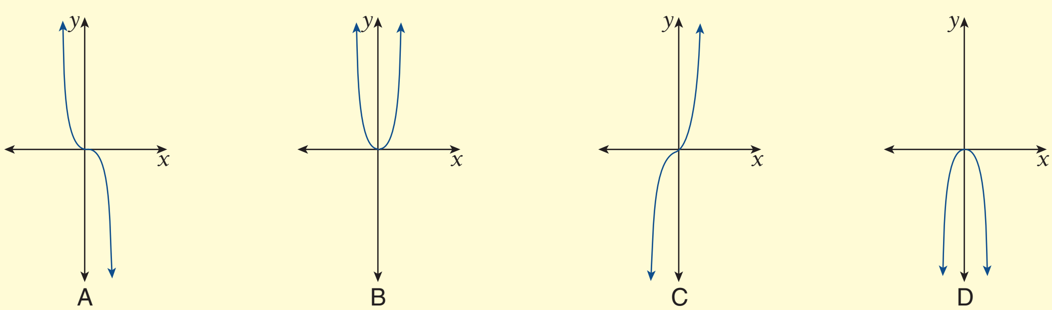


a $y = x^6$ **b** $y = -2x^4$ **c** $y = 3x^5$

2 Sketch the following curves on separate number planes.
a $y = x^4 - 4$ **b** $y = 2 - 4x^5$

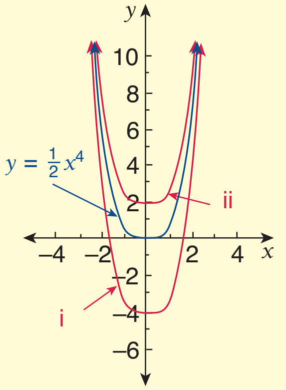
1 Choose which of the curves A, B, C or D below is the best sketch for each of the following.

a $y = x^4$ **b** $y = -x^5$
c $y = 2x^7$ **d** $y = -3x^6$
e $y = 4x^4$ **f** $y = -4x^5$
g $y = \dfrac{x^5}{10}$ **h** $y = \dfrac{-1}{10}x^7$
i $y = -2x^8$

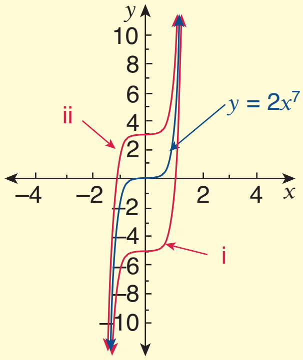


2 In each diagram, the two unlabelled curves were obtained by moving the other curve up or down. Give the equations of the unlabelled curves.

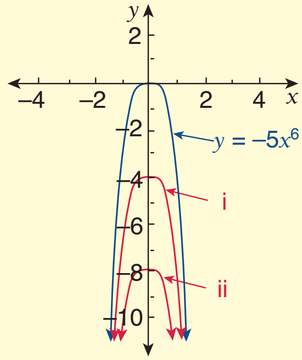
a



b

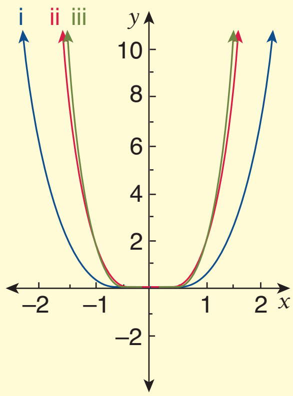


c



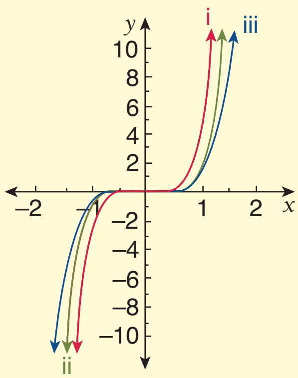
3 In each of the following, match the curves **i**, **ii** and **iii** with the equations A, B and C.

a



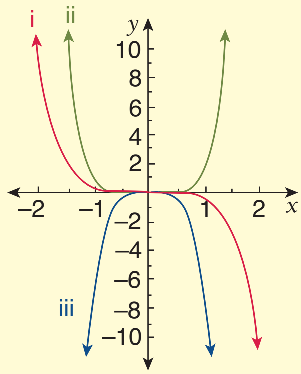
A $y = 2x^4$
B $y = 0{\cdot}5x^4$
C $y = 2x^6$

b



A $y = x^5$
B $y = x^7$
C $y = 4x^5$

c



A $y = -5x^4$
B $y = x^6$
C $y = -0{\cdot}25x^5$

4 Make sketches of the following on separate number planes.

a $y = \frac{1}{2}x^5$ **b** $y = 2x^4$ **c** $y = -2x^6$
d $y = x^5 - 1$ **e** $y = x^6 + 1$ **f** $y = -2x^4 + 4$

When graphing $y = ax^n + d$, look first at n then a, then d.

5 Make sketches of each pair of graphs on the same number plane.

a $y = x^4$, $y = x^6$
b $y = x^4$, $y = -x^4$
c $y = x^5$, $y = 1 + x^5$
d $y = x^6 + 1$, $y = x^6 - 1$
e $y = 2x^5 + 1$, $y = 2x^3 + 1$
f $y = 1 - x^6$, $y = -1 - x^6$

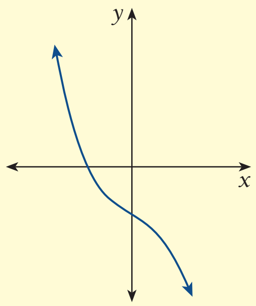
6 **a** How do each of the numbers in the equation $y = -2x^4 + 1$ determine the features of the curve?
b Describe the shape and features of the curve $y = 2x^3 - 3$.
c What is the relationship between the curves $y = 2x^5$ and $y = -2x^5$?

7 From the curves of the form $y = ax^n$, sketch a possible curve, given that:
- **a** a is positive and n is even
- **b** a is negative and n is even
- **c** a is positive and n is odd
- **d** a is negative and n is odd

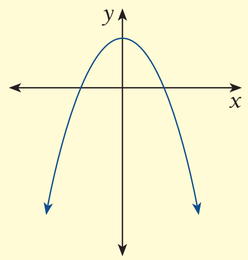
8 From the curves of the form $y = ax^n + d$, sketch a possible curve, given that:
- **a** a is positive, n is odd and d is negative
- **b** a is negative, n is even and d is positive

9 Each of the curves below has the form $y = ax^n + d$. For each curve, state what you can about the signs of a and d, and the value of n.

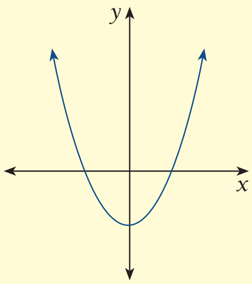
a



b



c



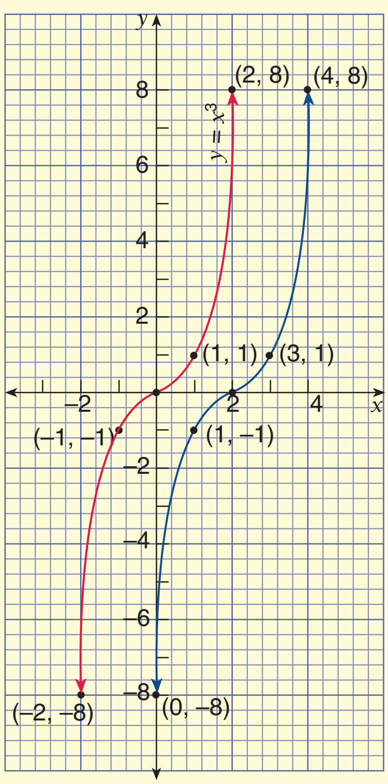
6:02 | Curves of the Form $y = ax^n$ and $y = a(x - r)^n$

In the diagram, the graph of $y = x^3$ has been moved (horizontally) 2 units to the right to produce the blue curve.

- A table of values for each of the curves is given below.
  $y = x^3$

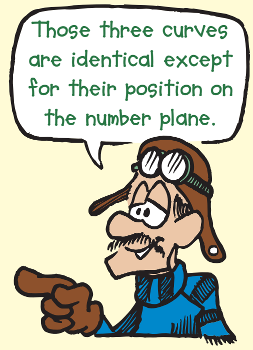
x	y
−2	−8
−1	−1
0	0
1	1
2	8

x	y
0	−8
1	−1
2	0
3	1
4	8



- As the tables suggest, if $y = x^3$ is moved horizontally 2 units to the right, then the x-coordinates will increase by 2 while the y-coordinates remain unchanged.
  What then is the equation of the blue curve?
- Reducing the x-coordinates by 2 will restore the cubic relationship. Hence the equation is $y = (x - 2)^3$.

The above case is an example of the general result.



If the curve $y = ax^n$ is moved (horizontally):
- **r units to the right, the equation of the new curve is $y = a(x - r)^n$**
- **r units to the left, the equation of the new curve is $y = a(x + r)^n$**

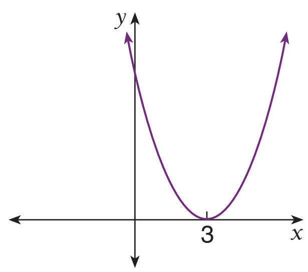
worked examples

Sketch each of the following curves.

1 $y = (x - 3)^2$ **2** $y = 2(x + 1)^4$ **3** $y = -\frac{1}{2}(x - 1)^5$

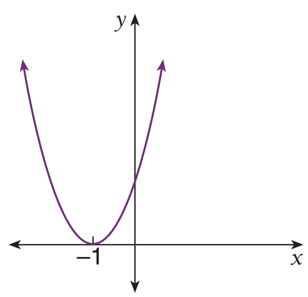
Solutions

1



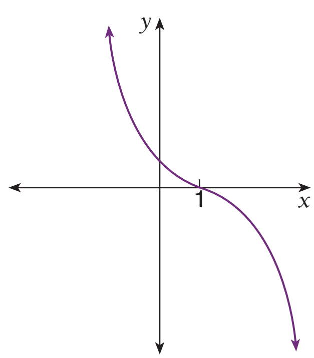
$y = (x - 3)^2$ has been produced by moving the curve $y = x^2$ three units to the right.

2



$y = 2(x + 1)^4$ has been produced by moving the curve $y = 2x^4$ one unit to the left.

3



$y = -\frac{1}{2}(x - 1)^5$ has been produced by moving the curve $y = -\frac{1}{2}x^5$ one unit to the right.

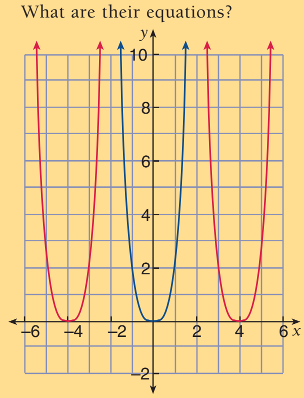
■ You must know how to graph $y = ax^n$ before you can graph $y = a(x - r)^n$ or $y = a(x + r)^n$.

Exercise 6:02

Foundation Worksheet 6:02

Curves of the form $y = ax^n$ and $y = a(x - r)^n$

1 The blue curve is $y = x^6$. It has been translated horizontally to produce the curves A and B. What are their equations?



2 Write down the equation of the curve that is obtained when $y = x^3$ is translated:

a 2 units to the left
b 1 unit to the right

1

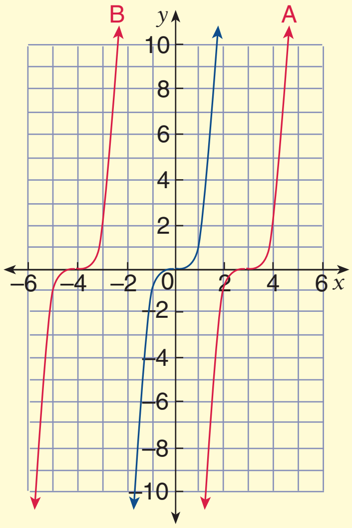


Diagram 1

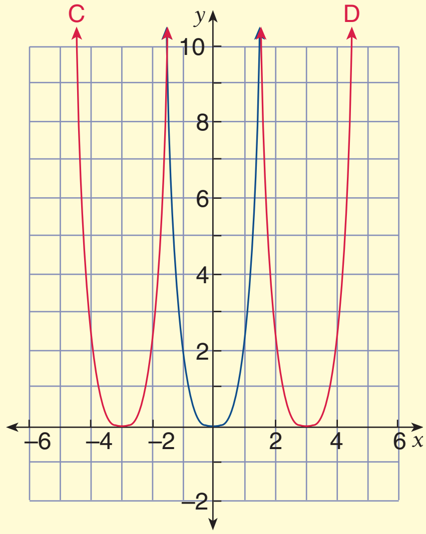


Diagram 2

a The blue curve in diagram 1 is $y = x^5$. It has been translated horizontally to produce curves A and B. What are their equations?

b The blue curve in diagram 2 is $y = 2x^4$. What curve has the equation $y = 2(x - 3)^4$?

2 Write down the equation of the curve that is obtained by translating the curve $y = x^4$ (horizontally):

a 4 units to the left **b** 4 units to the right
c 1 unit to the right **d** $\frac{1}{2}$ unit to the left

3 Describe how $y = -2x^5$ must be translated to produce curves with the equations:

a $y = -2(x - 1)^5$ **b** $y = -2(x + 3)^5$ **c** $y = -2(x - \frac{1}{2})^5$

4 Make sketches on separate number planes of:

a $y = -(x + 1)^4$ **b** $y = (x + 1)^4$ **c** $y = (x - 1)^4$ **d** $y = -(x - 1)^4$

5 Sketch on separate number planes the following curves.

a $y = 3(x + 2)^2$ **b** $y = 2(x - 3)^5$ **c** $y = -2(x - 4)^4$ **d** $y = -(x + 1)^6$

6 Make sketches of each pair of curves on the same number plane.

a $y = -(x - 1)^2$, $y = (x + 1)^2$ **b** $y = (x - 3)^3$, $y = -(x - 3)^3$ **c** $y = 2(x - 3)^3$, $y = 2(x - 3)^5$ **d** $y = 2(x + 1)^2$, $y = 2(x + 1)^4$

7 Make sketches of the following curves.

a $y = 2(x - 1)^2 + 1$ **b** $y = 3(x - 2)^3 + 1$
c $y = (x + 1)^4 - 1$ **d** $y = -2(x + 3)^6 - 2$

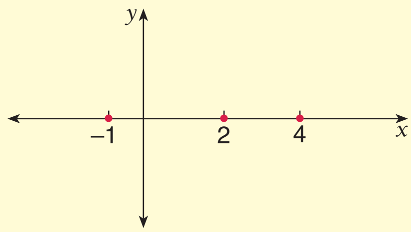
6:03 | Curves of the Form $y = a(x - r)(x - s)(x - t)$

In the work so far, the sign analysis for x and y has been of great importance. In this section, it is also important, though not as simple.

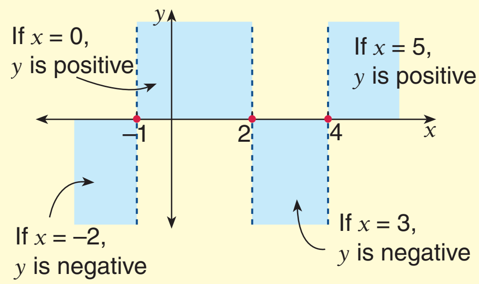
Discussion

Consider the example $y = 3(x - 2)(x + 1)(x - 4)$.

- Now, x can take any value and for every x value there is a corresponding y value. Hence, the curve has no gaps.
- First, find the x-intercepts of the curve.
  If $y = 0$ then:
  $3(x - 2)(x + 1)(x - 4) = 0$
  $\therefore\ x - 2 = 0$ or $x + 1 = 0$ or $x - 4 = 0$
  $\therefore\ x = 2$ or -1 or 4
  Plot these x-intercepts.

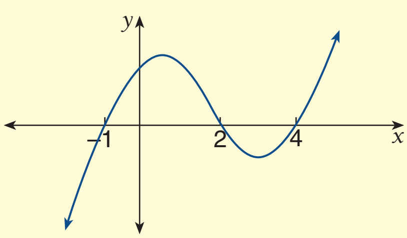


- Now, for all the other values of x, y is either positive or negative. The three x-intercepts divide the x-axis into four sections. In each of these sections the y values will be either positive or negative. Testing a point in each of the sections will determine whether that section is positive or negative.



- Also, as x increases past 4, y increases, and as x decreases below -1, y decreases.
- Putting all the information together allows us to sketch the curve.

Testing a value
When $x = 5$, $y = 3(+)(+)(+)$
$= \text{positive}$
When $x = 3$, $y = 3(+)(+)(-)$
$= \text{negative}$



These ideas are further explored in the following examples.

worked examples

Make sketches of the following curves.

1 $y = 2(x - 1)(x - 4)(x + 2)$

2 $y = -2(x - 1)(x - 4)(x + 2)$

3 $y = (x - 1)^2(x - 4)$

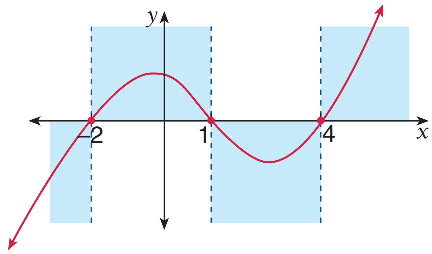
Solutions

1 $y = 2(x - 1)(x - 4)(x + 2)$

- The curve has x-intercepts at $x = 1$, $x = 4$ and $x = -2$.
- Testing values of x shows that the curve is in the coloured regions.

x	−3	−2	0	1	3	4	5
sign of y	(−)	0	(+)	0	(−)	0	(+)

- Considering the size of y when x is greater than 4 and less than −2 gives the correct shape.

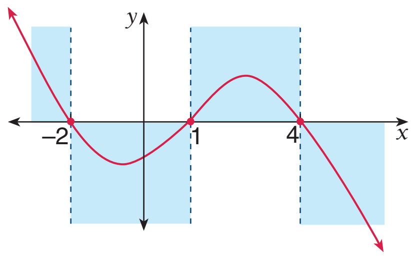


2 $y = -2(x - 1)(x - 4)(x + 2)$

- The curve has x-intercepts at $x = 1$, $x = 4$ and $x = -2$.
- Testing values of x shows that the curve is in the coloured regions.

x	−3	−2	0	1	3	4	5
y	(+)	0	(−)	0	(+)	0	(−)

- Considering the size of y when x is greater than 4 and less than −2 gives the correct shape.

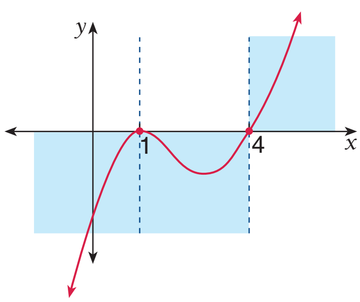


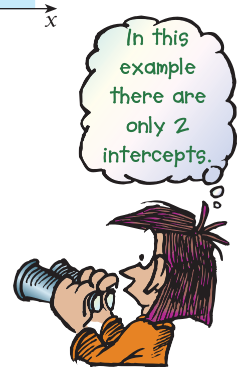
3 $y = (x - 1)^2(x - 4)$

- x-intercepts occur at $x = 1$ and $x = 4$.
- Sign analysis.

x	0	1	2	4	5
y	(−)	0	(−)	0	(+)

- Considering the size of y when x is greater than 4 and less than 1 gives the correct shape.





Exercise 6:03

Foundation Worksheet 6:03

Equations of the form $y = a(x - r)(x - s)(x - t)$

1 Match each of the curves A to D to the following equations:
- **a** $y = (x - 1)(x - 2)(x - 4)$
- **b** $y = (x + 1)(x + 2)(x + 4)$
- **c** $y = (x - 1)(x + 2)(x - 4)$
- **d** $y = (x + 1)(x + 2)(x - 4)$

2 Sketch the curves:
- **a** $y = (x - 3)(x + 2)(x - 1)$
- **b** $y = (x + 3)(x - 2)(x - 1)$

1 Complete the sign analysis tables for each of the given curves.

a $y = 2(x + 4)(x + 2)(x + 1)$

x	−5	−4	−3	−2	−1·5	−1	0
y		0		0		0	

b $y = -4(x + 2)(x - 2)(x - 5)$

x	−3	−2	0	2	4	5	6
y		0		0		0	

c $y = \frac{1}{2}(x - 1)^2(x - 4)$

x	0	1	2	4	5
y		0		0	

d $y = -(x + 1)(x - 2)^2$

x	−2	−1	0	2	3
y		0		0	

2 Find the x-intercepts for each of the following curves.
- **a** $y = (x - 1)(x - 2)(x - 4)$
- **b** $y = (1 - x)(x + 2)(4 - x)$
- **c** $y = (x - 2)^2(x + 5)$
- **d** $y = 2x(x + 4)(x - 2)$

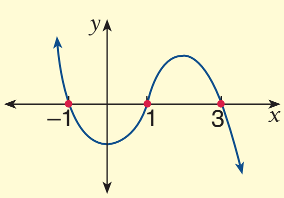
To find where a curve crosses the x-axis (ie x-intercepts), put $y = 0$.

3 Make sketches of the following curves on separate number planes.
- **a** $y = (x - 2)(x + 2)(x - 4)$
- **b** $y = -(x - 2)(x + 2)(x - 4)$
- **c** $y = 2(x - 1)(x - 3)^2$
- **d** $y = -2(x - 1)(x - 3)^2$

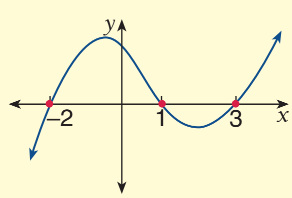
4 **a** Find the y-intercepts of the curves in question **3**.
b How can the y-intercepts be used to check your sketches?

5 Write down a possible equation for each of the curves shown.

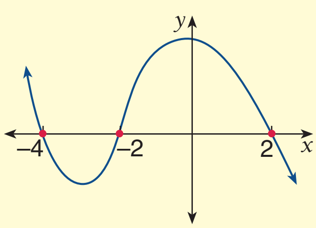
a



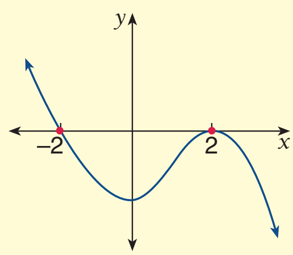
b



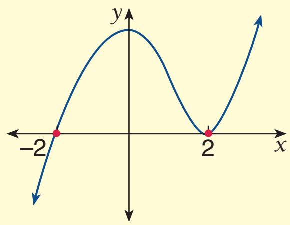
c



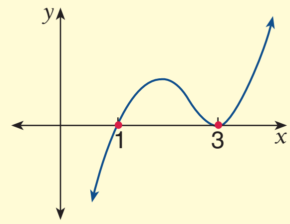
d



e



f



6 What additional information would have been needed in question **5** to find the actual equation of the curve?

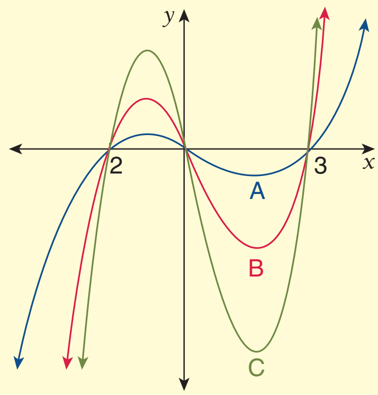
7 **a** Match each of the curves to the following equations.

i $y = x(x - 3)(x + 2)$

ii $y = 2x(x - 3)(x + 2)$

iii $y = 0{\cdot}25x(x - 3)(x + 2)$

b On the same number plane, sketch the curves $y = -x(x - 3)(x + 2)$ and $y = -2x(x - 3)(x + 2)$



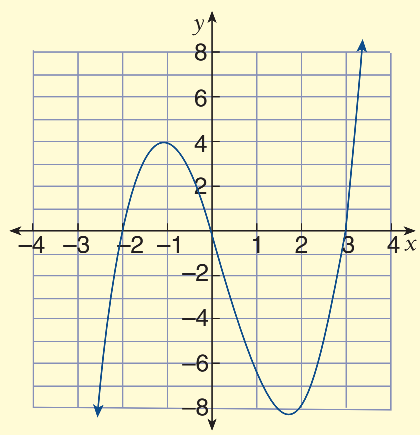
8 The equation of the curve shown is $y = x(x + 2)(x - 3)$. Give the equation of the curve that would result if this curve was:

a translated 1 unit to the right

b translated 2 units to the left

c reflected in the x-axis

d reflected in the y-axis



9 **a** A cubic curve is known to have x-intercepts of -1, 1 and 4 and y-intercept of 8. What is its equation?

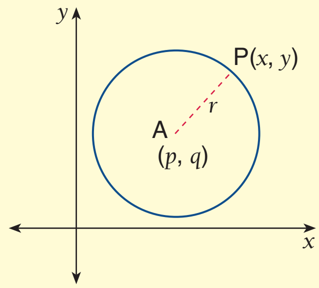
b A cubic curve has x-intercepts of 2 and -2 and passes through the point $(3, -10)$. What is its equation?

10 Sketch the graph of $y = (x - r)(x - s)(x - t)$ if $r = s = t$.

6:04 | Circles and their Equations

The fact that every point on a circle is the same distance away from the centre allows us to derive its equation.

Let $P(x, y)$ be any point on a circle that has its centre at $A(p, q)$ and a radius of r units.



Now, $AP = r$

Using coordinate geometry,

$$AP = \sqrt{(x-p)^2 + (y-q)^2}$$

$$\therefore \sqrt{(x-p)^2 + (y-q)^2} = r$$

$$(x-p)^2 + (y-q)^2 = r^2$$

The equation of a circle which has centre (p, q) and radius r is given by the equation: $(x - p)^2 + (y - q)^2 = r^2$.

worked examples

1 What is the equation of a circle that has centre (–3, 4) and radius 5?

2 Find the centre and radius of the circle $(x - 6)^2 + (y + 7)^2 = 9$.

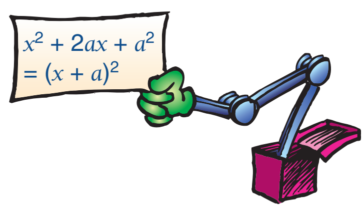
3 Find the centre and radius of the circle $x^2 + y^2 + 6x - 2y + 6 = 0$.

Solutions

1 The circle formula is $(x - p)^2 + (y - q)^2 = r^2$.
Here, (p, q) is (–3, 4) and $r = 5$.
$\therefore (x - (-3))^2 + (y - 4)^2 = 5^2$
The equation of the circle is $(x + 3)^2 + (y - 4)^2 = 25$

2 $(x - 6)^2 + (y + 7)^2 = 9$ can be written as $(x - 6)^2 + (y - (-7))^2 = 3^2$
This circle has centre (6, –7) and radius 3.

3 Here, we write the equation in the form $(x - p)^2 + (y - q)^2 = r^2$ by completing the squares.
$x^2 + y^2 + 6x - 2y + 6 = 0$
$x^2 + 6x + y^2 - 2y = -6$
Completing each square.
$(x^2 + 6x + 9) + (y^2 - 2y + 1) = -6 + 9 + 1$
$(x + 3)^2 + (y - 1)^2 = 4$
$(x - (-3))^2 + (y - 1)^2 = 2^2$
This circle has centre (–3, 1) and radius 2.



Exercise 6:04

1 Find the equation of the following circles.

a centre (1, 1), radius 7 units

b centre (5, 0), radius 2 units

c centre (–3, –5), radius 4 units

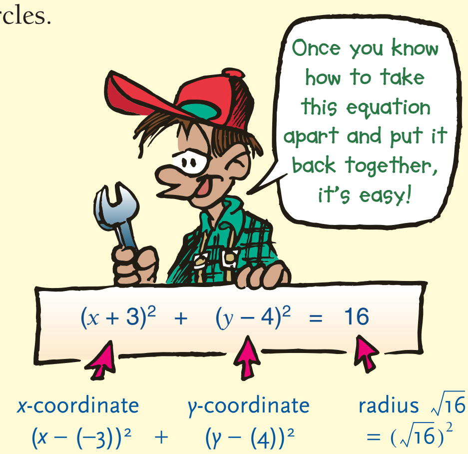
d centre (2, –5), radius 1 unit

e centre (0, 2), radius $\frac{1}{2}$ unit

f centre (0, 0), radius 3 units

2 Find the centre and radius of each of the following circles.

a $(x-2)^2+(y-3)^2=64$
b $(x+4)^2+(y-1)^2=4$
c $(x+3)^2+(y+3)^2=9$
d $(x-6)^2+(y-5)^2=100$
e $x^2+(y+5)^2=16$
f $(x-3)^2+y^2=1$
g $x^2+y^2=81$
h $x^2+y^2=49$
i $x^2+y^2=11$
j $(x-7)^2+(y-8)^2=2$



3 Find the centre and radius of each of the following circles.

a $x^2-10x+y^2+8y+32=0$
b $x^2+y^2+8x-14y=35$
c $x^2+y^2-18x-20y+60=0$
d $x^2+y^2-9x+13\frac{1}{4}=0$

4 On the one number plane, draw the graphs of:

a $x^2+y^2=4$
b $(x+3)^2+(y+3)^2=9$
c $(x-4)^2+y^2=1$
d $(x-1)^2+(y-1)^2=16$

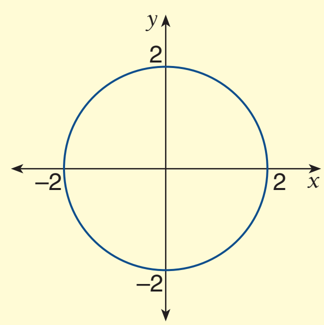
5 Find the x- and y-intercepts of the circles:

a $x^2+(y-2)^2=9$
b $(x-2)^2+(y-1)^2=16$

6 a A circle with its centre at (3, 4) passes through the origin. What is its equation?
b Which of the following represents the equation of a circle?
i $x^2-y^2=16$ ii $4x^2+4y^2=8$ iii $6x^2+3y^2=6$

7 The circle $x^2+y^2=4$ is translated to new positions. What is the equation of the circle if it is translated:

a (horizontally) 2 units to the right?
b (horizontally) 2 units to the left?
c (vertically) 2 units up?
d (vertically) 2 units down?



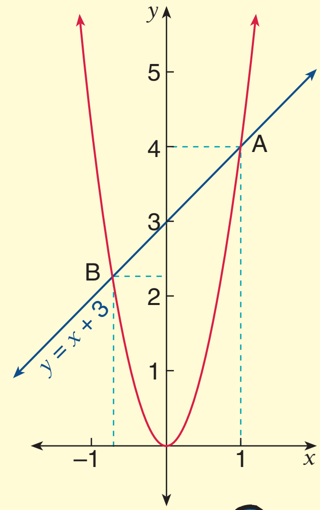
8 What is the equation of the circle that results from translating the circle $(x-2)^2+(y+2)^2=4$ (horizontally) 2 units to the right and then (vertically) 1 unit down?

6:05 | The Intersection of Graphs

In this section, both graphical and algebraic methods will be used to find the point or points of intersection of a line with a parabola, circle or hyperbola.

Graphical method

- In the diagram, the points of intersection of the parabola $y = 4x^2$ and the line $y = x + 3$ are A and B.
  It would appear from the graph that:
  A is the point (1, 4) and
  B is the point (–0·7, 2·2).
- However, the only way to be certain that these are the exact points of intersection is to substitute the coordinates into the original equations and see if true statements result.



Checking A(1, 4):

For $\quad y = 4x^2 \qquad$ and $\quad y = x + 3$

$\qquad 4 = 4 \times 1^2 \qquad\qquad 4 = 1 + 3$

Both of these statements are true.

∴ A(1, 4) is the exact point of intersection.

Checking B(–0·7, 2·2):

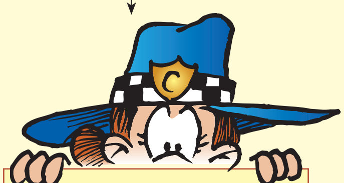
For $\quad y = 4x^2 \qquad$ and $\quad y = x + 3$

$\qquad 2{\cdot}2 = 4 \times (-0{\cdot}7)^2 \qquad\qquad 2{\cdot}2 = -0{\cdot}7 + 3$

Both of these statements are false.

∴ B(–0·7, 2·2) only approximates the point of intersection. The accuracy of the estimate can be improved by trial and error.

Even though the graphical method is slow and often cannot give us an exact solution, it is still a valuable technique.



- To find points of intersection graphically:
  1. Draw the graphs.
  2. Read off coordinates.
  3. Check the solutions by substitution.

Algebraic method

This method is based on the fact that at a point of intersection the *x*-coordinates on both curves are equal and the *y*-coordinates on both curves are equal.

So, for the curves $y = 4x^2$ and $y = x + 3$:

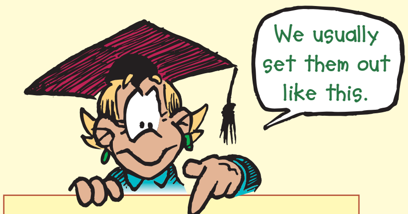
If the *y*-coordinates are equal, then:

$$4x^2 = x + 3$$
$$4x^2 - x - 3 = 0$$
$$(4x + 3)(x - 1) = 0$$
$$x = -\tfrac{3}{4} \text{ or } 1.$$

Substitution of these values into either $y = 4x^2$ or $y = x + 3$ gives:

$y = 4$ (when $x = 1$)

$y = 2\frac{1}{4}$ (when $x = -\frac{3}{4}$)



$y = 4x^2 \quad \ldots (1)$

$y = x + 3 \quad \ldots (2)$

Substitute (1) in (2).

$$4x^2 = x + 3$$
$$4x^2 - x - 3 = 0$$
$$(4x + 3)(x - 1) = 0$$
$$x = \frac{-3}{4} \text{ or } 1$$

Substitute in (2).

When $x = \frac{-3}{4}$, $\quad y = 2\frac{1}{4}$

$\qquad x = 1$, $\quad y = 4$

∴ Points of intersection are:

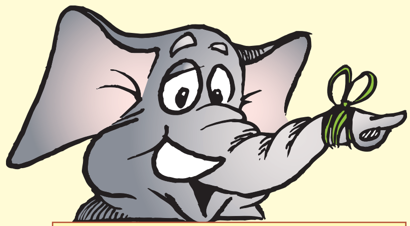
(1, 4) and $(-\frac{3}{4}, 2\frac{1}{4})$

$\therefore$ Points of intersection are:

$(1, 4)$ and $(-\frac{3}{4}, 2\frac{1}{4})$.

These should be checked by substituting each point into both equations.

The points of intersection can also be found using the fact that the *x*-coordinates are equal, but in this case it requires more work.



It is important to check all possible solutions in both of the original equations.

$y = 4x^2 \quad \ldots (1)$

$y = x + 3 \quad \ldots (2)$

Rearranging these equations gives:

$x^2 = \frac{y}{4} \quad \ldots (3)$

$x = y - 3 \quad \ldots (4)$

Substituting (4) into (3) gives:

$$(y - 3)^2 = \frac{y}{4}$$
$$4(y - 3)^2 = y$$
$$4(y^2 - 6y + 9) = y$$
$$4y^2 - 24y + 36 = y$$
$$4y^2 - 25y + 36 = 0$$
$$(4y - 9)(y - 4) = 0$$
$$y = \frac{9}{4} \text{ or } 4$$

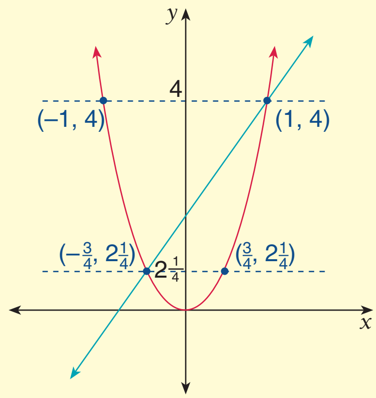
Substituting in (1)

When $y = \frac{9}{4}$, $\quad x = \pm\frac{3}{4}$

$y = 4$, $\quad x = \pm 1$

Checking in both (1) and (2) shows that the points of intersection are $(1, 4)$ and $(-\frac{3}{4}, 2\frac{1}{4})$.

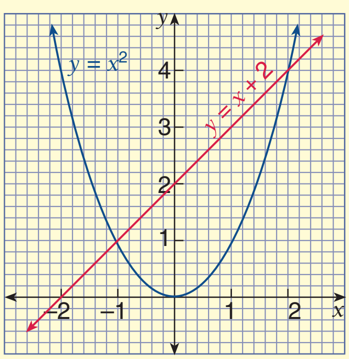
The sketch below shows that there are only two solutions



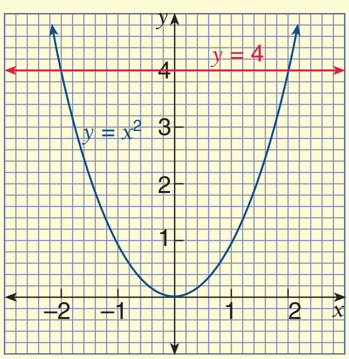
Exercise 6:05

1 Find the points of intersection of the graphs in each part. Check that the coordinates of each point of intersection satisfy both equations.

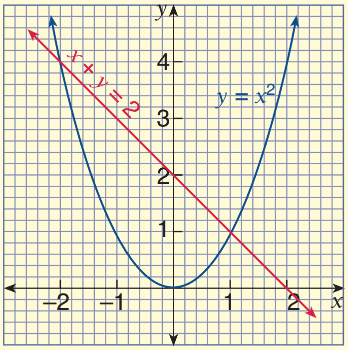
a



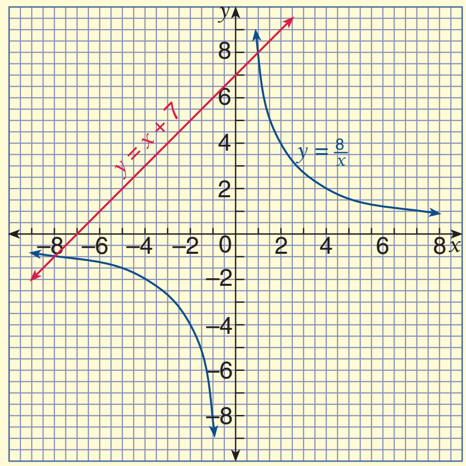
b



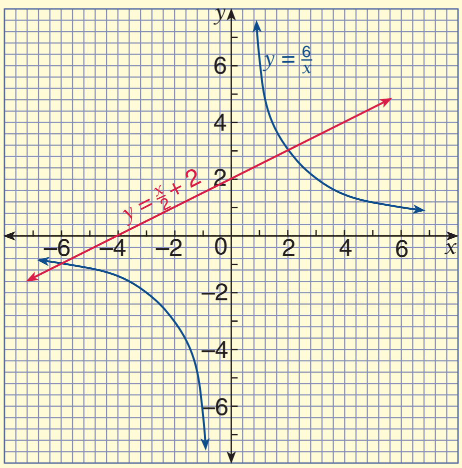
c



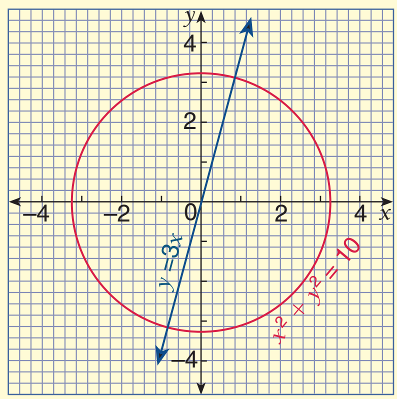
d



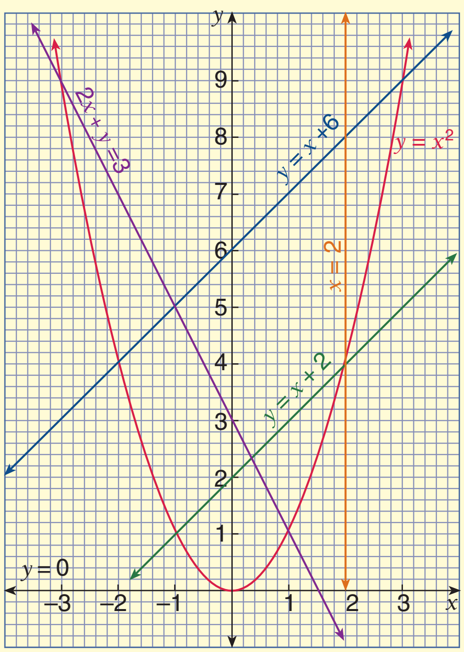
e



f



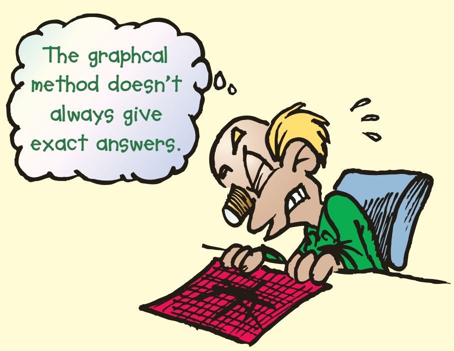
2



Use this diagram to find the point or points of intersection of the parabola $y = x^2$ with the line:

a $y = x + 6$ **b** $x = 2$

c $y = x + 2$ **d** $2x + y = 3$

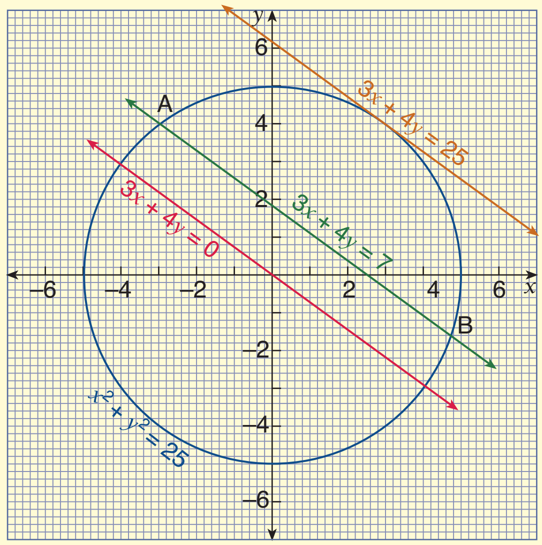


3 Use the diagram to answer the following questions.

a What are the points of intersection of the circle $x^2 + y^2 = 25$ and the line $3x + 4y = 0$?

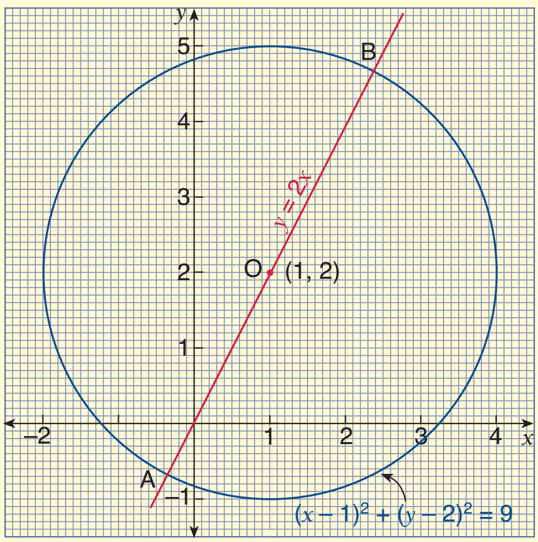
b The line $3x + 4y = 25$ is a tangent to the circle $x^2 + y^2 = 25$. What is the point of intersection?

c The line $3x + 4y = 7$ meets the circle at A(−3, 4) and at B. Use the graph to estimate the coordinates of B.

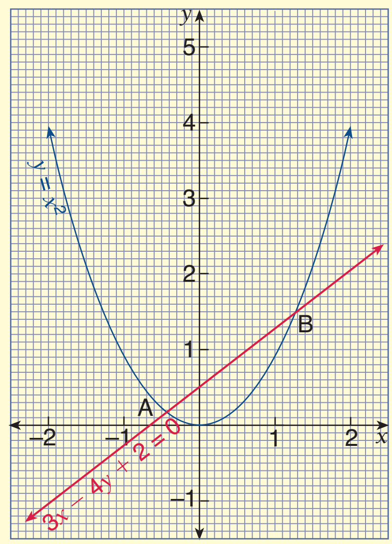


4 In each, estimate the coordinates of the points of intersection, A and B.

a



b



5 Find the points of intersection of the following graphs by graphical means.

a $y = x^2$
$y = 2x$

b $y = x^2$
$y - 2x = 3$

c $y = x$
$y = \frac{1}{x}$

d $y = x - 2$
$y = \frac{1}{x}$

e $x^2 + y^2 = 4$
$y = x$

f $x^2 + y^2 = 4$
$2x - y = 2$

6 Find the points of intersection of the graphs in question **5** using the algebraic method.

7 Find the points of intersection of the curves:

a $y = x + 1$ and $y = x^2 - 1$ **b** $xy = 2$ and $x + y = 4$

8 Use the algebraic method to find the coordinates of A and B in question **4**.

Investigation 6:05 | A parabola and a circle

investigation 6:05

Consider the parabola $y = k - x^2$ and the circle $x^2 + y^2 = 4$.

1 Use the graphical method to find the number of points of intersection if:
a $k = -2$ **b** $k = 0$ **c** $k = 2$ **d** $k = 4$

2 Use the algebraic method to find the points of intersection of $y = k - x^2$ and $x^2 + y^2 = 4$ when $k = 4$.

3 Determine the maximum value of k for which there is at least one point of intersection.

Mathematical Terms 6

mathematical terms 6

circle (equation of)
- The equation of a circle with its centre at the point (p, q) and a radius of r is $(x - p)^2 + (y - q)^2 = r^2$.

cubic
- A curve that contains an x^3 term as its highest power.

equation
- An algebraic statement that expresses the relationship between the x- and y-coordinates of every point (x, y) on the curve.

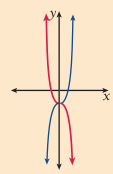
graph (of a curve)
- The line that results when the points that satisfy a curve's equation are plotted on a number plane.

horizontally
- In the direction of the x-axis.

hyperbola
- A curve with the equation $y = \frac{k}{x}$.

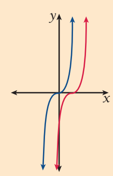
parabola
- A curve that contains an x^2 term as its highest power.

reflect (a curve)
- To flip a curve about a line usually the x- or y-axis. These curves are reflections of each other in the y-axis.



sketch (a curve)
- An approximation of a graph.

translate (a curve)
- To slide the curve in the direction of a line, usually the x- or y-axis. Either curve can be translated to produce the other.



vertically
- In the direction of the y-axis.

x-y intercepts
- The point(s) where a curve crosses the x- or y-axis.

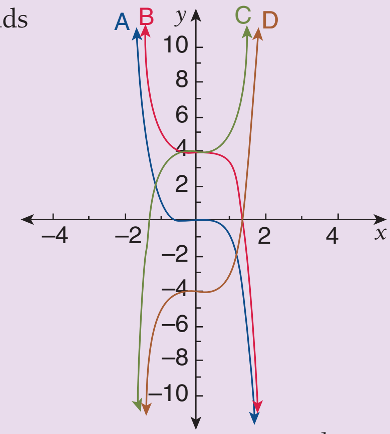
Mathematical terms 6

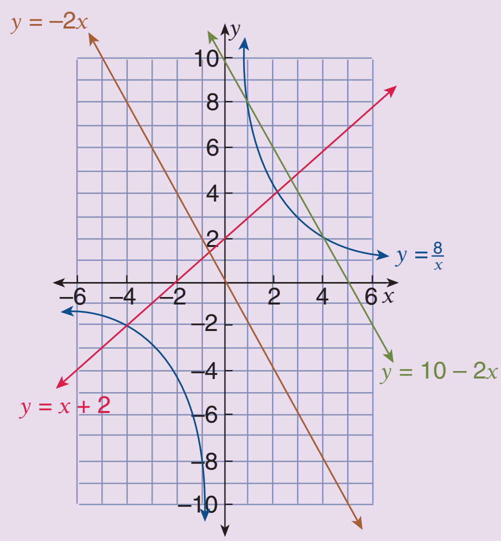
diagnostic test 6

Diagnostic Test 6 | Curve Sketching

- These questions reflect the important skills introduced in this chapter.
- Errors made will indicate an area of weakness.
- Each weakness should be treated by going back to the section listed.

Question	Section
1 Sketch the curves: **a** $y = 2x^5$ **b** $y = \frac{1}{2}x^6$ **c** $y = -2x^4$	6:01
2 Sketch the curves: **a** $y = \frac{1}{2}x^3 - 1$ **b** $y = -x^4 + 1$ **c** $y = -x^5 + 1$	6:01
3 Select the curve that corresponds to each equation: **a** $y = x^5 - 4$ **b** $y = -x^5 + 4$ **c** $y = -x^5$ **d** $y = x^5 + 4$	6:01
4 Sketch the curves: **a** $y = (x-3)^3$ **b** $y = 2(x+1)^4$ **c** $y = -\frac{1}{2}(x-1)^5$	6:02
5 Sketch the curves: **a** $y = x(x-2)(x-3)$ **b** $y = 2(x+1)(x-2)^2$ **c** $y = -(x+1)(x-2)(x+3)$	6:03
6 Sketch the circles: **a** $(x-1)^2 + y^2 = 4$ **b** $(x+1)^2 + y^2 = 9$ **c** $(x-2)^2 + (y+1)^2 = 16$	6:04
7 Use the graph to find the points of intersection of: **a** $y = \frac{8}{x}$ and $y = x + 2$ **b** $y = \frac{8}{x}$ and $y = -2x$ **c** $y = \frac{8}{x}$ and $y = 10 - 2x$	6:05
8 Find the points of intersection of: **a** $y = x^2$ and $y = 12 - x$ **b** $y = x - 3$ and $y = -\frac{2}{x}$ **c** $y = x + 2$ and $x^2 + y^2 = 10$	6:05





Chapter 6 | Revision Assignment

1 Make sketches of the curves:

a $y = 2x^7$ **b** $y = -2x^7$

c $y = 2x^7 - 1$ **d** $y = 1 - 2x^7$

e $y = 2(x - 1)^7$

2 What is the equation of the curve that results when the curve $y = -2x^8$ is translated:

a up 2 units? **b** down 2 units?

c left 2 units? **d** right 2 units?

3 What is the equation of the curve that results when the following curves are reflected in:

i the x-axis? **ii** the y-axis?

a $y = x^5$ **b** $y = x^5 - 2$

c $y = \frac{1}{2}x^6$ **d** $y = -3(x - 2)^3$

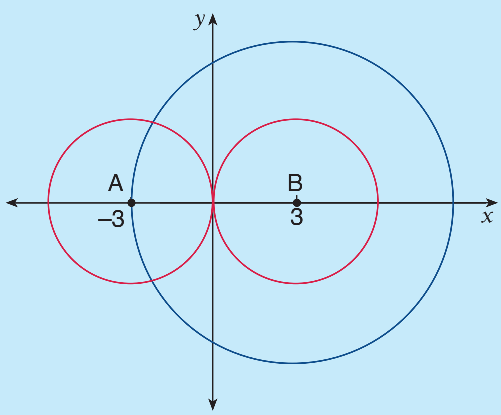
4 Sketch the following curves on the same number plane.

a $y = (x - 2)^2(x - 4)$

b $y = 2(x - 2)^2(x - 4)$

c $y = (x - 2)^2(4 - x)$

5

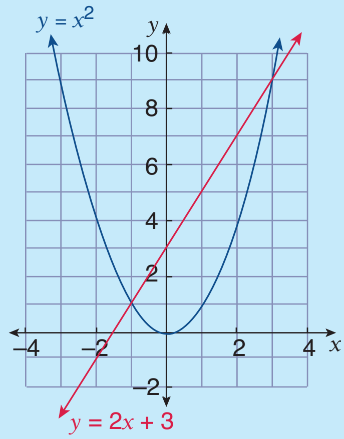


The two small circles have their centres at A and B. The larger circle's centre is also at B and it passes through A. What are the equations of:

a the circle centred at A?

b the larger circle with its centre at B?

6



a Use the graph to find the points of intersection of $y = x^2$ and $y = 2x + 3$.

b Estimate the points of intersection of the line through (0, 6) perpendicular to $y = 2x + 3$ with $y = x^2$.

c Use simultaneous equations to find the coordinates of the points of intersection correct to one decimal place.

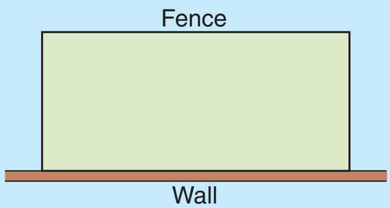
assignment 6A

assignment 6B

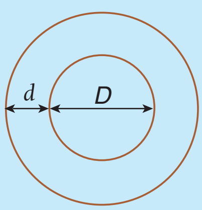
Chapter 6 | Working Mathematically

1 You are given an equation of the form $y = ax^n + d$. Describe how you would sketch the curve.

2 Describe briefly the features of curves with the following equations.
$y = -4x$
$y = 4x^2$
$y = -4x^2 - 4$
$y = (x - 4)^3$
$y = 4^x$
$y = x^4$
$x^2 + y^2 = 4$

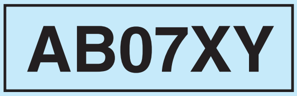
3


A rectangular pen is to be fenced on three sides with 20 metres of fencing material. What is the largest area that can be enclosed and what are its dimensions?

4


A pair of concentric circles is as shown. If you are given the circumference C, of the outer circle and the difference between the radii, d, find a formula for the inside diameter D in terms of C and d.

5 A new number plate consists of two letters, followed by 2 digits, followed by two letters as shown.

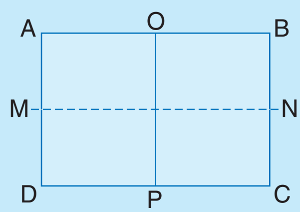


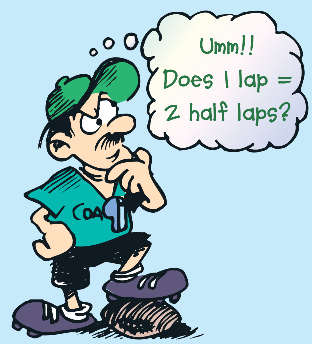
How many different number plates are possible?

6 A rugby field is rectangular in shape. It has a length of 90 m and a width of 60 m.

A coach arrived and told his team to run around the field 10 times to warm up. The team complained that this was too much, so the coach relented and told them to run around half of the field 15 times.

Was this a better deal for the team members?





Chapter Review Questions

Technology Applications Activities

1 Parabolas
2 Parabolas in real life

Transforming curves

Technology Applications Drag and Drops

7 Polynomials

Chapter Contents

Learning Outcomes

Students will be able to:

- Describe the properties of a polynomial.
- Perform operations with polynomials.
- Use the remainder and factor theorems.
- Solve polynomials.
- Sketch and perform basic transformations on the graphs of polynomials.

Areas of Interaction

Approaches to Learning (Knowledge Acquisition, Problem Solving, Communication, Logical Thinking, Information and Communication Technology Skills, Reflection), Human Ingenuity

7:01 | Polynomials

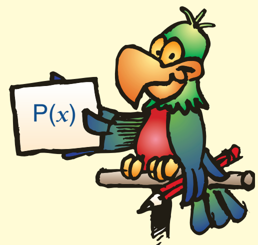
You should be familiar with the names **monomial**, **binomial** and **trinomial**, which are used to describe algebraic expressions with one, two and three terms, respectively.

- The name **polynomial** is used to describe an algebraic expression that is the sum of any number of terms, where each and every term is of the form ax^n (and n is a positive integer or zero).

  So, a polynomial $P(x)$ can be written in the following form:

> **$P(x) = a_nx^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \ldots + a_2x^2 + a_1x^1 + a_0x^0$**
> **where x is the variable**
> **$a_n, a_{n-1}, \ldots a_2, a_1, a_0$ are the coefficients**
> **a_nx^n is the leading term**
> **n is the degree of the polynomial (only if $a_n \neq 0$) (note that n is the highest power of x)**
> **a_n is the leading coefficient**
> **a_0 is the constant term (since $x^0 = 1$)**

- Note that expressions such as $x + \frac{1}{x}$, $x^2 + \sqrt{x}$ or $2^x + x$ are not polynomials in x, since each term is not a *positive integral* power of x or a constant.



- The notation $P(x)$ is used for polynomials. $P(c)$ refers to the value of $P(x)$ at $x = c$.
- If the leading coefficient is 1, the polynomial is said to be *monic*.
- *Linear*, *quadratic* and *cubic* expressions, which have been met in earlier chapters, are polynomials of degree 1, 2 and 3 respectively.

worked examples

1 For each polynomial, state the degree, leading coefficient, constant term and number of terms.

a $3x^5 - 7x^4 + 2x^2 - 5$

b $7x - 4x^2 + x^3$

2 $P(x)$ is a monic quadratic polynomial with a constant term of -2. If $P(2) = 8$, find the polynomial.

Solutions

1 a degree = 5
leading coefficient = 3
constant term = -5
number of terms = 4

b Rearranging in order: $x^3 - 4x^2 + 7x$
degree = 3
leading coefficient = 1 (monic)
constant term = 0
number of terms = 3

2 $P(x)$ is quadratic (degree 2), monic (leading coefficient = 1) and has a constant term equal to -2.

$\therefore P(x) = x^2 + bx - 2$

Now, $P(2) = 2^2 + b \times 2 - 2$

$\therefore 8 = 2b + 2$

$\therefore b = 3$

$\therefore P(x) = x^2 + 3x - 2$

Exercise 7:01

1 Which expression in each set is *not* a polynomial? (A, B or C)

	A	B	C
a	$x^2 + 7x - 3$	$5x + 10$	$3 - \frac{1}{x}$
b	$x\sqrt{x} + 4x$	$5x^2 - 7x^3$	$7x$
c	5	$2^x + 1$	$-2x^2$
d	$x^2 + x^{-2}$	$4x^3 - x + 3$	$9 - x^2 + x^4$
e	$9x - x^3 + x^5$	$9 - x^4$	$x^5 + 2x^{\frac{1}{2}}$

2 Which of the following are monic polynomials?

a $2x^2 - x + 7$
b $x^2 - 3x - 1$
c $x^5 - x^3 - 1$
d $9x^3 - 4x^4$
e $x^5 + 2x^6$
f $6 - x - x^2$
g $5 + x^3 + x^6$
h $x^7 + 3x^4 + x$
i $4x^2 + 2x^3 + x^4$

3 For each polynomial, state the degree, the leading coefficient and the constant term.

a $2x^3 + x^2 - x + 3$
b $x^5 - x^3 + x - 2$
c $9x^4 + x^2$
d $5 - 2x$
e $7x^6$
f $5 - 2x + x^2$
g $9 - x^2 + x^4 - x^6$
h 3
i $x^3 - 2x^5$
j $\sqrt{3}x^2 + \sqrt{2}x + 1$
k $5x^3 + \frac{3x^4}{2}$
l $\frac{x^4}{3} + \frac{x^2}{2} + \frac{1}{5}$

4 If $P(x) = x^2 + x - 3$, then $P(2)$ is found by substituting 2 for x. So the value of $P(2) = 3$. Similarly, find the value of $P(2)$ if:

a $P(x) = 3x^2 - x + 5$
b $P(x) = 5 - 3x + x^2$
c $P(x) = x^3 - x^2 + x - 2$
d $P(x) = \frac{x^4}{2} - \frac{x^2}{4} + 1$

5 For the polynomial functions below, determine the values indicated.

a $P(x) = 2x^2 - 3x + 4$ i $P(1)$ ii $P(-2)$ iii $P(0)$
b $P(x) = x^3 - 2x^2 + x - 3$ i $P(0)$ ii $P(10)$ iii $P(-2)$
c $P(x) = 6 - x^3 + x^6$ i $P(2)$ ii $P(5)$ iii $P(-1)$

6 Expand and simplify the following polynomials and state the degree, leading term and constant term for each.

a $(2x + 3)^2$
b $(x + 2)^2 - (x^2 + x + 1)$
c $(x + 3)^2 - (x - 3)^2$
d $2x(x^3 - x^2 + 1) + x^2(x^2 + 2x - 1)$
e $(3x - 6x^2)^2$
f $(x + 7)(2x - 3) - (x + 3)(2x - 7)$
g $(x + 1)(x + 2)(x + 3)$
h $(4x + 1)(3x - 1)^2$

7 a $P(x)$ is a quadratic polynomial with two terms. It is known that $P(0) = 3$ and $P(1) = 5$. What is the polynomial?

b $P(x)$ is a cubic polynomial with a constant term of zero. If all its coefficients are equal and $P(2) = 28$, what is the polynomial?

c A monic cubic polynomial $P(x)$, has a constant term equal to 4. If the polynomial only has three terms and $P(2) = 14$, give the possible solutions for $P(x)$.

7:02 | Sum and Difference of Polynomials

prep quiz 7:02

A: $3x^2 - x + 5$ **B:** $x^3 - 2x$ **C:** $6 - x^2 + x^4$

What is the degree of: **1** A? **2** B? **3** C?

What is the leading coefficient of: **4** A? **5** B? **6** C?

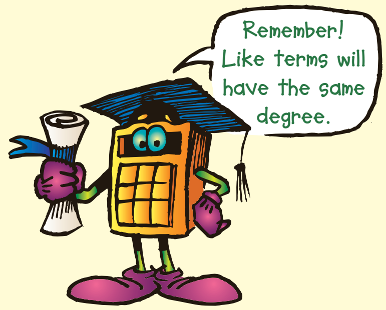
What is the constant term of: **7** A? **8** B? **9** C?

10 Which polynomials above are monic?

When adding or subtracting polynomials, simply collect like terms.

If we have two polynomials $P(x)$ and $Q(x)$, then the sum $P(x) + Q(x)$ will also be a polynomial and the difference $P(x) - Q(x)$ will likewise be a polynomial.

If many terms are involved, the following setting out may prove helpful.



worked examples

1 If $P(x) = 5x^4 + 2x^3 - x + 7$ and $Q(x) = 2x^4 - x^3 + 3x^2 - 1$, find:

a $P(x) + Q(x)$ **b** $P(x) - Q(x)$

2 If $A(x) = x^3 - x^2 + x - 2$ and $B(x) = x^3 + x^2 - x + 1$, determine the degree and leading term of:

a $A(x) + B(x)$ **b** $A(x) - B(x)$

Solutions

1 **a**

$$\begin{array}{rrrrr} 5x^4 + & 2x^3 & & -x & +7 \\ 2x^4 - & x^3 & +3x^2 & & -1 \\ \hline 7x^4 + & x^3 & +3x^2 & -x & +6 \end{array} \; +$$

b

$$\begin{array}{rrrrr} 5x^4 + & 2x^3 & & -x & +7 \\ 2x^4 - & x^3 & +3x^2 & & -1 \\ \hline 3x^4 + & 3x^3 & -3x^2 & -x & +8 \end{array} \; -$$

$\therefore\ P(x) + Q(x) = 7x^4 + x^3 + 3x^2 - x + 6$; $P(x) - Q(x) = 3x^4 + 3x^3 - 3x^2 - x + 8$

2 Of course, the process shown in example **1** can be carefully done mentally.

a $A(x) + B(x) = (x^3 - x^2 + x - 2) + (x^3 + x^2 - x + 1)$
$= 2x^3 - 1$

so the degree of $A(x) + B(x)$ is 3 and the leading term is $2x^3$.

b $A(x) - B(x) = (x^3 - x^2 + x - 2) - (x^3 + x^2 - x + 1)$
$= x^3 - x^2 + x - 2 - x^3 - x^2 + x - 1)$
$= -2x^2 + 2x - 3$

so the degree of $A(x) - B(x)$ is 2 and the leading term is $-2x^2$.

Exercise 7:02

1 If $P(x) = x^3 + 2x^2 - 4x + 1$, $Q(x) = x^2 + 3x - 2$ and $R(x) = x^3 + 3x - 1$, determine each of the following.

a $P(x) + Q(x)$ **b** $P(x) + R(x)$ **c** $Q(x) + R(x)$
d $P(x) - Q(x)$ **e** $P(x) - R(x)$ **f** $Q(x) - R(x)$
g $R(x) - P(x)$ **h** $P(x) + Q(x) + R(x)$ **i** $P(x) + Q(x) - R(x)$
j $P(x) - Q(x) + R(x)$ **k** $P(x) - Q(x) - R(x)$ **l** $R(x) + Q(x) - P(x)$

2 Simplify the following.

a $(5x^2 - 2x + 7) + (x^3 - 5x^2 - x + 3)$
b $(x^3 - x^2 + x + 1) + (x^3 + x^2 - x - 1)$
c $(2x^4 + x^2 - 1) + (x^4 - x^3 + x + 1)$
d $(x^4 - 2x^3 + 3x^2 - 4x + 5) + (5x^3 - x^2 + 7x - 3)$
e $(2x^3 + 3x^2 - 5x + 1) + (6 - 2x + 3x^2 - x^3)$
f $(4x^2 - x + 3) - (3x^2 + x + 1)$
g $(2x^3 + x^2 + 5x - 7) - (x^3 - 2x^2 + 5x + 4)$
h $(9x^4 + x^3 + 2x - 3) - (5x^4 + 7x^2 - 2x + 3)$
i $(5x^2 + 7x + 1) + (2x^2 + x - 3) + (x^2 - 10x + 7)$
j $(x^3 - x + 3) + (x^2 - 3x + 4) + (2x^3 - x^2 + 5)$
k $(x^4 + x^3 + x^2 - x - 1) + (2x^3 - x + 5) - (x^4 + 2x^2 - 7)$
l $(4x^5 + x^3 - 2x^2 + 7x) - (x^4 + 2x^3 - 7) + (2x^5 - x^4 + 3x^2 + 5)$

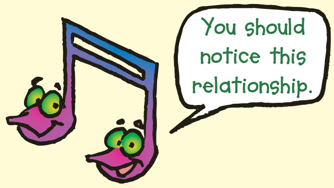
3 For each of the following, state the degree of $A(x) + B(x)$ and its leading term.

a $A(x) = x^3 + x^2 - x + 1$ $B(x) = x^2 - 2x + 7$
b $A(x) = 2x^4 + 1$ $B(x) = 2x^4 + 3x^2 - 7$
c $A(x) = 3x^3 - 2x^2 + x$ $B(x) = 5x^4 - 3x^3 + 2x^2 + 7$
d $A(x) = 7x + 1$ $B(x) = 3x^2 + 7x + 1$
e $A(x) = 2x^5 - x^3 + x$ $B(x) = 4x^3 + 2x - 1$
f $A(x) = 7 + 3x - x^2$ $B(x) = x^2 - 2x - 7$

What can be said about the degree of $A(x)$, $B(x)$ and $A(x) + B(x)$?

4 For each pair of polynomials $A(x)$ and $B(x)$ in question **3**, determine the degree of $A(x) - B(x)$. What can be said about the degree of $A(x)$, $B(x)$ and $A(x) - B(x)$?

5 **a** $P(x) = x^3 + x^2 - x + 1$ and $Q(x) = x^2 + 2x + 3$.
Determine $A(x)$ and $B(x)$ if $A(x) = P(x) + Q(x)$ and $B(x) = P(x) - Q(x)$.
b Evaluate $A(2)$, $B(2)$, $P(2)$ and $Q(2)$.
c Evaluate $A(-1)$, $B(-1)$, $P(-1)$ and $Q(-1)$.



If $A(x) = P(x) \pm Q(x)$ then for any x value, a, $A(a) = P(a) \pm Q(a)$.

7:03 | Multiplying and Dividing Polynomials by Linear Expressions

A linear expression is of the form $ax + b$, where a and b are constants. It is a polynomial of degree 1.

Multiplication

To multiply polynomials by a linear expression, we extend the procedure used in binomial expansions.

worked example

Expand and simplify $(x + 5)(x^3 + 2x^2 - x + 1)$.

Solution

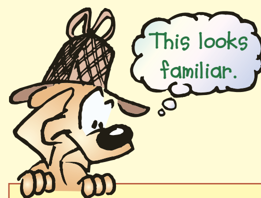
$$\begin{aligned}(x + 5)(x^3 + 2x^2 - x + 1) &= x(x^3 + 2x^2 - x + 1) + 5(x^3 + 2x^2 - x + 1)\\ &= x^4 + 2x^3 - x^2 + x + 5x^3 + 10x^2 - 5x + 5\\ &= x^4 + 7x^3 + 9x^2 - 4x + 5\end{aligned}$$

Division

When dividing a polynomial by another polynomial, a process is used that is similar to that known as 'long division' for integers.

This is shown to the right for the case 9736 ÷ 27.

9736 is called the *dividend*
27 is called the *divisor*
360 is called the *quotient*
16 is called the *remainder*
Note that:
Dividend = quotient × divisor + remainder
These terms are also used in polynomial division



$$\begin{array}{r} 360 \\ 27\overline{)\,9736} \\ \underline{81} \\ 163 \\ \underline{162} \\ 16 \\ \underline{0} \\ 16 \end{array}$$

$\therefore$ 9736 = 360 × 27 + 16

worked example

Find the quotient and remainder when $2x^2 + 5x - 7$ is divided by $x + 2$.

Solution

- At each step, we simply divide leading terms. $2x^2 \div x = 2x$ → $2x + 1$
- Then multiply $(x + 2)$ by $2x$ and subtract. → $2x^2 + 4x$ → $x - 7$
- Now, repeat the above procedure, this time dividing $(x - 7)$ by $(x + 2)$.

$$\begin{array}{r} 2x + 1 \\ x + 2\overline{)\,2x^2 + 5x - 7} \\ \underline{2x^2 + 4x} \\ x - 7 \\ \underline{x + 2} \\ -9 \end{array}$$

Bring down next term.

So the quotient for this division is $2x + 1$ and the remainder is -9.

Note that:

- the procedure is an iterative one: divide, multiply, subtract, divide, multiply, subtract, . . . until the division process can no longer be carried out.
- The degree of the remainder must always be less than the degree of the divisor.
- If one polynomial is exactly divisible by another, the remainder will be zero. Thus, the divisor and quotient will be factors of the dividend.

worked examples

$$
\begin{array}{r|l}
 & x^2 + 10x + 49 \\
\hline
x - 5 & x^3 + 5x^2 - x + 7 \\
 & x^3 - 5x^2 \\
 & \quad 10x^2 - x \\
 & \quad 10x^2 - 50x \\
 & \qquad 49x + 7 \\
 & \qquad 49x - 245 \\
 & \qquad\quad 252
\end{array}
$$

- Divide x^3 by x; write x^2 on top.
- Multiply $(x - 5)$ by x^2 and subtract.
- Divide $10x^2$ by x; write $10x$ on top.
- Multiply $(x - 5)$ by $10x$ and subtract.
- Divide $49x$ by x; write 49 on top.
- Multiply $(x - 5)$ by 49 and subtract.

The result of this division may be written in the following way.

$x^3 + 5x^2 - x + 7 = (x - 5)(x^2 + 10x + 49) + 252$

dividend = divisor × quotient + remainder

Exercise 7:03

1 Expand and simplify the following.

a $(x + 2)(x^2 + 3x + 1)$ **b** $(x + 4)(x^2 - 2x + 3)$ **c** $(x - 2)(x^2 + 5x - 4)$

d $(2x - 1)(x^2 - 2x + 5)$ **e** $(3x - 2)(x^3 - x + 1)$ **f** $(5 - x)(x^3 - 2x^2 + 1)$

2 By doing a multiplication, check that the following are true.

a $\dfrac{x^2 - 2x - 3}{x + 1} = x - 3$ **b** $\dfrac{x^3 - 3x + 2}{x + 2} = x^2 - 2x + 1$

3 **a** If $P(x)$ is of degree 3, what is the degree of $(x - 3) \times P(x)$?

b If $P(x)$ is of degree 3, what is the degree of $Q(x)$ if $Q(x) = \dfrac{P(x)}{x - 3}$?

4 **a** If $P(x) = x^3 + 2x^2 + x - 1$ and $Q(x) = x + 4$, find $R(x)$ if $R(x) = P(x) \,.\, Q(x)$.

b Evaluate $P(1)$, $Q(1)$ and $R(1)$.

c Evaluate $P(-2)$, $Q(-2)$ and $R(-2)$.

d What relationship exists between $P(a)$, $Q(a)$ and $R(a)$?

> A dot is sometimes used for *multiplication*.

5 Complete the following divisions, expressing each result in the form dividend = divisor × quotient + remainder.

a $(x^2 + 2x - 3) \div (x + 1)$ **b** $(x^2 - 5x + 1) \div (x - 3)$

c $(3x^2 + x - 2) \div (x + 3)$ **d** $(2x^2 - 4x + 3) \div (x - 5)$

e $(x^3 + 2x^2 + x - 7) \div (x + 1)$ **f** $(3x^3 - 2x + 6) \div (x - 3)$

g $(2x^3 - x^2 + x - 1) \div (x - 2)$ **h** $(3x^3 + 2x - 5) \div (x - 1)$

6 Determine the quotient and remainder for the following divisions.

a $(6x^2 - x + 2) \div (x - 3)$
b $(x^3 - 2x^2 + x + 1) \div (x + 2)$
c $(2x^3 - x^2 + 3x + 3) \div (2x + 1)$
d $(6x^3 - 5x^2 - 8x + 3) \div (2x - 3)$
e $(x^5 - x^4 + 2x^3 - x^2 + x + 3) \div (x + 1)$
f $(2x^6 + 2x^4 - 3x^2 + 4) \div (x - 1)$

7 Show that the remainder is zero in each of the following, and hence write the dividend as the product of two factors.

a $(x^3 + 2x^2 + 3x + 2) \div (x + 1)$
b $(2x^3 - 3x^2 + 4x + 3) \div (2x + 1)$
c $(2x^3 - 3x^2 + 1) \div (x - 1)$
d $(2x^3 + x^2 + 1) \div (x + 1)$

7:04 | Remainder and Factor Theorems

- In the last section, it was seen that if $P(x)$ is divided by $(x - a)$, the result can be expressed in the form:
  $P(x) = (x - a)\,Q(x) + r$
  where $Q(x)$ is the quotient and r is the remainder.
  Now, if we let x equal a in this statement, it becomes:
  $$P(a) = (a - a)\,Q(a) + r$$
  $$= 0 \times Q(a) + r$$
  so, $P(a) = r$

> $Q(x)$ ← quotient
> $x - a\,)\,P(x)$ ← dividend
> divisor ↑
> ⋮
> r ← remainder
> $P(x) = (x - a)\,Q(x) + r$

- This means that if a given polynomial $P(x)$ is divided by $(x - a)$, then the remainder will be $P(a)$, ie the value of $P(x)$ when a is substituted for x. This is called the **remainder theorem**.

Remainder theorem: **If a given polynomial $P(x)$ is divided by $(x - a)$, then the remainder is $P(a)$.**

- From the remainder theorem, it can be seen that if $r = 0$ then $P(x) = (x - a)\,Q(x)$. This means that $(x - a)$ is a factor of $P(x)$, and leads to the **factor theorem**.

Factor theorem:
$(x - a)$ divides $P(x)$ if and only if $P(a) = 0$.

> *Note:* 'if and only if' means that the statement is also true in reverse, that is:
> If $(x - a)$ divides $P(x)$, then $P(a) = 0$
> and
> If $P(a) = 0$, then $(x - a)$ divides $P(x)$.

worked examples

1 Determine the remainder when $x^3 - 2x^2 + x - 1$ is divided by $x - 2$. Is $x - 2$ a factor of $x^3 - 2x^2 + x - 1$?

2 Show that $x - 3$ is a factor of $P(x) = x^3 - 2x^2 - 5x + 6$, and hence express $P(x)$ as a product of its factors.

3 If $f(x) = x^3 + ax + b$ is divisible by both $x + 2$ and $x - 3$, find the values of a and b.

Solutions

1 Divisor is $x - 2$, so remainder $r = P(2)$.

$$\begin{aligned}
\text{Let}\quad P(x) &= x^3 - 2x^2 + x - 1\\
\therefore\quad P(2) &= 2^3 - 2(2)^2 + 2 - 1\\
&= 8 - 8 + 2 - 1\\
&= 1
\end{aligned}$$

$\therefore$ Remainder = 1

$x - 2$ is not a factor since the remainder is not zero.

2 If $(x - 3)$ is a factor of $P(x)$, then $P(3)$ will equal zero.

$$\begin{aligned}
\text{Now}\quad P(3) &= (3)^3 - 2(3)^2 - 5(3) + 6\\
&= 27 - 18 - 15 + 6\\
&= 0
\end{aligned}$$

$\therefore$ $(x - 3)$ must be a factor of $P(x)$.

If $P(x)$ is now divided by $x - 3$, this will enable the other factors to be found.

$$\begin{array}{r l}
 & x^2 + x - 2\\
x - 3 \,\big)\!\!\! & \overline{x^3 - 2x^2 - 5x + 6}\\
 & \underline{x^3 - 3x^2}\\
 & \quad x^2 - 5x\\
 & \quad \underline{x^2 - 3x}\\
 & \qquad -2x + 6\\
 & \qquad \underline{-2x + 6}\\
 & \qquad\qquad 0
\end{array}$$

$\therefore$ $P(x) = (x - 3)(x^2 + x - 2)$

Further factorising $x^2 + x - 2$ gives:

$\therefore$ $P(x) = (x - 3)(x - 1)(x + 2)$

3 If $f(x) = x^3 + ax + b$ is divisible by $x + 2$ and $x - 3$,

$$\begin{array}{r l r}
\text{then}\quad f(-2) = 0 & \text{and} & f(3) = 0.\\
(-2)^3 + a(-2) + b = 0 & & (3)^3 + a(3) + b = 0\\
-8 - 2a + b = 0 & & 27 + 3a + b = 0\\
2a - b = -8 & & 3a + b = -27
\end{array}$$

Solving simultaneously:

$$\begin{aligned}
2a - b &= -8 \quad +\\
3a + b &= -27\\
\hline
5a \quad &= -35\\
\therefore\quad a &= -7\\
b &= -6
\end{aligned}$$

$\therefore$ The values of a and b are $a = -7$, $b = -6$.
($\therefore$ $f(x) = x^3 - 7x - 6$)

Exercise 7:04

1 Use the remainder theorem to find the remainder for the following divisions. Check your answer by carrying out the division.

a $(x^2 + 7x - 5) \div (x - 1)$
b $(x^2 - 3x - 10) \div (x + 2)$
c $(2x^2 + 3x + 7) \div (x + 1)$
d $(5x^2 + 4x - 10) \div (x - 2)$
e $(x^3 - 3x^2 + 7x - 5) \div (x - 3)$
f $(x^4 + 5x^2 - 6) \div (x - 1)$

2 Find the remainder for each division when $P(x)$ is divided by $A(x)$.

a $P(x) = x^2 + 5x + 7$, $A(x) = x - 1$ **b** $P(x) = x^2 - 2x - 9$, $A(x) = x + 4$
c $P(x) = 2x^2 + x - 11$, $A(x) = x - 3$ **d** $P(x) = 3x^2 - 4x + 1$, $A(x) = x + 3$
e $P(x) = 2x^3 + 7x - 13$, $A(x) = x - 1$ **f** $P(x) = 5x^3 + x^2 - 4x$, $A(x) = x - 2$
g $P(x) = x^4 - x^2 + 1$, $A(x) = x - 2$ **h** $P(x) = x^5 - x^3 + 4x^2 + 7$, $A(x) = x + 2$

3 Determine the remainder when $x^4 + 2x^3 - 5x^2 - 7x + 2$ is divided by:

a $(x + 1)$ **b** $(x - 1)$ **c** $(x - 3)$ **d** $(x + 3)$

4 **a** If $x^3 - 4x^2 - 7x + k$ is divided by $(x - 4)$, the remainder is 2. Find the value of k.
b When $x^4 - 3kx^2 + 5k$ is divided by $(x + 1)$, the remainder is 5. Find the value of k.
c When $x^2 + x - 1$ is divided by $(x - k)$, the remainder is 5. Find all possible values of k.

5 Find whether each polynomial $P(x)$ has the linear polynomial indicated as a factor.

a $P(x) = x^2 + 7x - 18;\ x + 9$ **b** $P(x) = 5x^2 - 9x - 2;\ x + 2$
c $P(x) = x^3 + x - 2;\ x + 1$ **d** $P(x) = x^3 - 2x^2 + 1;\ x - 1$
e $P(x) = 2x^3 + 3x^2 - 1;\ x + 1$ **f** $P(x) = x^3 - x^2 - 10x - 8;\ x + 2$

6 Show that the first polynomial is a factor of $P(x)$ and then determine all the factors of $P(x)$.

a $x - 1$, $P(x) = x^3 + 4x^2 + x - 6$ **b** $x + 1$, $P(x) = x^3 - 6x^2 + 5x + 12$
c $x + 1$, $P(x) = x^3 + 4x^2 + 5x + 2$ **d** $x - 2$, $P(x) = x^3 - 5x^2 + 8x - 4$

7 Determine the factors of the following polynomials.

a $x^3 - 2x^2 - 5x + 6$ **b** $x^3 - 6x^2 - 13x + 42$ **c** $x^3 - 4x^2 + x + 6$
d $x^3 - 21x + 20$ **e** $x^3 + 4x^2 - 15x - 18$ **f** $2 + 3x - 14x^2 - 15x^3$

8 For what value of k is:

a $x - 1$ a factor of $x^2 - 5x + k$? **b** $x + 2$ a factor of $2x^2 - x + k$?
c $x + 1$ a factor of $x^2 + kx - 7$? **d** $x - 4$ a factor of $3x^2 - kx - 32$?

9 If $(x + 1)$ and $(x + 2)$ are both factors of $x^3 + ax^2 + bx - 10$, find the values of a and b.

10 If $x^3 + px + q$ is divisible by both $x - 3$ and $x + 2$, find the values of p and q.

7:05 | Solving Polynomial Equations

- A polynomial $P(x)$ can take many values. The values of x which make $P(x)$ zero are called, appropriately enough, **zeros**. These zeros must also be the solutions to the equation $P(x) = 0$. As the solutions to equations are often called the roots of the equation, the words *zero* and *root* are almost synonymous.
- The factor theorem is the basis for solving equations of the form $P(x) = 0$.

The following example shows the steps involved in solving equations of this type.

worked example

Solve the equation $x^3 - x^2 - 10x - 8 = 0$.

Solution

Step 1 *Find a value of x for which P(x) = 0*

Let $P(x) = x^3 - x^2 - 10x - 8 = 0$

The values of x to try are the factors of the constant term, -8, namely $\{1, -1, 2, -2, 4, -4, 8, -8\}$.

Starting with the smallest value, we use trial and error to find one that makes $P(x)$ equal zero.

Now $\quad P(1) = 1 - 1 - 10 - 8 \neq 0$

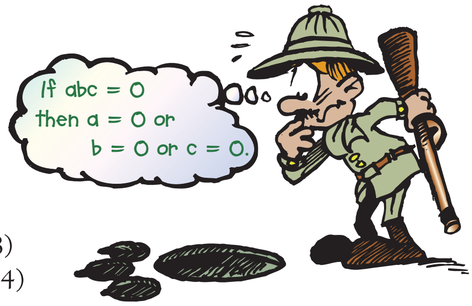
but $\quad P(-1) = -1 - 1 + 10 - 8 = 0$.

Step 2 *Use the factor theorem.*

Since $P(-1) = 0$, then $x + 1$ is a factor.

Step 3 *Divide P(x) by the factor.*

$$\begin{array}{r l}
 & x^2 - 2x - 8 \\
x + 1 \,\big) & x^3 - x^2 - 10x - 8 \\
 & \underline{x^3 + x^2} \\
 & \quad -2x^2 - 10x \\
 & \quad \underline{-2x^2 - \;\, 2x} \\
 & \qquad\qquad -8x - 8 \\
 & \qquad\qquad \underline{-8x - 8} \\
 & \qquad\qquad\qquad 0
\end{array}$$



Step 4 *Write P(x) as a product of its factors.*

$$\begin{aligned} x^3 - x^2 - 10x - 8 &= (x + 1)(x^2 + 2x - 8) \\ &= (x + 1)(x + 2)(x - 4) \end{aligned}$$

Step 5 *Write down the solutions.*

$$x^3 - x^2 - 10x - 8 = 0$$

$$\therefore\ (x + 1)(x + 2)(x - 4) = 0$$

$$\therefore\ x = -1, -2 \text{ or } 4$$

Exercise 7:05

1 The first step in solving the equation $x^3 - 2x^2 - 11x + 12 = 0$ is to find a value of x that solves the equation. What values of x should be tried?

2 You have found that $x = 3$ is a solution to the equation $x^3 - 2x^2 - 11x + 12 = 0$. How do you use this to factorise $x^3 - 2x^2 - 11x + 12$?

3 Factorise:

a $x^2 - x - 12$ **b** $2x^2 - 5x + 2$

4 Solve:

a $(x - 3)(x - 2)(x + 1) = 0$ **b** $(x + 1)(x - 4)^2 = 0$

c $(2x - 1)(x + 4)(x - 1)(x + 2) = 0$ **d** $(x - 2)^2(x + 3)^2 = 0$

5 Factorise completely and solve:

a $(x - 2)(x^2 - 2x - 8) = 0$ **b** $(p + 3)(2p^2 + 5p - 3) = 0$

c $(x - 1)(x^3 + 4x^2 + x - 6) = 0$ **d** $(p + 2)(2p^3 - p^2 - 13p - 6) = 0$

6 These questions refer to the equation $2x^4 - 7x^3 - 2x^2 + 13x + 6 = 0$.
- **a** Show that $x = -1$ is a solution.
- **b** If $Q(x) = (2x^4 - 7x^3 - 2x^2 + 13x + 6) \div (x + 1)$, find $Q(x)$.
- **c** Show that $x - 2$ is a factor of $Q(x)$.
- **d** Write $2x^4 - 7x^3 - 2x^2 + 13x + 6$ as a product of its linear factors.
- **e** Using your answer to **d**, write down the roots of the equation $2x^4 - 7x^3 - 2x^2 + 13x + 6 = 0$.

7 Find all real roots of these equations.
- **a** $x^3 - 4x^2 + x + 6 = 0$
- **b** $x^3 + 3x^2 + 7x + 10 = 0$
- **c** $x^3 - x^2 - x + 1 = 0$
- **d** $x^3 + 2x^2 - 9x - 18 = 0$
- **e** $x^3 - 3x^2 + 3x - 1 = 0$
- **f** $x^3 + 2x^2 + 4x + 21 = 0$

8
- **a** Solve the equation $x^4 - 4x^3 - 7x^2 + 22x + 24 = 0$.
- **b** Solve the equation $y^2(y - 3) = 6y - 8$.
- **c** A box is to have a square base and a height that is 10 cm longer than the length of the base. If the volume of the box is to be 2000 cm^3, write a polynomial equation to represent this information and, by solving the equation, find the dimensions of the box.

7:06 | Sketching Polynomials

prep quiz 7:06

What is the degree of the polynomial:

1 $2x - 3$?
2 $x^2 + 5x - 6$
3 $4x^3 - 10$?

4 Expand $x(x - 1)(x + 2)$.

5 What are the zeros of the polynomial $x(x - 1)(x + 2)$?

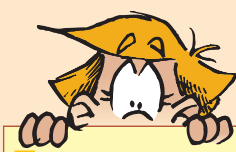
6 Where will the curve $y = x(x - 1)(x + 2)$ cut the x-axis?

7 What is the y-intercept of the curve $y = x(x - 1)(x + 2)$?

8 Is $x - 1$ a factor of $x^3 + 2x^2 - 3$?

9 If $x = 1\,000\,000$, which term of the polynomial $x^4 + 100x^3 + 1000x^2 + 1\,000\,000$ has the greatest value?

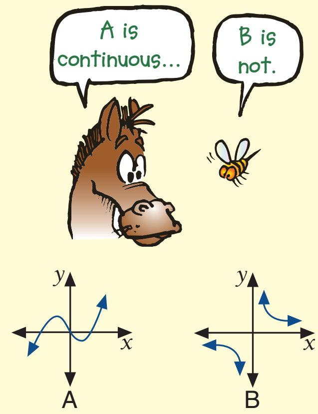
10 For the polynomial $y = x^3 - 2x^2 + x - 6$, which term will have the biggest influence on the value of y when x is large?



Polynomials
Linear: degree 1
Quadratic: degree 2
Cubic: degree 3

- The definition of the polynomial in section 7:01 reminds us that linear, quadratic and cubic expressions are in fact all polynomials. Hence, we already know how to sketch a considerable number of polynomials.
  All of the curves sketched in Exercises 6:01 to 6:03 were polynomials and all of the techniques learnt there are applicable in this section of work.
- In this section, we will consider only the graphs of polynomials that can be factorised.

- Polynomials are examples of *continuous functions*. This means they have no gaps. Polynomials can have any x value (from negative infinity $[-\infty]$ to positive infinity $[+\infty]$) and every x value has a corresponding y value.
- Questions **9** and **10** in the Prep Quiz illustrate that as x becomes very large or very small (ie as x approaches positive or negative infinity), the sign of $P(x)$ is the same as the sign of the leading term, ax^n. In symbols, we write:



For the polynomial, $y = P(x)$, as $x \to \pm\infty$, $P(x) \to ax^n$

- If a polynomial is of degree n, it can have at most n factors,

  ie $a_nx^n + a_{n-1}x^{n-1} + \ldots + a_1x + a_0 = \underbrace{(x - \alpha_1)(x - \alpha_2)(x - \alpha_3)\ldots(x - \alpha_n)}_{n \text{ factors}}$

  This means that a polynomial of degree n can have a maximum of n zeros (or roots) and that the curve $y = P(x)$ will have a maximum of n x-intercepts.

We will now see how we use these ideas in conjunction with what we have learnt earlier in the chapter to sketch the graphs of polynomials.

worked examples

Sketch the graphs of the following polynomials.

1 $y = x(x + 1)(x - 2)(x + 3)$

2 $y = (x - 2)^2(x + 2)^2$

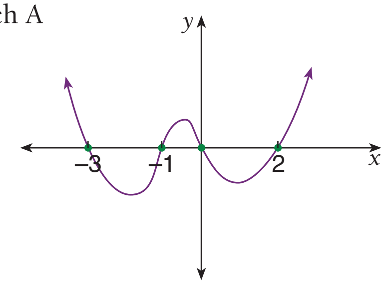
3 $y = -(x + 1)(2x - 5)^3$

4 $y = -x^4 + 2x^3 + x^2 - 2x$

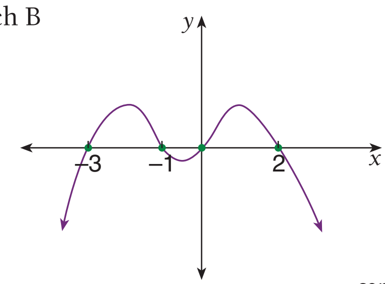
Solutions

1 $y = x(x + 1)(x - 2)(x + 3)$
- When $y = 0$,
  $x(x + 1)(x - 2)(x + 3) = 0$
  $\therefore\ x = 0, -1, 2, -3$
  The curve has 4 x-intercepts.
- As the curve is continuous, there are only two possible shapes, ie A or B.

Sketch A



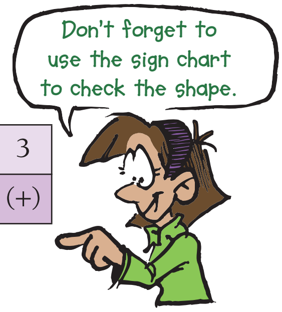
Sketch B



continued →→→

- If the polynomial were expanded it would be of degree 4, with a leading term of x^4. Hence, as $x \to \pm\infty$ it will become like x^4. Hence, sketch A is the correct one.

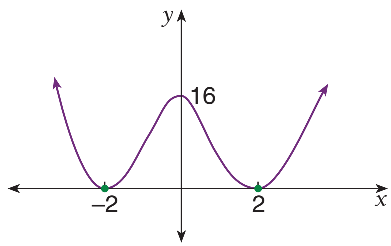
x	-4	-3	-2	-1	$-\frac{1}{2}$	0	1	2	3
y	(+)	0	(–)	0	(+)	0	(–)	0	(+)



2 $y = (x - 2)^2(x + 2)^2$

- If this were expanded, it would be of degree 4 with leading term x^4.
- It has only two x-intercepts, namely at $x = 2$ and -2. Each of these is said to be a double root.
- A sign analysis, the y-intercept and analysis of the leading term as $x \to \pm\infty$ will confirm the shape of the curve.

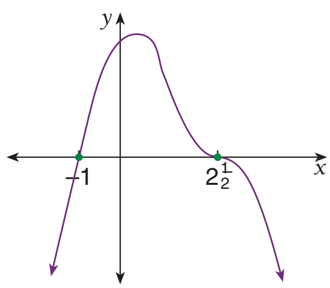
Don't forget to check the y-intercept. When $x = 0$, $y = 16$.



x	-3	-2	0	2	3
y	(+)	0	(+)	0	(+)

3 $y = -(x + 1)(2x - 5)^3$

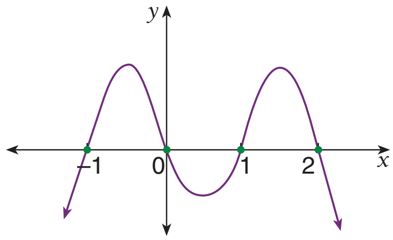
- If expanded it would be of degree 4 with leading term $-x^4$.
- It has only two x-intercepts, at -1 and $2\frac{1}{2}$. The polynomial has a single root at $x = -1$ and a triple root at $x = 2\frac{1}{2}$. The triple root has the effect of turning the curve over, as shown in the sketch.
- Sign analysis and the y-intercept will confirm the rest of the shape.



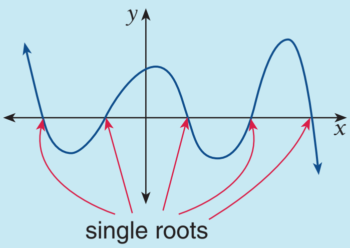
If $x = 3$,
$y = -(3 + 1)(6 - 5)^3$
$= -(4)(1)^3$
$= -4$

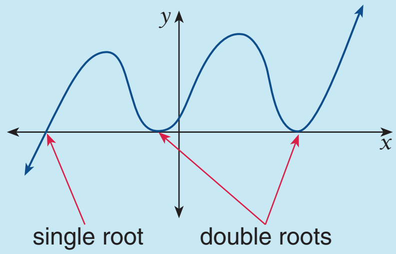
4 $y = -x^4 + 2x^3 + x^2 - 2x$

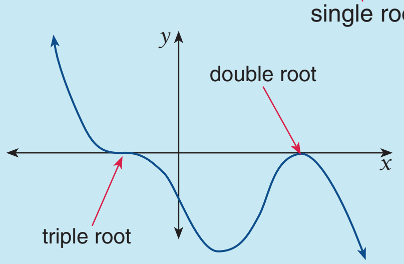
- First, we need to factorise $-x^4 + 2x^3 + x^2 - 2x$.
  $-x^4 + 2x^3 + x^2 - 2x = -x(x^3 - 2x^2 - x + 2)$
  Using the factor theorem on $(x^3 - 2x^2 - x + 2)$ shows that $x - 1$ is a factor.
  Dividing gives $\quad y = -x(x - 1)(x^2 - x - 2)$
  $\therefore\ y = -x(x - 1)(x - 2)(x + 1)$
  $\therefore$ The polynomial has 4 single roots: 0, 1, 2 and −1.
- As the leading term is $-x^4$, y behaves like $-x^4$ as $x \to \pm\infty$.
  As $x \to \pm\infty$, $y \to -\infty$.
  Hence, the sketch is as shown.



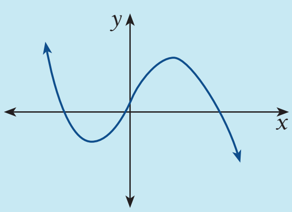
- When sketching $y = P(x)$, the nature of the roots will control the shape of the curve.

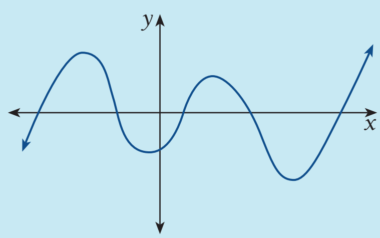




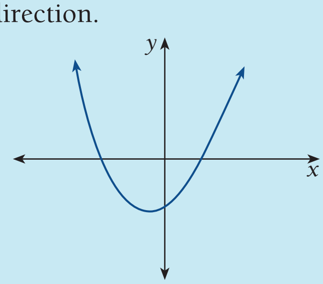


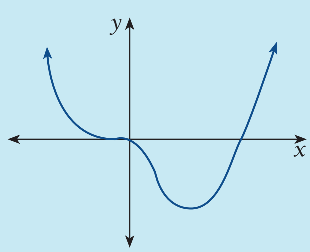
- If the degree of the polynomials is *odd*, the arrows on the ends of the curve will point in opposite directions.





- If the degree of the polynomial is *even*, the arrows on the ends of the curve will point in the same direction.





Exercise 7:06

1 The following polynomials are given in factored form. Determine their zeros and sketch their graphs, indicating clearly where they cut the x-axis.

a $y = (x + 1)(x - 3)$
b $y = 2x(x - 5)$
c $y = x(x + 1)(x - 1)$
d $y = (x - 2)(x - 1)(x + 1)$
e $y = (x + 4)(x + 1)(x - 2)$
f $y = x(x - 3)(x + 5)$
g $y = -x(x + 2)(x - 1)$
h $y = (3 - x)(x + 1)(x + 2)$
i $y = (x - 2)(x + 1)(1 - x)$
j $y = (2 - x)(x + 1)(x + 4)$
k $y = x(x + 2)(x + 1)(x - 3)$
l $y = (x + 3)(x + 1)(x - 1)(3 - x)$

2 Each polynomial function has a repeated root. Sketch the graph of each neatly.

a $y = (x - 1)^2$
b $y = -(x + 2)^2$
c $y = x(x + 2)^2$
d $y = (x + 1)(x - 2)^2$
e $y = (x + 3)^2(x - 1)$
f $y = -x(x - 1)^2$
g $y = x^2(5 - x)$
h $y = (x + 3)^3$
i $y = -(x + 3)(x - 5)^2$
j $y = (5 - x)^3$
k $y = x(x + 1)(x - 2)^2$
l $y = (x + 1)(x - 2)^3$

3 Use the factor theorem to factorise the following and hence sketch their graphs, showing clearly the zeros of each function.

a $y = x^3 + 4x^2 + x - 6$
b $y = x^3 + 4x^2 + 5x + 2$
c $y = x^3 - 6x^2 + 5x + 12$
d $y = x^3 - 5x^2 + 8x - 4$
e $y = x^3 + 8x^2 + 17x + 10$
f $y = x^3 - 3x^2 + 3x - 1$
g $y = -x^3 + 2x^2 + 5x - 6$
h $y = -x^3 + 4x^2 - x - 6$
i $y = x^4 - 17x^2 + 16$
j $y = x^4 - x^3 - 10x^2 - 8x$

4 Write an equation that could represent each of the following and sketch the curve.

a $P(x)$ is of degree 3 and has a double root at 2 and a single root at -1. It has a negative y-intercept.
b The polynomial $y = P(x)$ has a double root at -2 and a triple root at 1. It is of degree 5 with a leading term of $-2x^5$.
c The polynomial is quartic (of degree 4). It has roots at -3, -1 and 4 and a positive y-intercept. The leading term is either x^4 or $-x^4$.
d The polynomial is quartic (of degree 4). It has roots at -3, -1 and 4 and a negative y-intercept. The leading term is either x^4 or $-x^4$.

5 Can a cubic equation have no roots? Give examples of cubic equations that have 1, 2 and 3 roots.

6 A polynomial is monic of degree 5. It has two double roots and a single root. Sketch the possible shapes of this polynomial.

Fun Spot 7:06 | How do you find a missing hairdresser?

Answer each question and put the letter for that part in the box that is above the correct answer.

For the polynomial $P(x) = x^3 - 2x^2 - x + 2$:

E What is the degree?
N What is the leading coefficient?
A What is the constant term?
O Evaluate $P(1)$.
Y What is the remainder when $P(x)$ is divided by $x + 2$?

For the circle $(x - 2)^2 + (y - 3)^2 = 16$, find:

B the radius **I** the coordinates of the centre

For the circle $(x + 2)^2 + (y + 3)^2 = 25$, find:

A the radius **E** the coordinates of the centre

M What is the y-intercept of the curve $y = x^3 + 8$?
E Find a if the curve $y = ax^3 - 5$ passes through the point (1, 10).
T Find k if the curve $y = k(x + 1)(x - 1)(x - 2)^2$ passes through (3, −16).

What would be the equation of the curve that is obtained if $y = x^3$ is translated:

H up 1 unit? **O** down 1 unit? **E** 1 unit to the right? **R** 1 unit to the left?

Solve the equations:

T $(x - 1)(x + 2)(x - 3) = 0$ **U** $(x + 1)(x - 2)(x + 3) = 0$
R $x^3 - 2x^2 - x + 2 = 0$ **C** $x^3 + 2x^2 - x - 2 = 0$

−12	$y = x^3 - 1$	−1, 2, −3	0	8	4	−8	$y = x^3 + 1$	3	$y = (x - 1)^3$	1	1, −2, 3	(2, 3)	$y = (x + 1)^3$	15	2	−1, 1, 2	(−2, −3)	5

7:07 | Sketching Curves Related to $y = P(x)$

Using $y = P(x)$ to sketch $y = P(x) + c$

Earlier in the chapter, we saw how $y = ax^n + d$ was obtained by moving the curve $y = ax^n$ vertically by d units.

The tables below show that the same results hold for the curves $y = P(x) + c$ and $y = P(x)$. Clearly, $y = P(x) + c$ is obtained by moving $y = P(x)$ vertically c units.

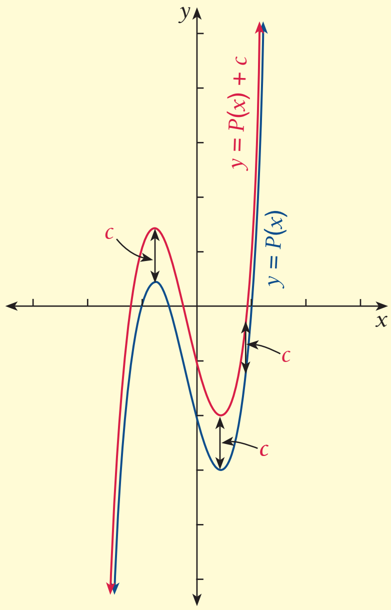
$y = P(x)$

x	−2	−1	0	1	2
y	$P(-2)$	$P(-1)$	$P(0)$	$P(1)$	$P(2)$

$y = P(x) + c$

x	−2	−1	0	1	2
y	$P(-2) + c$	$P(-1) + c$	$P(0) + c$	$P(1) + c$	$P(2) + c$

The tables show that if c is added to each of the y values in the table for $y = P(x)$, the values in the table for $y = P(x) + c$ are obtained. If $y = P(x)$ is translated vertically by c units, it produces the curve $y = P(x) + c$.



■ $y = P(x) + c$ is obtained by translating the curve $y = P(x)$ vertically by c units.
If c is positive it moves up.
If c is negative it moves down.

Using $y = P(x)$ to sketch $y = -P(x)$

$y = P(x)$

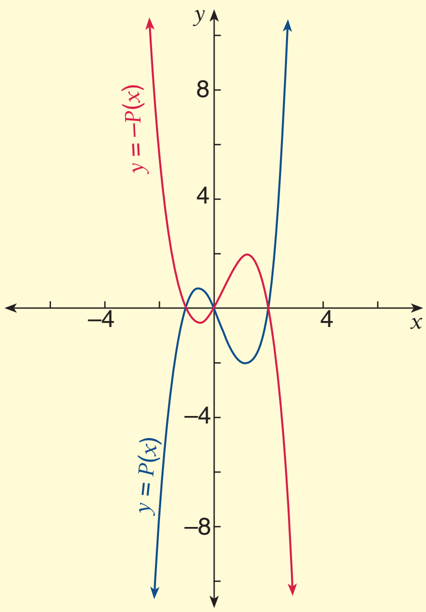
x	-2	-1	0	1	2
y	$P(-2)$	$P(-1)$	$P(0)$	$P(1)$	$P(2)$

$y = -P(x)$

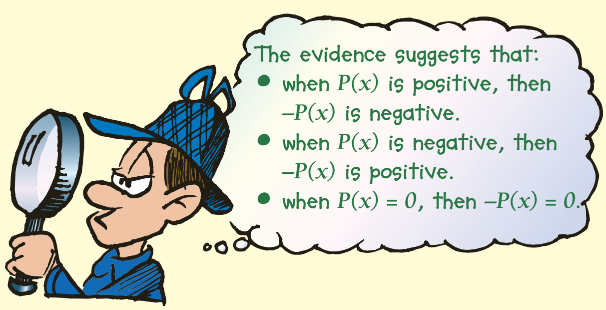
x	-2	-1	0	1	2
y	$-P(-2)$	$-P(-1)$	$-P(0)$	$-P(1)$	$-P(2)$

The tables show that the curves have the same x values but opposite y values. Hence, to obtain the curve $y = -P(x)$ from $y = P(x)$, simply replace all the y values in the table for $y = P(x)$ with their opposites.

Now, by keeping the x values the same and changing the y values to their opposites, we are in fact producing a curve that is the reflection of the first curve in the x-axis.



■ $y = P(x)$ is obtained by reflecting the curve $y = P(x)$ in the x-axis.



Using $y = P(x)$ to sketch $y = aP(x)$

$y = P(x)$

x	-2	-1	0	1	2
y	$P(-2)$	$P(-1)$	$P(0)$	$P(1)$	$P(2)$

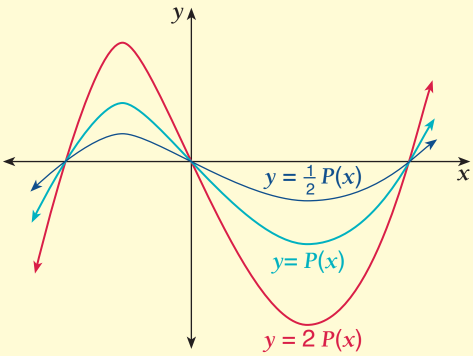
$y = aP(x)$

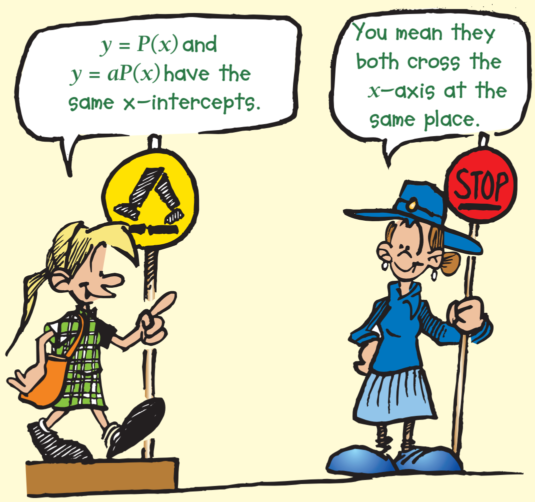
x	-2	-1	0	1	2
y	$aP(-2)$	$aP(-1)$	$aP(0)$	$aP(1)$	$aP(2)$

To obtain the values for $y = aP(x)$ from the values for $y = P(x)$, we simply multiply all the y values by a.

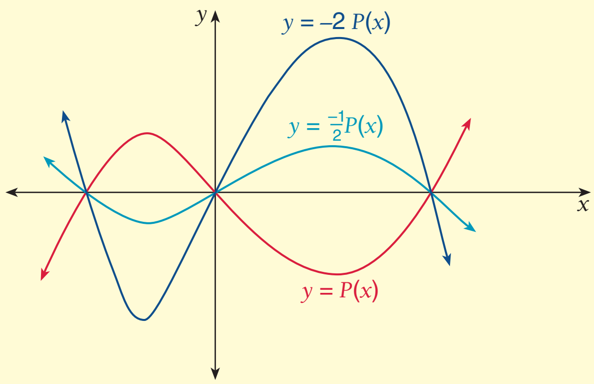
The size and sign of a are both important in determining the shape of $y = aP(x)$.

For positive a:
If $a > 1$, $y = aP(x)$ is obtained by stretching $y = P(x)$.
If $a < 1$, $y = aP(x)$ is obtained by squashing $y = P(x)$.
If $y = 0$ on $y = P(x)$, then $y = 0$ on $y = aP(x)$.





For negative a:
If $a < -1$, $y = aP(x)$ is obtained by reflecting $y = P(x)$ in the x-axis and then stretching it.
If $-1 < a < 0$, (ie a is a negative fraction), $y = aP(x)$ is obtained by reflecting $y = P(x)$ and then squashing it.



Using $y = P(x)$ to sketch $y = P(-x)$

The table for $y = P(x)$ is shown on the right.

Consider $y = P(-x)$.

As x and $-x$ are opposites, swapping each x value in Table 1 with its opposite $(-x)$ (ie -2 with 2, -1 with 1, 1 with -1 and 2 with -2) will give you the values in Table 2.

Rearranging the values in Table 2, so that the x values are in ascending order, produces Table 3, which is the table for $y = P(-x)$.

Hence, the points on $y = P(-x)$ can be obtained by moving each point on $y = P(x)$ horizontally from one side of the y-axis to an equal distance on the other side.

Geometrically, this means that $y = P(-x)$ can be obtained by reflecting $y = P(x)$ in the y-axis.

The diagram below shows this procedure.

$y = P(x)$

x	-2	-1	0	1	2
y	$P(-2)$	$P(-1)$	$P(0)$	$P(1)$	$P(2)$

Table 1

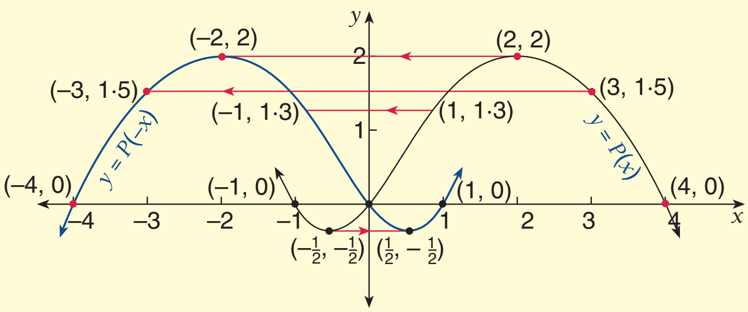
x	2	1	0	-1	-2
y	$P(-2)$	$P(-1)$	$P(0)$	$P(1)$	$P(2)$

Table 2

$y = P(-x)$

x	-2	-1	0	1	2
y	$P(2)$	$P(1)$	$P(0)$	$P(-1)$	$P(-2)$

Table 3

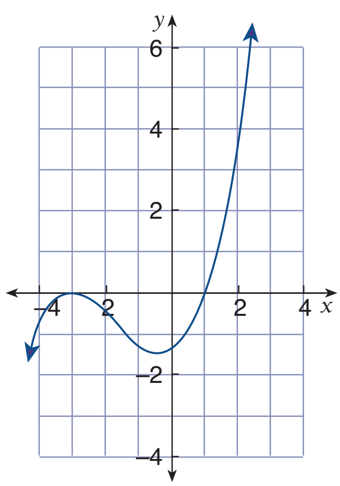


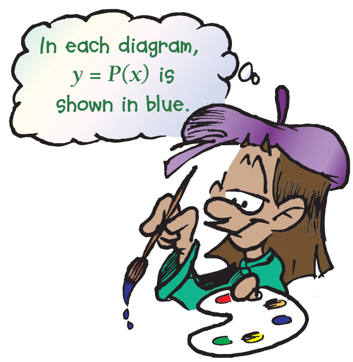
■ $y = P(-x)$ is obtained by reflecting the curve $y = P(x)$ in the y-axis.

worked examples

Use this graph of $y = P(x)$ to make sketch graphs of:

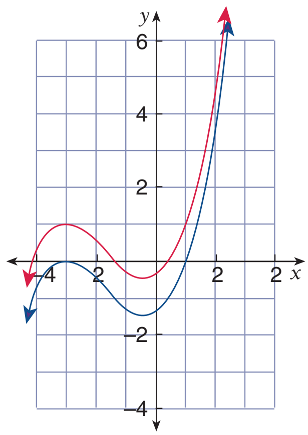
1. $y = P(x) + 1$
2. $y = P(x) - 2$
3. $y = \frac{1}{2}P(x)$
4. $y = -P(x)$
5. $y = P(-x)$
6. $y = 1 - \frac{1}{2}P(-x)$





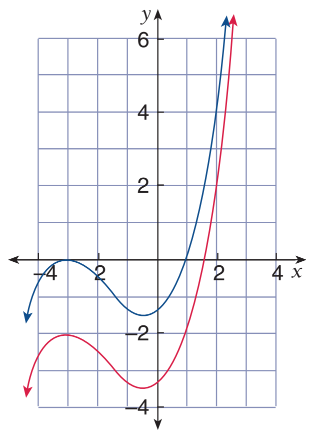
Solutions

1



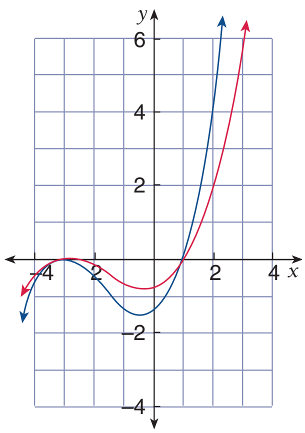
$y = P(x) + 1$ is obtained by moving each point on $y = P(x)$ up 1 unit.

2



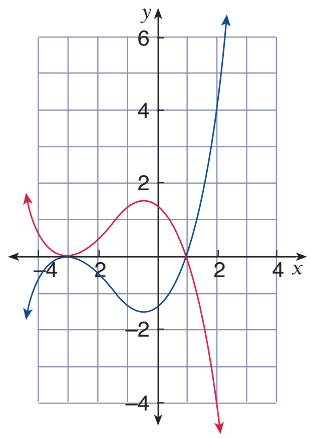
$y = P(x) - 2$ is obtained by moving each point on $y = P(x)$ down 2 units.

3



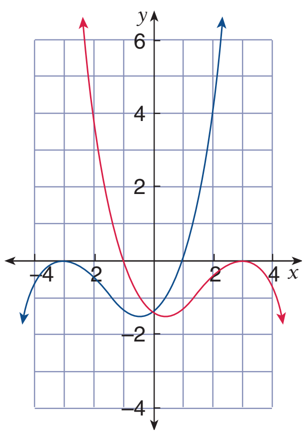
$y = \frac{1}{2}P(x)$ is obtained by halving the y-coordinate of each point on $y = P(x)$.

4



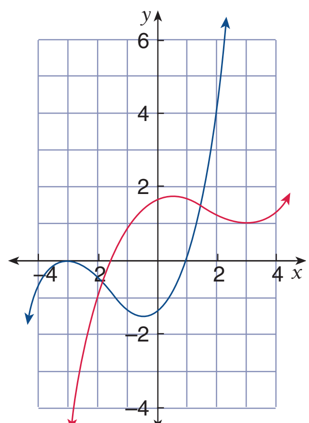
$y = -P(x)$ is the reflection of $y = P(x)$ in the x-axis.

5



$y = P(-x)$ is the reflection of $y = P(x)$ in the y-axis.

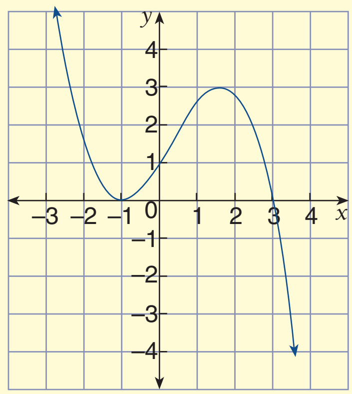
6



$y = 1 - \frac{1}{2}P(-x)$ is built from $y = P(x)$ in stages. $P(x)$ is reflected in the y-axis and then the x-axis to give $-P(-x)$. This is then halved and moved up 1 unit.

Exercise 7:07

1 Use the graph of $y = P(x)$ shown to sketch:



a $y = P(x) + 1$

b $y = P(x) - 2$

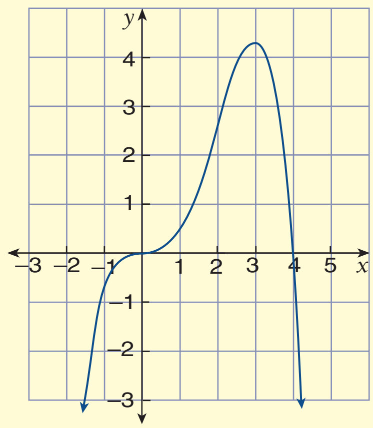
c $y = \frac{1}{2}P(x)$

d $y = 2P(x)$

e $y = -P(x)$

f $y = P(-x)$

2 Use the graph of $y = H(x)$ shown to sketch:



a $y = H(x) + 1$

b $y = H(x) - 1$

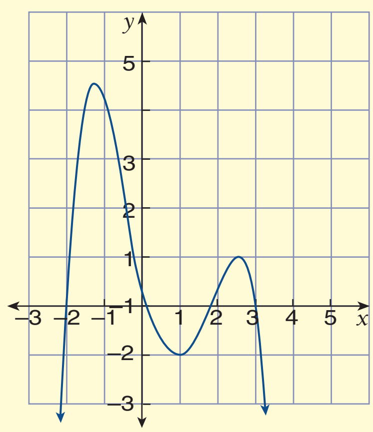
c $y = \frac{1}{2}H(x)$

d $y = 2H(x)$

e $y = -H(x)$

f $y = H(-x)$

3 Use the graph of $y = Q(x)$ shown to sketch:



a $y = Q(x) + 1$

b $y = Q(x) - 1$

c $y = \frac{1}{2}Q(x)$

d $y = 2Q(x)$

e $y = -Q(x)$

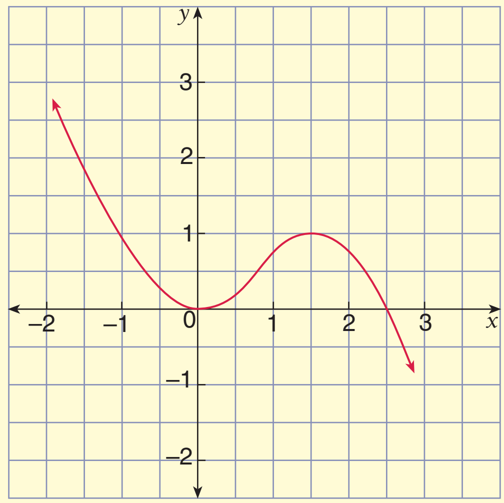
f $y = Q(-x)$

4 Make a sketch of the polynomial $y = x(x - 1)(x + 2)$. Use this sketch to make sketches of:

a $y = \frac{1}{2}x(x - 1)(x + 2)$
b $y = 2x(x - 1)(x + 2)$
c $y = -\frac{1}{2}x(x - 1)(x + 2)$
d $y = -2x(x - 1)(x + 2)$

5 The polynomial $y = P(x)$ is shown in the diagram.

a Sketch $y = -P(x)$ and then use this to produce a sketch of $y = 1 - P(x)$.
b Sketch $y = P(-x)$ and then use this to produce a sketch of $y = P(-x) + 1$.



6 Using the sketch of $y = H(x)$ in question **2**, make sketches of:

a $y = \frac{1}{2}H(x) - 1$
b $y = \frac{1}{2}H(-x)$
c $y = 2 - H(x)$

7 If $P(x) = (x - 3)^2(x + 1)^3$, sketch:

a $y = P(x)$
b $y = P(-x)$
c $y = -P(x)$
d $y = 1 - P(x)$
e $y = P(-x) + 1$

• A time for reflection.

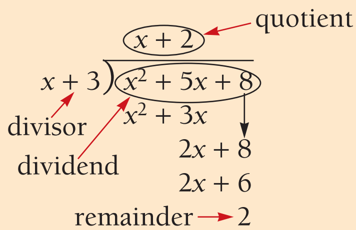
Mathematical Terms 7

constant term
- The term in a polynomial that does not change in value as the variable changes.
  eg In $P(x) = x^2 + 6x - 2$, the constant term is -2.

continuous (curve)
- A curve with no gaps.
- No matter what x value is chosen, a corresponding y value can always be found.

degree
- The highest power of any term in a polynomial.

dividing (a polynomial)
- The divisor, dividend quotient and remainder are names given to the different parts of the division process.
  eg



factor (of a polynomial)
- An expression that will divide into a polynomial and leave a remainder of zero.

factor theorem
- A theorem that is used to find factors of a polynomial.

leading coefficient
- The coefficient of the leading term.

leading term
- The term that has the highest power of the variable.

monic
- A polynomial that has a leading coefficient of 1.

polynomial
- An expression that is the sum of any number of terms of the form ax^n (where n is a positive integer or zero).
  eg $P(x) = x^4 + 3x^2 - 6x + 8$
  $Q(x) = 2x^2 - x^3$

remainder theorem
- A theorem that is used to find the remainder when a polynomial is divided by $(x - a)$.

root
- A solution of the polynomial equation $P(x) = 0$.
- The degree of the polynomial gives the maximum number of roots that are possible. A polynomial of degree 3, for example, can have at most 3 roots.
- Roots can be called single, double or triple roots
  eg $y = (x - 1)(x - 2)(x - 3)$
  Three single roots at $x = 1, 2, 3$
  $y = (x - 1)(x - 2)^2$
  Single root at $x = 1$, double root at $x = 2$
  $y = (x - 1)^3$
  Triple root at $x = 1$

zero
- A value of the variable that gives the polynomial a value of zero.

Mathematical terms 7

Diagnostic Test 7 | Polynomials

- Each part of this test has similar items that test a certain type of question.
- Errors made will indicate an area of weakness.
- Each weakness should be treated by going back to the section listed.

	Section
1 What is the degree of each polynomial? **a** $5x^2 + 7x + 3$ **b** $x^7 - x^5 + x^3 + x$ **c** $9 - 2x + x^2 - x^3$	7:01
2 What is the leading term for each polynomial? **a** $x^2 - 5x^3 + 3$ **b** $9x^4 + 7x^2 - 1$ **c** $5 + 3x + x^2$	7:01
3 Which expression, A or B, is *not* a polynomial? **a** A: $x^3 - x + 1$ B: $x^2 + x^{-1} + 3$ **b** A: $x^2 + \sqrt{x}$ B: $1 - 5x$ **c** A: $2 - 3x + x^3$ B: $x^2 + 2^x$	7:01
Questions 4 to 6 refer to the following polynomials: $P(x) = x + 2$, $Q(x) = 2x^2 + 3x - 2$, $R(x) = 2x^3 + x^2 - 5x + 2$.	
4 Determine the following. **a** $Q(x) + R(x)$ **b** $R(x) - P(x)$ **c** $R(x) - Q(x)$	7:02
5 Find the following products. **a** $P(x)\ .\ Q(x)$ **b** $P(x)\ .\ R(x)$ **c** $(2x - 1)\ .\ P(x)\ .\ R(x)$	7:03
6 Find: **a** $Q(x) \div P(x)$ **b** $R(x) \div P(x)$ **c** $R(x) \div (x - 2)$	7:03
7 Find the remainder for the following divisions. **a** $(x^2 + 3x - 1) \div (x - 2)$ **b** $(x^3 + 2x^2 - 7) \div (x + 3)$ **c** $(2x^3 - x^2 + x - 4) \div (x - 5)$	7:04
8 For the following, state whether $A(x)$ is a factor of $B(x)$ or not. **a** $A(x) = x - 3$, $B(x) = x^2 - 5x + 3$ **b** $A(x) = x + 1$, $B(x) = x^3 + x^2 - x - 1$ **c** $A(x) = x - 2$, $B(x) = x^4 - 2x^2 - x - 6$	7:04
9 Solve the equations: **a** $x(x - 1)(x + 2) = 0$ **b** $x^3 - 3x - 2 = 0$ **c** $x^3 - 4x^2 + x + 6 = 0$	7:05
10 Make sketch graphs of the following. **a** $y = -x(x + 1)(x - 2)$ **b** $y = (x + 1)^2(x - 1)(x + 3)$ **c** $y = (x - 1)(x + 2)^3$	7:06

11 The graph represents the polynomial $y = P(x)$.

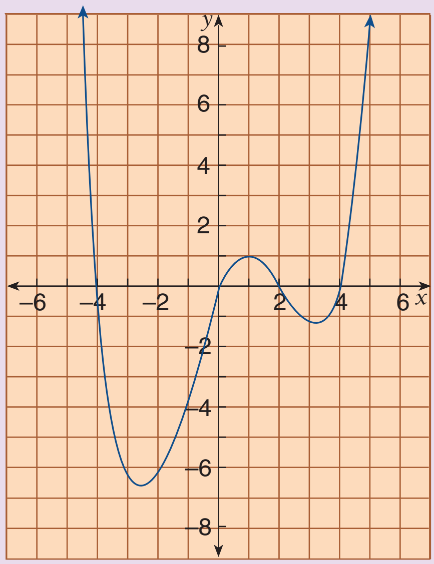
Use the graph to make sketches of:

a $y = -P(x)$

b $y = P(x) + 1$

c $y = P(-x)$

d $y = \frac{1}{2}P(x)$



Section 7:07

• Many mathematical curves can be seen in bridges.

Chapter 7 | Revision Assignment

7A

1 $A(x) = 5x^2 - 3x^3 + 5x - 3$.

a State the degree, leading term and constant term of $A(x)$.

b Evaluate $A(-1)$.

c If $A(x)(ax + b)$ is monic and has a constant term of –6, what is the value of a and b?

d What will the remainder be when $A(x)$ is divided by $(x + 2)$?

2 Solve these polynomial equations.

a $2x(x - 3)(x + 4) = 0$

b $(x - 3)(2x + 3)(3x - 1) = 0$

c $x^3 - 4x^2 + 4x = 0$

d $x^4 - 8x^2 + 16 = 0$

3 Sketch the graphs of the following polynomials.

a $y = (x - 1)(x + 1)(x - 4)$

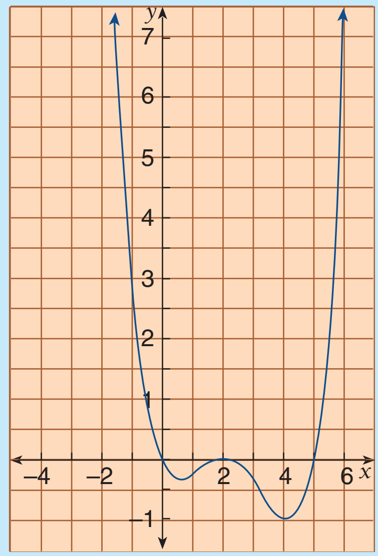
b $y = (x + 1)^2(x - 4)$

c $y = -(x + 1)^2(x - 4)^2$

4 A polynomial has zeros at 1, 2 and 4 and a y-intercept of 5. Make a possible sketch of the polynomial if it is of:

a degree 3 b degree 4

5



The graph of $y = P(x)$ is shown above. Use the graph to sketch:

a $y = P(x) + 1$ b $y = -0{\cdot}5P(x)$

c $y = P(-x)$

6 a Give an equation of a polynomial of degree 4 that has:

i no zeros ii 1 zero

iii 2 zeros iv 3 zeros

v 4 zeros

b Sketch the polynomial $y = x(x - 2)^2$. How many zeros will each of the following polynomials have?

i $y = x(x - 2)^2 + 1$

ii $y = x(x - 2)^2 - 1$

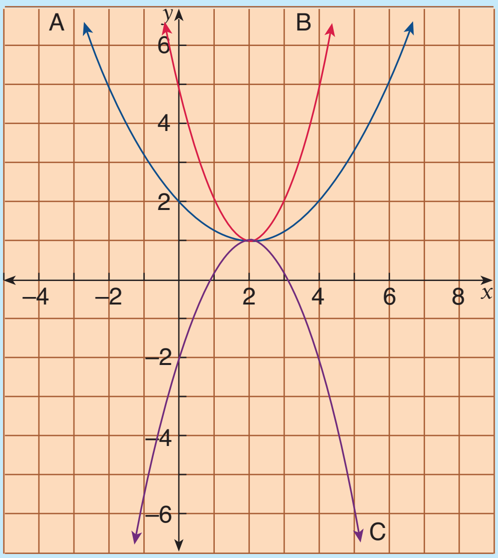
• Rivers can look like polynomial curves winding their way to the sea.

7B

Chapter 7 | Working Mathematically

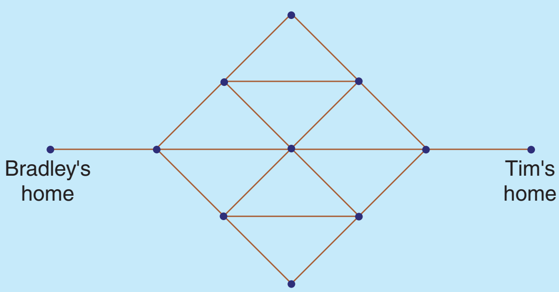
1 What can be learned about the shape of a polynomial from its leading term?

2 Find the equations of the parabolas A, B and C.



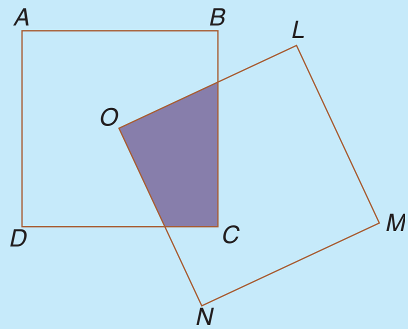
3 Bradley must travel from his home to Tim's home regularly. He decides to travel a different route each day but, while travelling, he always chooses a way that will take him closer to Tim's home (see the diagram below).

How many days can he travel before he has to repeat an earlier route?



4 A tank can be filled by a tap in 3 hours, and it can be emptied via a plug hole in 7 hours. If the tank is empty, how long will it take to fill if the tap is turned on and the plug hole is left open? (Assume that water drains out at a constant rate.)

5 Two identical squares are overlapped so that the vertex of one square is at the centre of the other. What fraction is the overlapping area of either square?



6 If the line $ax + by + c = 0$ has a negative gradient, what does this say about a, b and c?

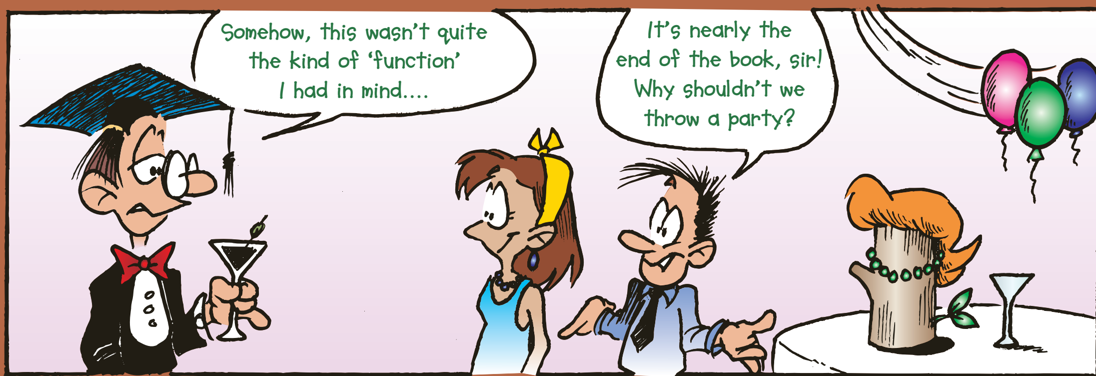
Chapter Review Questions

Technology Applications Drag and Drops

Polynomials

8

Functions and Logarithms



Chapter Contents

Learning Outcomes

Students will be able to:

- Identify a function.
- Find the inverse of a function.
- Convert between the index and logarithmic form of a number.
- Solve for unknowns in logarithmic expressions.
- Draw graphs of logarithmic and exponential functions and understand the connection between them.
- Use the laws of logarithms.
- Solve exponential equations.

Areas of Interaction

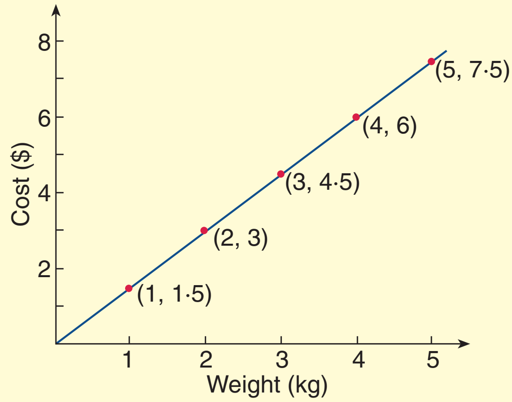
Approaches to Learning (Knowledge Acquisition, Problem Solving, Communication, Logical Thinking, Information and Communication, Technology Skills, Reflection), Human Ingenuity

8:01 | Functions

- Graphs are used to show the relationship between variables.
- Graphs originate from ordered pairs, which in turn come from tables, as shown below.

Weight of butter (kg)	*Cost ($)*
1	$1.50
2	$3
3	$4.50
4	$6
5	$7.50

Order pairs (W, C)
(1, 1·5)
(2, 3)
(3, 4·5)
(4, 6)
(5, 7·5)



- As tables cannot include every possible ordered pair, the relationship between variables is usually expressed as a rule or formula.

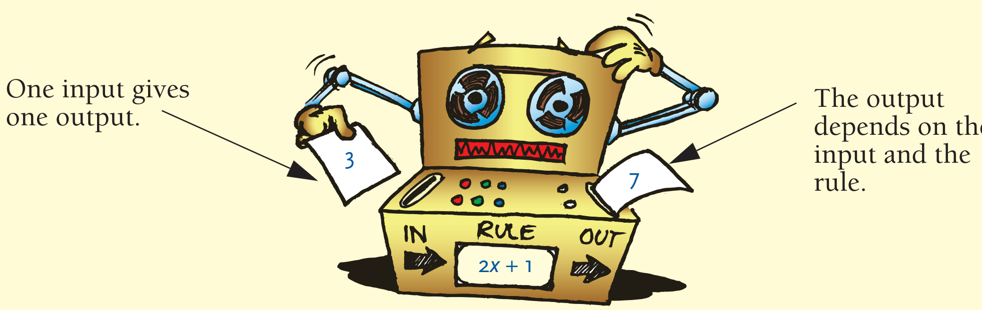
  In this case, $C = 1{\cdot}5W$.

  As the value of C depends on the value of W, it is usually called the **dependent variable** while W is called the **independent variable**.
- It should be clear that the rule $C = 1{\cdot}5W$ will produce an infinite set of ordered pairs (W, C) and that every value of W will produce only one value of C.

  When a set of ordered pairs has this property it is called a **function**.
- Functions can be thought of as being produced by a machine. The 'function machine' is given an input value and using some rule produces a unique output value.

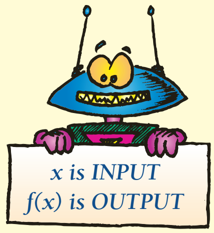
  The output value is dependent on both the input value and the rule being used.

A function is a special rule or relationship that assigns to every input value a unique output value.



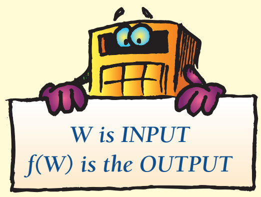
Function notation

- The relationship between the input value and the output value is represented by a statement such as $f(x) = 2x + 1$
  This tells us that the function f turns the input value x into the output value $2x + 1$.
  Hence, $f(1) = 2 \times 1 + 1$ and $f(2) = 2 \times 2 + 1$
  $= 3$ $= 5$



- Different letters can be used to represent different functions.
  eg $g(x) = x^2$ $H(x) = (1 + x)^2$
  $g(3) = 3^2$ $H(4) = (1 + 4)^2$
  $= 9$ $= 25$

- While it has been usual to represent the input value by x, other pronumerals can be used. The example concerning the cost of butter mentioned earlier is typical.
  eg $f(W) = 1{\cdot}5W$
  $f(3) = 1{\cdot}5 \times 3$
  $= 4{\cdot}5$



- Comparing $f(x) = 2x + 1$ and $y = 2x + 1$ suggests that $y = f(x)$ and, hence, there is an immediate connection with the number plane as (x, y) becomes an (input value, output value) pair.

Graphs of functions

As stated earlier, a function is a set of ordered pairs, where each input value produces a unique output value.

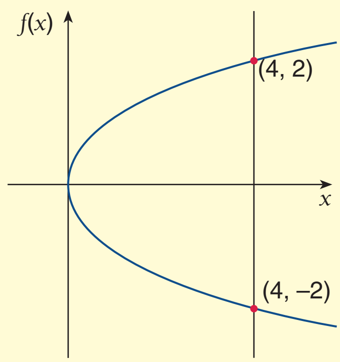
This means that no two ordered pairs can have the same input value, or the same input value cannot produce different output values.

This gives rise to an easy test for deciding whether a graph is the graph of a function.

In the graph on the right, the vertical line shown cuts the graph at the points (4, 2) and (4, –2).

Because the graph has two ordered pairs with the same input value, it cannot represent the graph of a function.

This gives us the vertical line test.



If a vertical line can be drawn to cut a graph in two or more places then the graph is not a function.

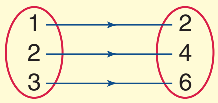
Exercise 8:01

1 In each of the following, state whether the set of ordered pairs represents a function or not.

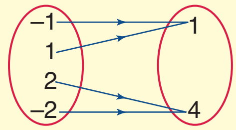
a (1, 5), (2, 6), (3, 7), (4, 8)
b (8, 4), (7, 5), (6, 6), (5, 4)
c (9, 3), (16, 4), (25, 5), (36, 6)
d (1, 3), (1, 4), (0, 3), (2, 4)
e (−1, 1), (1, 1), (2, 4), (−2, 4)

2 In each of the following diagrams, an input value is joined to an output value by an arrow. State whether each diagram represents a function.

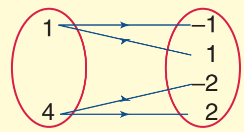
a



b



c



3 If the arrows in the diagrams of question **2** are reversed, state whether each diagram represents a function.

4 If $f(x) = 2x - 5$, find:

a $f(0)$ **b** $f(2)$ **c** $f(-2)$

5 If $H(x) = 2^x + 1$, find:

a $H(1)$ **b** $H(0)$ **c** $H(-1)$

6 If $g(x) = \dfrac{x^2 + 1}{x}$, calculate:

a $g(1)$ **b** $g(2)$ **c** $g(\frac{3}{4})$

7 It is known that $H(x) = 3x - 2$. Find x if:

a $H(x) = 13$ **b** $H(x) = -14$

8 Given that $f(x) = 2x - 5$, write an expression for:

a $f(2a)$ **b** $f(a + 1)$ **c** $f\left(\dfrac{1}{a}\right)$

9 It is known that $C(W) = W^2 - 2W + 5$.

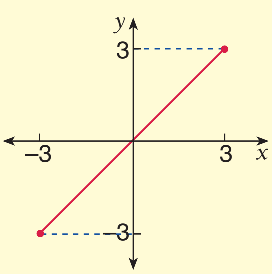
a Find $C(-1)$ **b** Find $C(1) - C(-1)$ **c** Find $C(W + 1)$

10 If $F(p) = p^2$, find an expression for $\dfrac{F(p+h) - F(p)}{h}$.

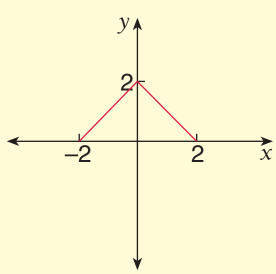
11 **a** If $f(x) = x^2$, show that $f(-x) = f(x)$.
b If $f(x) = x^3$, show that $f(-x) = -f(x)$.

12 Which of the following are graphs of functions?

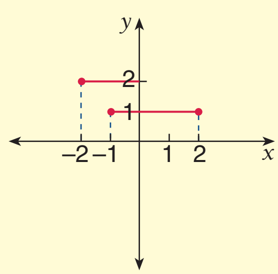
a



b



c



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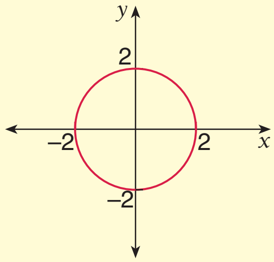

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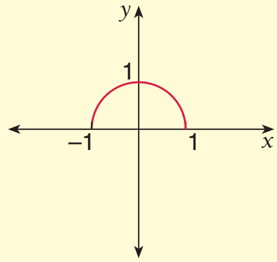

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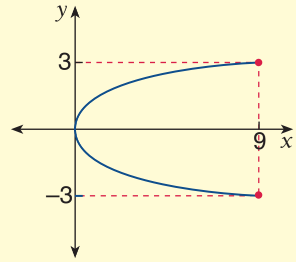

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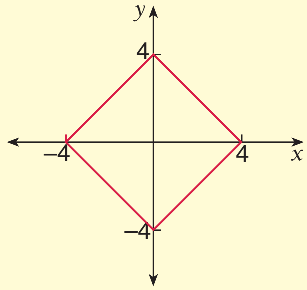

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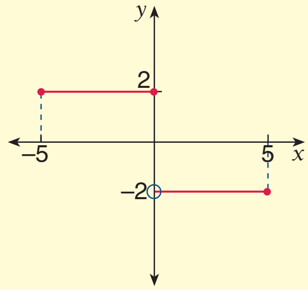
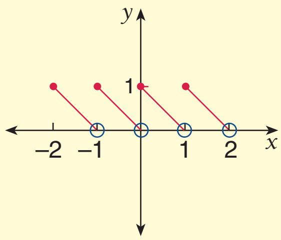


i



13 Does a straight line graph always represent a function?

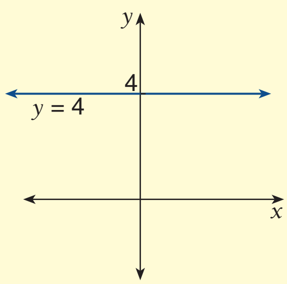
14 Mathematicians are often interested in the permissible x and y values that a function can take. For example, in question **12a**, the possible x values run from -3 to 3 inclusive. We write this as $\{-3 \leqslant x \leqslant 3\}$.

Similarly, the y values run from -3 to 3 inclusive. This is written as: $\{-3 \leqslant y \leqslant 3\}$.

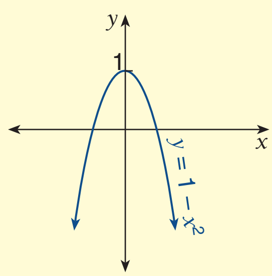
Use this notation described above to describe the permissible x and y values for each of the graphs in question **12**.

15 Write down the possible y values for each of the following functions.

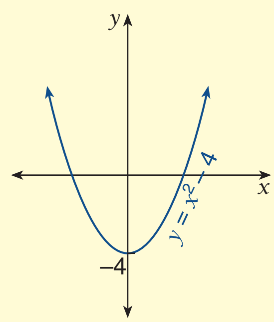
a



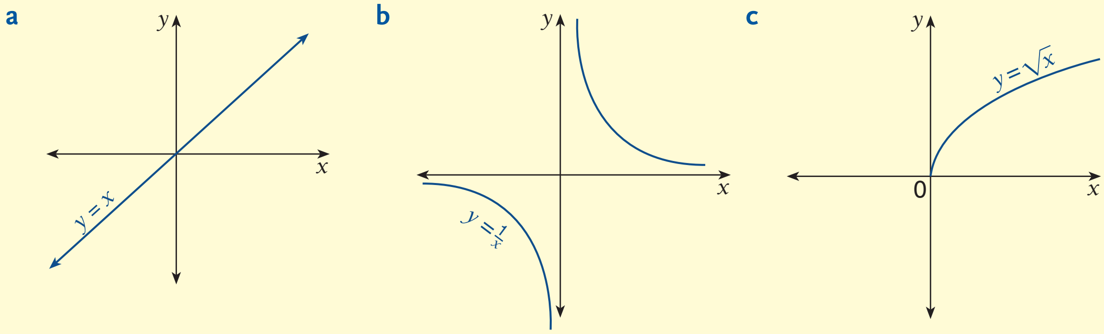
b



c



16 Write down the possible x values for each of the following functions.



17 By graphing each of the following or otherwise, write down the permissible x and y values for each function.

a $y = 2x - 1$ **b** $y = x^2 + 1$ **c** $y = \dfrac{1}{x-1}$

d $y = 2^x$ **e** $y = (1 + x)^2$

18 **a** The distance fallen by a stone dropped from a cliff is a function of the time that has elapsed since being dropped. The function is given by $f(t) = 4{\cdot}9t^2$, where t is the time in seconds and $f(t)$ is the distance in metres. What is meant by:

i $f(1{\cdot}5)$? **ii** $f(1{\cdot}6)$? **iii** $f(1{\cdot}6) - f(1{\cdot}5)$?

iv $\dfrac{f(1{\cdot}6) - f(1{\cdot}5)}{1{\cdot}6 - 1{\cdot}5}$? (Answer in words.)

b The period of a pendulum, T, is known to be a function of the length, L, where T is measured in seconds and L is measured in metres.

If $f(L) = 2\pi\sqrt{\dfrac{L}{g}}$ where $g = 9{\cdot}8$:

i find $f(9{\cdot}8)$

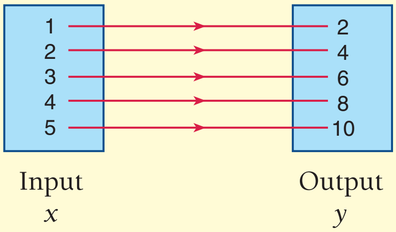
ii find the value of the period when $L = 4{\cdot}9$

iii what is the physical meaning of $f(1{\cdot}5) - f(1)$?

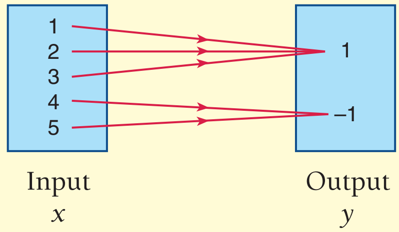
8:02 | Inverse Functions

In a function, an input value produces a unique output value. The following diagrams represent two different types of functions. Example 1 is called a **one-to-one function** because each input produces one output and each output is produced from only one input. In example 2, which is called a **many-to-one function**, this does not happen. Each of the output values is produced by more than one input value.

Example 1

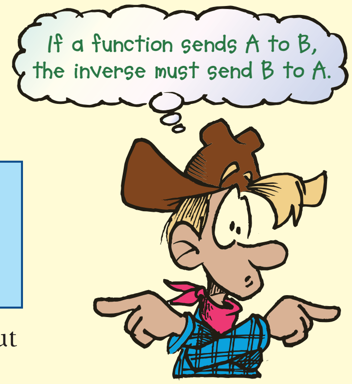


Example 2

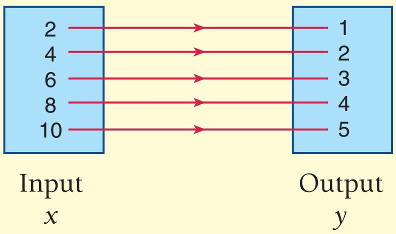


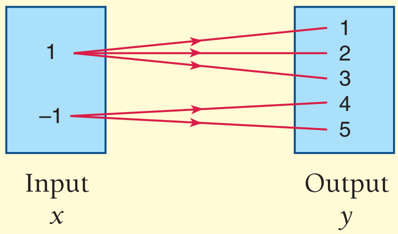
Given a certain function, f, we are now interested in finding a function that reverses f. If we can find such a function, it is called the **inverse function** and is denoted by f^{-1}.

To find the inverse function, it would seem that all we have to do is swap the input and output values or interchange the x and y in the original function.



If this is done in the example above, the following results.





Inspection of the above shows that while the reversing has produced a function in example **1**, it has not produced a function in example **2**.

It should be clear from the above that the reversing process will only produce a function if the original function is one-to-one.

When the function is given in notation form rather than as a set of ordered pairs, the inverse function is produced by simply interchanging the x and y in the original equation and then making y the subject.

This is shown in the example below.

worked example

Find the inverse function of the function $y = 3x + 1$.

Solution

$y = 3x + 1$ (where $f(x) = 3x + 1$)

Interchange the x and y to obtain the inverse.

$x = 3y + 1$

$x - 1 = 3y$

$y = \frac{x-1}{3}$ (where $f^{-1}(x) = \frac{x-1}{3}$)

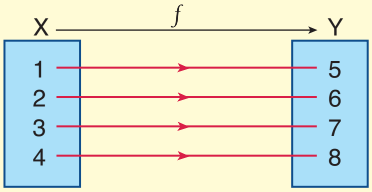
Hence, $y = \frac{x-1}{3}$ is the inverse function of $y = 3x + 1$.

If $y = f(x)$ is a one-to-one function, then the inverse function is found by:
1 interchanging x and y and then
2 making y the subject

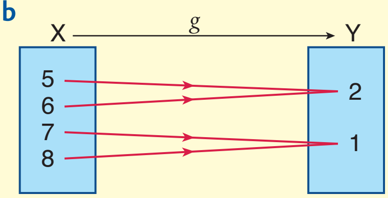
Exercise 8:02

1 Which of the following functions will have an inverse function?

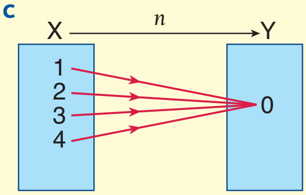
a



b



c



2 **a** A function H is represented by the ordered pairs {(0, 2), (2, 4), (3, 6), (4, 8)}. Does H have an inverse function? If so, list the ordered pairs of the inverse function.

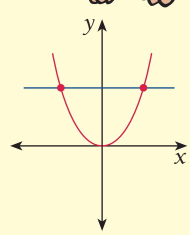
b A function f is represented by the ordered pairs {(−1, 1), (1, 1), (0, 0), (−2, 2), (2, 2)}. Does f have an inverse function? If so, list the ordered pairs of the inverse function.

3 Find the inverse function of each of the following.

a $y = 2x$ **b** $y = 3x + 5$

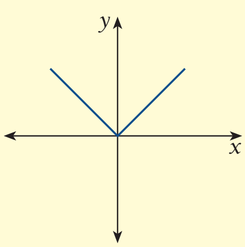
c $y = \dfrac{x-1}{3}$

4 The function $y = x^2$ is an example of a many-to-one function as it has more than one point with the same y value. As such it does not have an inverse function.

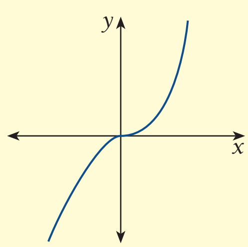


Use the above to state whether the functions shown in the graphs below are one-to-one or many-to-one.

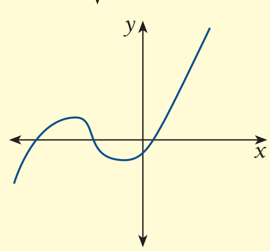
a



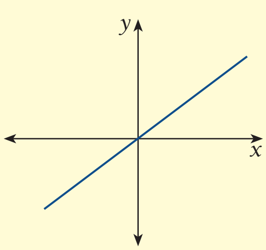
b



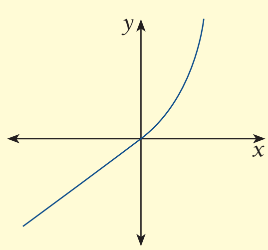
c



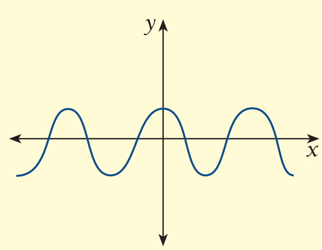
d



e



f



5 Do all straight lines that are functions have inverses? Explain your answer.

6 Can you think of a rule (similar to the vertical line test for functions) that could be used with the graph of the function to determine whether the function has an inverse or not?

7 Find the inverse function for each of these linear functions.

a $f(x) = 4x$ **b** $f(x) = 1 - x$ **c** $f(x) = \frac{2 - x}{3}$

8 **a** What is the inverse function of the line $y = x$?

b The inverse of the line $y = 2x$ is $y = \frac{1}{2}x$. Graph both lines and the line $y = x$ on the same number plane. What do you notice?

c On the same number plane, graph $y = x + 1$, its inverse $y = x - 1$ and $y = x$. What do you notice?

9 On the same number plane, graph the lines $y = x$ and the line through the points A(1, 1) and B(3, 2).

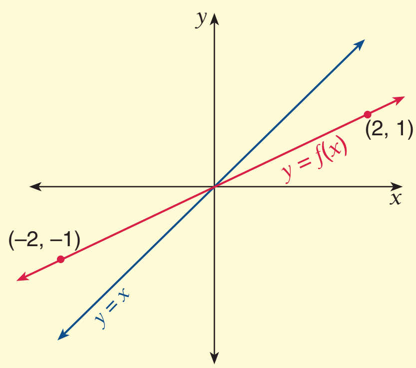
a Why do the points (1, 1) and (2, 3) lie on the inverse of line AB?

b Find the equation of the line AB and hence find the equation of its inverse.

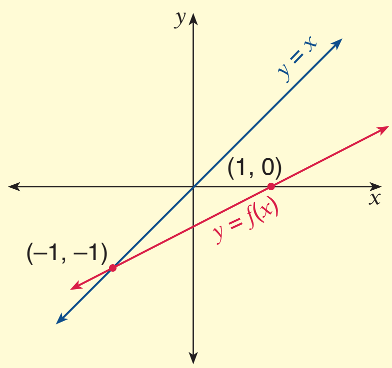
c Show that the points (1, 1) and (2, 3) satisfy the equation of the inverse.

10 Copy the graphs below and add to each the graph of the inverse function $y = f^{-1}(x)$.

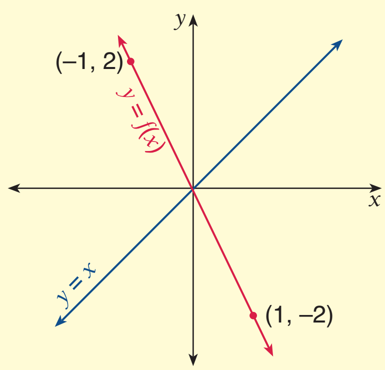
a



b



c



11 The function $y = x^2$ does not have an inverse as it is a many-to-one function. If it is divided into two parts (each of which is one-to-one) it is then possible to find the inverse function.

a Sketch $y = x^2$, $x \geqslant 0$ (ie the part of $y = x^2$ which has zero or positive x values).

b Use the fact that the inverse function is a reflection in the line $y = x$ to sketch the inverse function.

c Find the equation of the inverse function.

d Sketch the function $y = x^2$, $x \leqslant 0$ and its inverse. What is the equation of the inverse function?

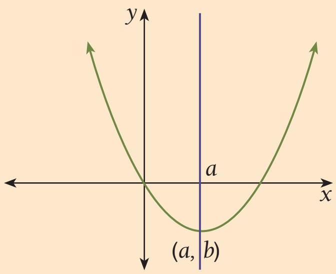
12 Show on a graph the function $y = x^3$. Find the equation of its inverse and graph it on the same diagram.

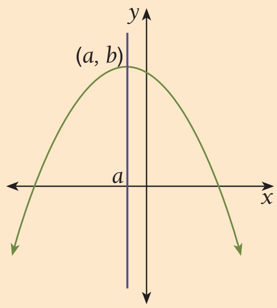
Investigation 8:02 | Quadratic functions and inverses

Please use the Assessment Grid on the following page to help you understand what is required for this Investigation.

In question **11** of exercise 8:02, we considered $y = x^2$ and why it didn't have an inverse function. We now extend these ideas to any quadratic function.

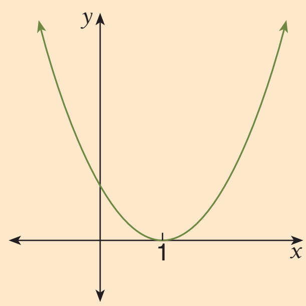
1 The two quadratic shapes we have met are given below.





a Why don't the parabolas have inverse functions?

b How can the parabolas be divided to give two parts, each of which will have an inverse function?



2 This graph shows $y = (x - 1)^2$.

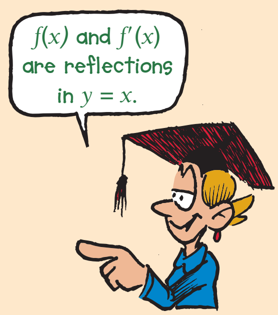
a What are the coordinates of its vertex?

b Where must the parabola be divided to give two parts that will have inverse functions?

c Sketch the function $y = (x - 1)^2,\ x \geqslant 1$.

d Sketch the inverse of the function $y = (x - 1)^2,\ x \geqslant 1$.

e What is the equation of the inverse function?



3 **a** Sketch the function $y = 4 - x^2,\ x \geqslant 0$.

b Sketch the inverse of the function and find its equation.

4 Suppose the height of an object dropped from a tall building is modelled by the quadratic function $y = 180 - 9{\cdot}8x^2$, where y is the height after x seconds. Find the equation of the inverse of this function and use it to find:

a when the height is 100 m.

b how long the object is in the air.

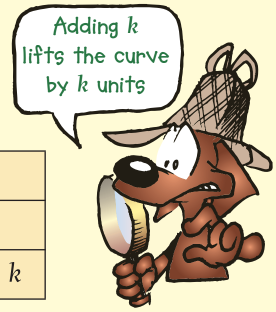
Assessment Grid for Investigation 8:02 | Quadratic functions and inverses

The following is a sample assessment grid for this investigation. You should carefully read the criteria *before* beginning the investigation so that you know what is required.

Assessment Criteria (B, C, D) for this investigation				Achieved ✓
Criterion B Investigating Patterns	a	None of the following descriptors have been achieved.	0	
	b	Some help was needed to attempt to explain why parabolas don't have inverse functions and how they can be divided to have one.	1	
			2	
	c	The student is able to suggest why parabolas don't have inverse functions and how they can be divided to have one. Some work is done towards parts 2, 3 and 4.	3	
			4	
	d	The student is able to describe why parabolas don't have inverse functions and how they can be divided in two to have one, as well as finding equations of the inverse functions needed and answering part 4.	5	
			6	
	e	The student has completed all parts of the investigation using graphical support with thorough explanations and justification throughout.	7	
			8	
Criterion C Communication in Mathematics	a	None of the following descriptors have been achieved.	0	
	b	There is a basic use of mathematical language and representation. Lines of reasoning are insufficient.	1	
			2	
	c	There is satisfactory use of mathematical language and representation. Graphs and explanations are clear but not always logical or complete.	3	
			4	
	d	A good use of mathematical language and representation. Graphs are accurate, to scale and fully labeled. Explanations are complete and concise.	5	
			6	
Criterion D Reflection in Mathematics	a	None of the following descriptors have been achieved.	0	
	b	An attempt has been made to explain whether the results make sense and are consistent. An attempt has been made to answer part 4.	1	
			2	
	c	There is a correct but brief explanation of whether results make sense and how they were found. Part 4 is answered well.	3	
			4	
	d	Part 4 is used to show the importance of the findings with a real life example. There is a critical explanation of whether the results make sense and are consistent throughout. The accuracy of results and possible improvement to the method are considered.	5	
			6	

8:03 | The Graphs of $y = f(x)$, $y = f(x) + k$ and $y = f(x - a)$

A relationship exists between these three curves that allows the graphs of $y = f(x) + k$ and $y = f(x - a)$ to be graphed given the shape of $y = f(x)$.



Graphs of $y = f(x)$ and $y = f(x) + k$

Study the table below.

x	1	2	3	-1	a
$f(x)$	$f(1)$	$f(2)$	$f(3)$	$f(-1)$	$f(a)$
$f(x) + k$	$f(1) + k$	$f(2) + k$	$f(3) + k$	$f(-1) + k$	$f(a) + k$

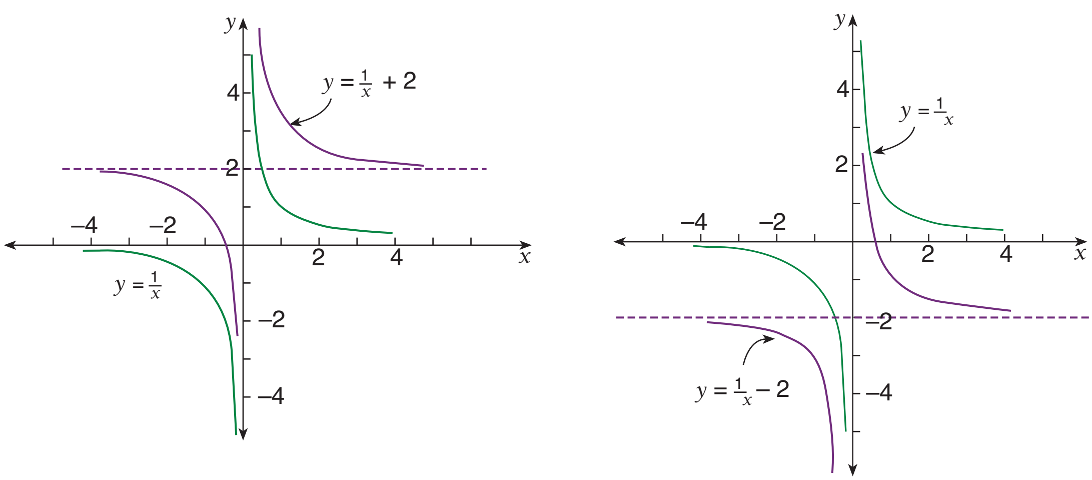
The table shows that for the same x values the y values on the curve $y = f(x) + k$ are k more (or less) than the y values on the curve $y = f(x)$.

This means that the points on the curve $y = f(x) + k$ can be obtained by moving all the points on the curve $y = f(x)$ by k units vertically.

worked example

Sketch the curve $y = \dfrac{1}{x}$ and use it to sketch the curves $y = \dfrac{1}{x} + 2$ and $y = \dfrac{1}{x} - 2$.

Solution



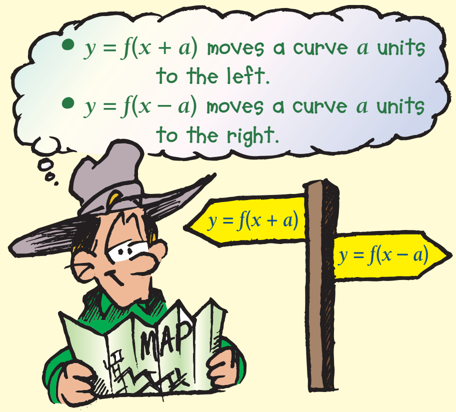
The curve $y = f(x) + k$ is obtained by moving the curve $y = f(x)$:
- **up k units if k is positive**
- **down k units if k is negative.**

Graphs of $y = f(x)$ and $y = f(x - a)$

Study the tables below:

x	0	1	2	−1
$f(x)$	$f(0)$	$f(1)$	$f(2)$	$f(-1)$

x	a	$a + 1$	$a + 2$	$a - 1$
$f(x - a)$	$f(0)$	$f(1)$	$f(2)$	$f(-1)$



The tables show that when the y values on $y = f(x)$ and $y = f(x - a)$ are equal, the x values differ by a units.

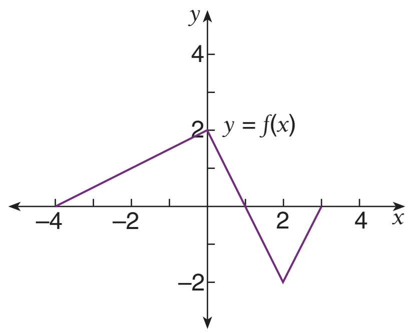
Hence, the points on the curve $y = f(x - a)$ can be obtained by moving all the points on the curve $y = f(x)$ a units horizontally.

worked example

Use the graph of $y = f(x)$, which is given, to sketch the following:

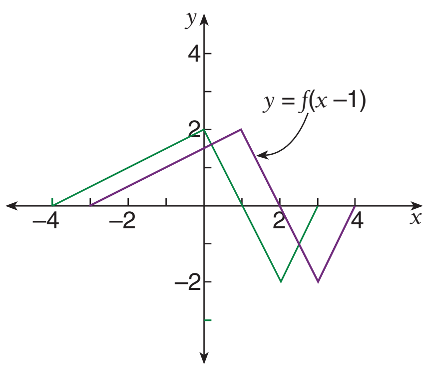
a $y = f(x - 1)$

b $y = f(x + 1)$

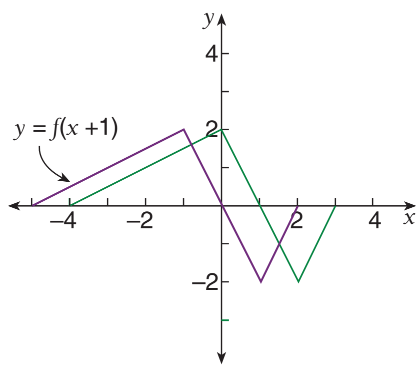


Solution

a To obtain $y = f(x - 1)$, move $y = f(x)$ one unit to the right.



b To obtain $y = f(x + 1)$, move $y = f(x)$ one unit to the left.



The curve $y = f(x - a)$ is obtained by moving the curve $y = f(x)$:
- ***a* units to the right if *a* is positive**
- ***a* units to the left if *a* is negative.**

Exercise 8:03

1 State how the curve $y = x^2$ could be moved to produce each of the following curves.

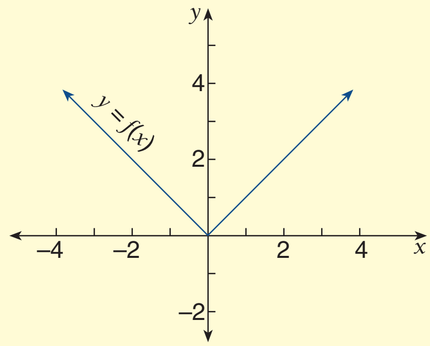
a $y = (x + 1)^2$ **b** $y = x^2 + 1$
c $y = x^2 - 1$ **d** $y = (x - 1)^2$

2 Give the equation of the curve that would result if the curve $y = 2^x$ was moved:

a 1 unit up **b** 1 unit down
c 1 unit to the right **d** 1 unit to the left

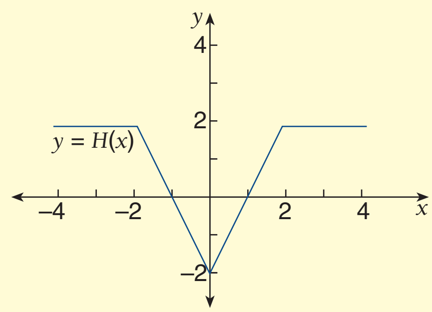
3 Use the given graph of $y = f(x)$ to sketch the following functions.

a $y = f(x) + 1$ **b** $y = f(x) - 2$
c $y = f(x - 1)$ **d** $y = f(x + 2)$



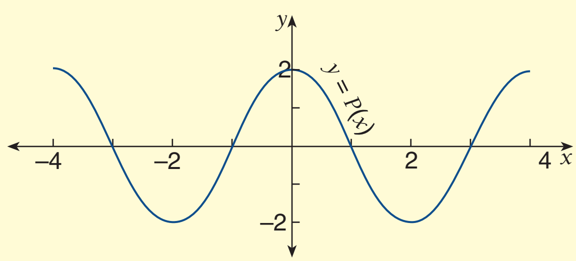
4 Use the given graph of $y = H(x)$ to sketch the following functions.

a $y = H(x - 2)$ **b** $y = H(x) + 2$
c $y = H(x) - 1$ **d** $y = H(x + 1)$



5 Use the graph of $y = P(x)$, which is given, to sketch the following functions.

a $y = P(x) + 2$
b $y = P(x + 1)$
c $y = P(x - 2)$
d $y = P(x) - 1$



6 Sketch $y = F(x) + 1$ and $y = F(x + 1)$ if:

a $F(x) = x^3$ **b** $F(x) = x^2 - 1$ **c** $y = \frac{1}{x}$

Fun Spot 8:03 | Where would you get a job playing a rubber trumpet?

Work out the answer to each part and put the letter for that part in the box that is above the correct answer.

If $f(x) = x^2$ and $g(x) = (x + 1)^2$, evaluate:

N $f(2)$ **T** $g(-2)$ **I** $f(g(2))$

State the permissible y values for each of the functions:

C $y = 4 - x^2$ **B** $y = x^2 + 1$

A $y = (x + 1)^2$ **I** $y = x^2 - 4$

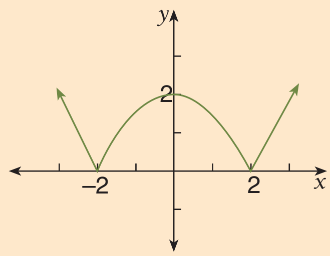
What is the inverse function of:

N $y = 2x - 1$? **D** $y = \frac{1}{2}x + 1$?

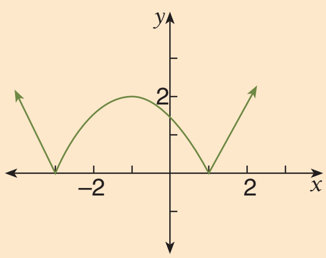
A If $y = 2x$ is moved up 1 unit, what is the equation of the resulting curve?

L If $y = 2x$ is moved 1 unit to the left, what is the equation of the resulting curve?

The function $y = H(x)$ is shown on the right. Moving this curve produces the four curves below. What is the equation of each curve?



A



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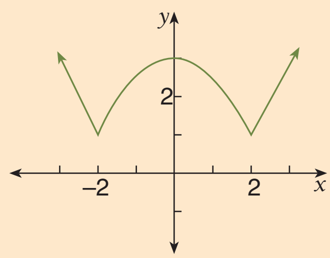
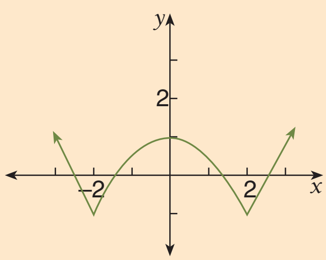
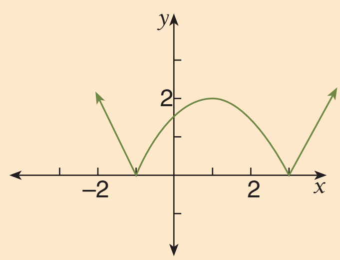


E



N



$y \geqslant -4$	4	$y \geqslant 0$	$y = \frac{1}{2}x + \frac{1}{2}$	$y = H(x) - 1$	$y = 2x + 2$	$y = 2x + 1$	$y = H(x) + 1$	1	81	$y \leqslant 4$	$y \geqslant 1$	$y = H(x + 1)$	$y = H(x - 1)$	$y = 2x - 2$

8:04 | Logarithms

prep quiz 8:04

Evaluate:	**1** 8^2	**2** 4^3	**3** 2^6	
Solve the following:	**4** $64 = 8^x$	**5** $64 = 4^x$	**6** $64 = 2^x$	
Write as powers of 2:	**7** 4	**8** 8	**9** $\frac{1}{2}$	**10** $\sqrt{2}$

In the Prep Quiz, we saw that:

- the same *number* could be expressed as a *power* of a different number or base

  eg $64 = 8^2$ $64 = 4^3$ $64 = 2^6$

- different *numbers* could be expressed as *powers* of the same number or *base*.

  $4 = 2^2$ $8 = 2^3$ $\frac{1}{2} = 2^{-1}$

When we look at how different numbers can be expressed as powers of another number (called the base), we are talking about logarithms.

Logarithms are indices. More specifically:

The logarithm of a number to any base is the index when the number is expressed as a power of the base.
In symbols, if $y = a^x$ then $\log_a y = x$.

If $64 = 8^2$, then $\log_8 64 = 2$.
If $64 = 4^3$, then $\log_4 64 = 3$.
Also if $\log_2 64 = 6$, then $2^6 = 64$.

worked examples

1 Evaluate: **a** $\log_4 16$ **b** $\log_2 1$ **c** $\log_9 (\frac{1}{27})$

2 Solve the following logarithmic equations.
a $\log_{27} 3 = x$ **b** $\log_4 x = -2$ **c** $\log_x 8 = 1{\cdot}5$

Solutions

1 **a** To evaluate an expression such as $\log_4 16$, we need to ask this question: '4 raised to what power will equal 16?'
Thus, since $4^2 = 16$, then $\log_4 16 = 2$.

b If $\log_2 1 = x$, then
$2^x = 1$
Since $2^0 = 1$, then $\log_2 1 = 0$.

c If $\log_9 (\frac{1}{27}) = x$, then
$9^x = \frac{1}{27}$
Solving: $3^{2x} = 3^{-3}$
so $x = -\frac{3}{2}$
$\therefore \log_9 (\frac{1}{27}) = -\frac{3}{2}$

2 **a** If $\log_{27} 3 = x$
then $27^x = 3$
$3^{3x} = 3^1$
$3x = 1$
$\therefore x = \frac{1}{3}$

b If $\log_4 x = -2$
then $4^{-2} = x$
$\therefore x = \frac{1}{16}$

c If $\log_x 8 = 1{\cdot}5$
then $x^{1\cdot5} = 8$
ie $x^{\frac{3}{2}} = 2^3$
so $x^{\frac{1}{2}} = 2$
$\therefore x = 4$

$y = a^x \Leftrightarrow \log_a y = x$

Exercise 8:04

1 Write each of these expressions in logarithmic form.

a $8 = 2^3$ **b** $16 = 4^2$ **c** $7 = 7^1$ **d** $64 = 2^6$
e $3^4 = 81$ **f** $4^4 = 256$ **g** $2^5 = 32$ **h** $3^0 = 1$
i $\frac{1}{2} = 2^{-1}$ **j** $\frac{1}{9} = 3^{-2}$ **k** $\sqrt{5} = 5^{\frac{1}{2}}$ **l** $9 = 27^{\frac{2}{3}}$
m $25^{\frac{3}{2}} = 125$ **n** $36^{-\frac{1}{2}} = \frac{1}{6}$ **o** $16^{\frac{3}{4}} = 8$ **p** $8^{-\frac{5}{3}} = \frac{1}{32}$

2 Rewrite each of these expressions in index form.

a $\log_2 4 = 2$ **b** $\log_3 9 = 2$ **c** $\log_5 1 = 0$ **d** $\log_4 4 = 1$
e $\log_{10} 1000 = 3$ **f** $\log_3 27 = 3$ **g** $\log_2 16 = 4$ **h** $\log_4 16 = 2$
i $\log_7 243 = 3$ **j** $\log_5 625 = 4$ **k** $\log_2 128 = 7$ **l** $\log_6 6 = 1$
m $\log_2 \sqrt{2} = \frac{1}{2}$ **n** $\log_3 (\frac{1}{3}) = -1$ **o** $\log_2 (\frac{1}{4}) = -2$ **p** $\log_4 8 = \frac{3}{2}$

3 Evaluate the following.

a $\log_3 9$ **b** $\log_2 8$ **c** $\log_5 25$ **d** $\log_7 1$
e $\log_3 3$ **f** $\log_2 32$ **g** $\log_4 64$ **h** $\log_8 64$
i $\log_6 216$ **j** $\log_{10} 1$ **k** $\log_{10} 10\,000$ **l** $\log_3 81$
m $\log_4 2$ **n** $\log_9 3$ **o** $\log_2 (\frac{1}{2})$ **p** $\log_4 8$

4 Solve for x.

a $\log_2 16 = x$ **b** $\log_3 27 = x$ **c** $\log_5 625 = x$ **d** $\log_7 49 = x$
e $\log_4 1 = x$ **f** $\log_9 3 = x$ **g** $\log_6 6 = x$ **h** $\log_{10} \sqrt{10} = x$
i $\log_9 (\frac{1}{9}) = x$ **j** $\log_2 (\frac{1}{4}) = x$ **k** $\log_8 4 = x$ **l** $\log_{16} 64 = x$
m $\log_2 (0{\cdot}5) = x$ **n** $\log_5 (0{\cdot}04) = x$ **o** $\log_8 \sqrt{2} = x$ **p** $\log_{100} 1000 = x$

5 Find x.

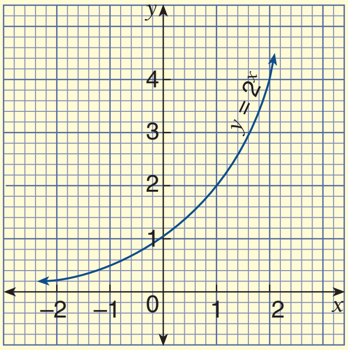
a $\log_2 x = 5$ **b** $\log_3 x = 3$ **c** $\log_2 x = 7$ **d** $\log_5 x = 0$
e $\log_7 x = 2$ **f** $\log_6 x = 3$ **g** $\log_3 x = -1$ **h** $\log_4 x = \frac{1}{2}$
i $\log_9 x = \frac{3}{2}$ **j** $\log_4 x = -2$ **k** $\log_2 x = -3$ **l** $\log_8 x = -\frac{1}{3}$
m $\log_{25} x = 1{\cdot}5$ **n** $\log_{10} x = -1$ **o** $\log_5 x = -\frac{1}{2}$ **p** $\log_{16} x = \frac{3}{4}$

6 Solve the following.

a $\log_x 16 = 4$ **b** $\log_x 9 = 2$ **c** $\log_x 5 = 1$ **d** $\log_x 81 = 4$
e $\log_x 125 = 3$ **f** $\log_x 100 = 2$ **g** $\log_x 8 = 3$ **h** $\log_x 729 = 6$
i $\log_x \sqrt{5} = \frac{1}{2}$ **j** $\log_x (\frac{1}{3}) = -1$ **k** $\log_x 2 = -1$ **l** $\log_x 8 = \frac{3}{2}$
m $\log_x 3 = \frac{1}{3}$ **n** $\log_x 4 = \frac{1}{2}$ **o** $\log_x 4 = \frac{1}{3}$ **p** $\log_x 27 = \frac{3}{4}$

8:05 | Logarithmic and Exponential Graphs

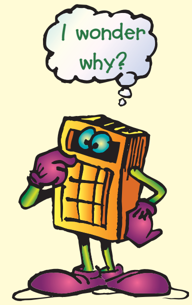
You should remember what an exponential graph such as $y = 2^x$ looks like.



If the points in the table below are plotted, the graph shown is obtained.

$y = 2^x$

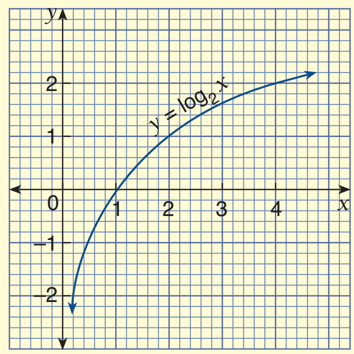
x	−2	−1	0	$\frac{1}{2}$	1	$1\frac{1}{2}$	2
y	$\frac{1}{4}$	$\frac{1}{2}$	1	1·4	2	2·8	4



■ All exponential graphs cut the y-axis at 1.

- To draw the graph of a logarithmic function such as $y = \log_2 x$, we can use the fact that this is equivalent to $x = 2^y$. (The x and y values have changed places from $y = 2^x$).
  Thus, a table of values and the graph would look like this.
  $y = \log_2 x$

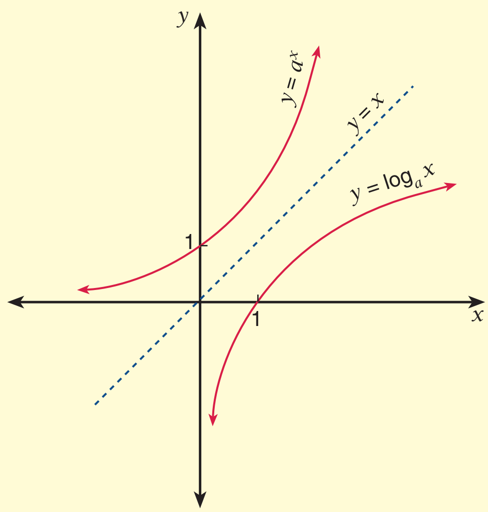
x	$\frac{1}{4}$	$\frac{1}{2}$	1	1·4	2	2·8	4
y	−2	−1	0	$\frac{1}{2}$	1	$1\frac{1}{2}$	2



■ A logarithmic graph always cuts the x-axis at 1.

- Now we have seen that $y = \log_a x$ is the same as $x = a^y$.
  Interchanging the x and y in $x = a^y$ produces its inverse function, $y = a^x$.

$y = \log_a x$ and $y = a^x$ (for $a > 0$) are inverse functions and hence their graphs are reflections of each other in the line $y = x$.



Exercise 8:05

1 Copy and complete these tables of values and then draw their graphs on the same number plane. (Approximate to one decimal place when necessary.)

$y = 3^x$

x	-2	-1	0	$\frac{1}{2}$	1	$1\frac{1}{2}$	2
y							

$y = \log_3 x$, $(x = 3^y)$

x							
y	-2	-1	0	$\frac{1}{2}$	1	$1\frac{1}{2}$	2

2 **a** Using the fact that $y = \log_2 x$ and $x = 2^y$ are equivalent functions, complete the table of values below and use it to construct an accurate graph of $y = \log_2 x$. (Approximate correct to two decimal places.)

$y = \log_2 x$

x													
y	-3	$-2\frac{1}{2}$	-2	$-1\frac{1}{2}$	-1	$-\frac{1}{2}$	0	$\frac{1}{2}$	1	$1\frac{1}{2}$	2	$2\frac{1}{2}$	3

b From your graph, estimate correct to one decimal place the value of:

i $\log_2 3$ **ii** $\log_2 5$ **iii** $\log_2 1{\cdot}5$ **iv** $\log_2 0{\cdot}8$

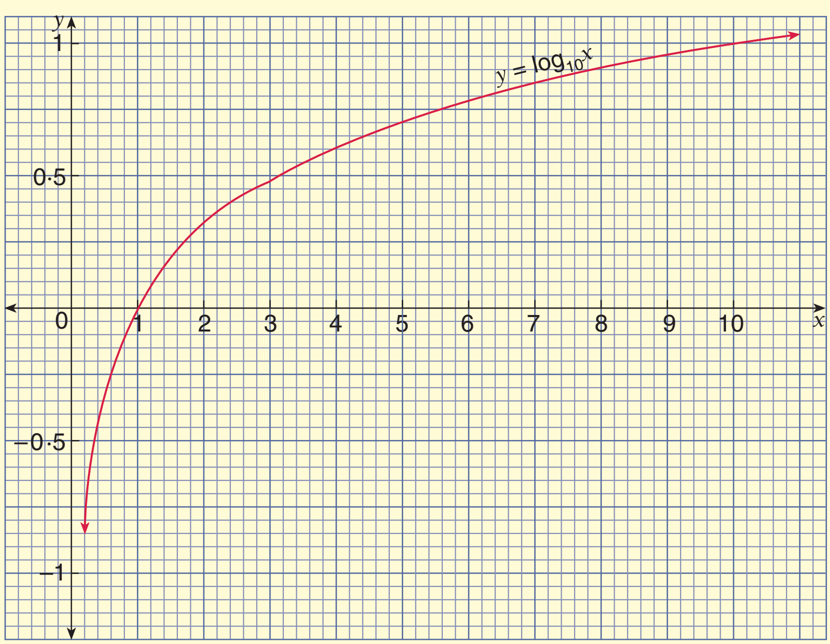
3 The graphing of $y = \log_{10} x$ is aided by the use of the log key on your calculator.

a Complete the table of values below using your calculator (approximating to two decimal places).

$y = \log_{10} x$

x	0·2	0·4	0·6	0·8	1	2	3	4	5	6	7	8	9	10
y														

b Using your table of values, draw a graph of $y = \log_{10} x$ like the one below. Use scales similar to those shown for each axis.



c Using your graph, find approximate solutions for the following equations.
i $\log_{10} x = 0{\cdot}7$ **ii** $\log_{10} x = 0{\cdot}25$ **iii** $\log_{10} x = -0{\cdot}5$

4 On the same number plane, *sketch* the graphs of the following, showing their positions relative to one another.
a $y = \log_2 x$ **b** $y = \log_3 x$ **c** $y = \log_5 x$

8:06 | Laws of Logarithms

prep quiz 8:06

Complete:

1. If $8 = 2^3$ then $\log_2 8 = \ldots$
2. If $m = 2^x$ then $\log_2 m = \ldots$
3. If $m = a^x$ then $\log_a m = \ldots$
4. If $n = a^y$ then $\log_a n = \ldots$
5. If $p = a^{x+y}$ then $\log_a p = \ldots$

Simplify: **6** $a^x \times a^y$ **7** $a^x \div a^y$ **8** $(a^x)^m$

9. Write $\dfrac{1}{x}$ in index form.
10. Since $3^0 = 1$ then $\log_3 1 = \ldots$

Since logarithms are indices, the index laws can be used to deduce the logarithm laws.

$$\mathbf{\log_a xy = \log_a x + \log_a y}$$

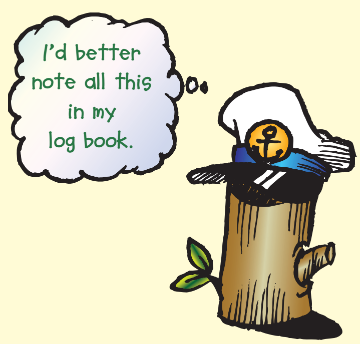
Proof: Let $x = a^m$ and $y = a^n$ (ie $\log_a x = m$ and $\log_a y = n$)

$$\begin{aligned} \text{Now, } xy &= a^m \times a^n \\ &= a^{m+n} \\ \therefore \log_a xy &= m + n \\ &= \log_a x + \log_a n \end{aligned}$$

$$\mathbf{\log_a \left(\frac{x}{y}\right) = \log_a x - \log_a y}$$

Proof: Let $x = a^m$ and $y = a^n$ (ie $\log_a x = m$ and $\log_a y = n$)

$$\begin{aligned} \text{Now } \frac{x}{y} &= a^m \div a^n \\ &= a^{m-n} \\ \log_a \left(\frac{x}{y}\right) &= m - n \\ &= \log_a x - \log_a y \end{aligned}$$



$$\mathbf{\log_a x^p = p \log_a x}$$

Proof: Let $x = a^m$ (ie $\log_a x = m$)

$$\begin{aligned} \text{Now } x^p &= (a^m)^p \\ &= a^{mp} \\ \therefore \log_a x^p &= mp \\ &= p \log_a x \end{aligned}$$

$\log_a 1 = 0 \qquad \log_a a = 1 \qquad \log_a a^x = x$

These three results follow from applying the definition of a logarithm to the three results $a^0 = 1$, $a^1 = a$ and $a^x = a^x$.

worked examples

The two main logarithm laws can be used to expand or contract expressions involving logarithms. Both processes are well illustrated in the following examples.

Contraction ←

$\log xy = \log x + \log y$

→ Expansion

1 Simplify:

a $\log_{10} 25 + \log_{10} 4$ **b** $\log_3 54 - \log_3 18$

2 Simplify:

a $5 \log_a a - \log_a a^4$ **b** $\dfrac{\log_a x^3}{\log_a \sqrt{x}}$

3 If $\log_a 3 = 1{\cdot}4$ and $\log_a 4 = 1{\cdot}6$, evaluate:

a $\log_a 12$ **b** $\log_a (1\frac{1}{3})$ **c** $\log_a 9$ **d** $\log_a 2$

4 Solve:

a $\log_{10} 2x + \log_{10} 5 = 3$ **b** $\log_5 x - \log_5 (x - 1) = 1$

5 Use the logarithm laws to expand $\log_a \left(\dfrac{x^2}{y}\right)$.

6 Use the logarithm laws to contract $\log_a x + \log_a y - 2 \log_a z$

Solutions

1 **a** $\log_{10} 25 + \log_{10} 4 = \log_{10} (25 \times 4)$
$= \log_{10} 100$
$= 2$

b $\log_3 54 - \log_3 18 = \log_3 (\frac{54}{18})$
$= \log_3 3$
$= 1$

2 **a** $5 \log_a a - \log_a a^4 = 5 \log_a a - 4 \log_a a$
$= \log_a a$
$= 1$

b $\dfrac{\log_a x^3}{\log_a \sqrt{x}} = \dfrac{\log_a x^3}{\log_a x^{\frac{1}{2}}}$
$= \dfrac{3 \log_a x}{\frac{1}{2} \log_a x}$
$= 6$

3 **a** $\log_a 12 = \log_a 3 + \log_a 4$
$= 1{\cdot}4 + 1{\cdot}6$
$= 3{\cdot}0$

b $\log_a (1\frac{1}{3}) = \log_a 4 - \log_a 3$
$= 1{\cdot}6 - 1{\cdot}4$
$= 0{\cdot}2$

$1\frac{1}{3} = \frac{4}{3}$

c $\log_a 9 = \log_a 3^2$
$= 2 \log_a 3$
$= 2 \times 1{\cdot}4$
$= 2{\cdot}8$

d $\log_a 2 = \log_a 4^{\frac{1}{2}}$
$= \frac{1}{2} \log_a 4$
$= \frac{1}{2} \times 1{\cdot}6$
$= 0{\cdot}8$

continued →→→

4 a $\log_{10} 2x + \log_{10} 5 = 3$

ie $\log_{10} (2x \times 5) = 3$

$\log_{10} 10x = 3$

so $10x = 10^3$

$= 1000$

$\therefore x = 100$

b $\log_5 x - \log_5 (x - 1) = 1$

ie $\log_5 \left(\frac{x}{x-1}\right) = 1$

so $\frac{x}{x-1} = 5$

$x = 5x - 5$

$\therefore x = \frac{5}{4}$

5 $\log_a \frac{x^2}{y} = \log_a x^2 - \log_a y$

$= 2 \log_a x - \log_a y$

6 $\log_a x + \log_a y - 2 \log_a z = \log_a xy - \log_a z^2$

$= \log_a \frac{xy}{z^2}$

Exercise 8:06

1 Evaluate:

- a $\log_6 9 + \log_6 4$
- b $\log_5 20 - \log_5 4$
- c $\log_2 48 - \log_2 3$
- d $\log_{10} 2 + \log_{10} 5$
- e $\log_a 4 + \log_a (\frac{1}{4})$
- f $\log_5 1000 - \log_5 8$
- g $\log_2 18 - 2 \log_2 3$
- h $2 \log_{10} 5 + \log_{10} 40$
- i $\log_3 24 - 3 \log_3 2$
- j $\log_{10} 125 - \log_{10} 4 + \log_{10} 32$
- k $3 \log_2 4 + \frac{1}{2} \log_2 81 - \log_2 18$

2 If $\log_a 2 = 0{\cdot}301$ and $\log_a 3 = 0{\cdot}477$, evaluate:

- a $\log_a 6$
- b $\log_a 1{\cdot}5$
- c $\log_a 9$
- d $\log_a 0{\cdot}5$
- e $\log_a \sqrt{3}$
- f $\log_a 18$
- g $\log_a 8$
- h $\log_a 24$
- i $\log_a 36$
- j $\log_a (\frac{2}{3})$
- k $\log_a (\frac{1}{4})$
- l $\log_a \sqrt{12}$

3 If $\log_x 3 = 1{\cdot}5$, $\log_x 5 = 1{\cdot}8$ and $\log_x 10 = 3{\cdot}0$, evaluate:

- a $\log_x 15$
- b $\log_x 2$
- c $\log_x 9$
- d $\log_x \sqrt{5}$
- e $\log_x 150$
- f $\log_x 250$
- g $\log_x 45$
- h $\log_x (\frac{1}{2})$
- i $\log_x 1000$
- j $\log_x 6$
- k $\log_x \sqrt{50}$
- l $\log_x 1{\cdot}5$

4 Simplify:

- a $\log_a (a^2)$
- b $\log_a 5a - \log_a 5$
- c $4 \log_a a - \log_a (a^4)$
- d $\frac{\log_a (x^5)}{\log_a x}$
- e $\log_a \left(\frac{x}{y}\right) + \log_a \left(\frac{y}{x}\right)$
- f $\frac{\log_a (x^3) - \log_a (x^2)}{\log_a \sqrt{x}}$

5 Use the logarithm laws to expand the following expressions.

- a $\log_a \frac{xy}{z}$
- b $\log_a \frac{xa}{z^2}$
- c $\log_a \frac{x}{(x+1)^2}$
- d $\log_a x\sqrt{x+1}$
- e $\log_y \sqrt{\frac{xy}{y+1}}$
- f $\log_y \frac{x(a+b)}{y^2}$

6 Use the logarithm laws to contract the following expressions.

a $\log_a x - \log_a y + 2\log_a z$

b $2\log_a x - \frac{1}{2}\log_a y + \log_a z$

c $\frac{1}{2}\log_a x + \frac{1}{2}\log_a y$

d $\log_a (x+1) - \log_a x + \log_a (x+2)$

e $2\log_y x - \log_y z + \log_y y^3$

f $2\log_a x + \log_a y - 3$

7 Find a relationship between x and y that does not involve logarithms.

a $\log_a x + \log_a y = \log_a (x+y)$

b $2\log_a x + \log_a 5 = \log_a y$

c $2\log_2 x + \log_2 y = 3$

d $\log_5 x = 3 + \log_5 y$

e $\log_a (1+x) - \log_a (1-x) = \log_a y$

f $\log_a y = \log_a 5 + 7\log_a x$

8 Solve the following equations for x.

a $\log_a x = \log_a 5 + \log_a 2$

b $\log_a x = \log_a 10 - \log_a 5$

c $\log_a x = 3\log_a 2 + \log_a 4$

d $\log_a x = \frac{1}{2}\log_a 9 - \log_a 2$

e $\log_a 2 + \log_a x = \log_a 8$

f $\log_a 10 - \log_a x = \log_a 5$

g $\log_a x + \log_a 3 = \log_a (x+1)$

h $\log_a (4x) - \log_a 3 = \log_a (x+4)$

9 Find values for x if:

a $\log_{10} 2 + \log_{10} x = 2$

b $\log_2 x - \log_2 3 = 3$

c $\log_{10} x = \dfrac{\log_{10} 8}{\log_{10} 2}$

d $\log_{10} x + 3 = 4\log_{10} x$

e $\log_{10} x - \log_{10} (x-1) = 1$

f $\log_{10} x = \log_{10} 5 - \log_{10} 2 - 1$

g $\log_2 x + \log_2 (x+1) = 1$

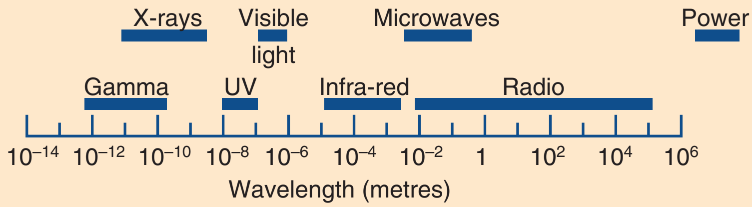
h $\log_3 2x + \log_3 (x-1) = 1$

investigation 8:06

Investigation 8:06 | Logarithmic scales

In most measurement situations, the measuring scales are linear. In some situations, however, a linear scale would produce a large range of measurements. To overcome this, scales based on indices are used. These are called logarithmic scales.

1 Electromagnetic spectrum



The diagram above shows the variation in wavelength for different types of electromagnetic radiation. The scale is a logarithmic scale.

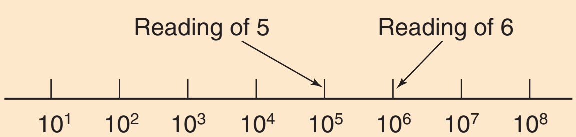
a Copy the scale and mark, approximately, the position on the scale of the numbers:
i 1000 **ii** 10 **iii** 0·1 **iv** 0·000 005

b The scale is marked with equally spaced markings, but the difference in value between markings is not equal. What is the difference in value of:
i 1 and 10^2? **ii** 10^4 and 10^6? **iii** 10^{-2} and 1?

c Where would zero be in relation to the numbers on this scale?

d What is the range in wavelength of:
i radio waves? **ii** visible light?

2 Richter scale
Another example of a logarithmic scale is the Richter scale, which measures the magnitude of earthquakes. In this question, we will assume that it is the same as the scale in the last question, in that every unit is ten times bigger than the unit before it. The scale is shown below.



a How many times bigger is:
i a reading of 6 than a reading of 5?
ii a reading of 7 than a reading of 5?

b Where would a reading of 5·5 be on the scale?

c How many times bigger is a reading of 5·5 than a reading of 5?

3 pH scale
The concentration of an acid is measured by its pH. (See page 132, question 6.)

8:07 | Simple Exponential Equations

prep quiz 8:07

Write as powers of 2:

1 4 **2** 8 **3** $\frac{1}{2}$ **4** $\sqrt{2}$

5 Write 125 as a power of 5.

6 Write 343 as a power of 3.

7 Write $\frac{1}{\sqrt{2}}$ as a power of 2.

8 Is $4^{-\frac{3}{2}} = \frac{1}{8}$?

9 If $x^3 = 4^3$, what is x?

10 If $2^x = 2^5$, find x.

An exponential equation is one in which the pronumeral to be found is involved in the 'exponent' or 'power'.

eg $2^x = 16$ or $5^{x-1} = 125$

For simple equations of this type, both sides of the equation are written as powers of the same number or base and then a comparison of powers is used to find the solution.

worked examples

Find the pronumeral in each of the following.

1 $2^x = 32$ **2** $3^{2x-1} = 27$ **3** $4^x = 8$ **4** $2^{x-1} = \frac{1}{4}$

Solutions

1 $2^x = 32$
ie $2^x = 2^5$
$\therefore x = 5$

2 $3^{2x-1} = 27$
ie $3^{2x-1} = 3^3$
so $2x - 1 = 3$
$\therefore x = 2$

3 $4^x = 8$
so $(2^2)^x = 2^3$
$2^{2x} = 2^3$
$2x = 3$
$\therefore x = \frac{3}{2}$ or $1\frac{1}{2}$

4 $2^{x-1} = \frac{1}{4}$
so $2^{x-1} = 2^{-2}$
$x - 1 = -2$
$\therefore x = -1$

Each side of the equation must be written with the same base; then the indices must be equal.

To solve simple exponential equations, rewrite each side of the equation with the same base; then equate the indices.

Exercise 8:07

1 Solve these equations.

a $2^x = 16$ **b** $2^x = 128$ **c** $3^x = 81$ **d** $3^x = 1$
e $4^x = 64$ **f** $5^x = 5$ **g** $6^x = 216$ **h** $2^x = 512$
i $3^x = 243$ **j** $4^x = 256$ **k** $10^x = 10\,000$ **l** $9^x = 729$
m $8^x = 4096$ **n** $11^x = 1331$ **o** $7^x = 16\,807$ **p** $6^x = 1296$

2 Find x.

a $4^x = 2$ **b** $9^x = 3$ **c** $16^x = 2$ **d** $25^x = 5$
e $27^x = 3$ **f** $32^x = 2$ **g** $16^x = 4$ **h** $100^x = 10$
i $2^x = \frac{1}{2}$ **j** $3^x = \frac{1}{9}$ **k** $4^x = \frac{1}{4}$ **l** $2^x = \frac{1}{4}$
m $5^x = \frac{1}{25}$ **n** $3^x = \frac{1}{27}$ **o** $4^x = \frac{1}{2}$ **p** $9^x = \frac{1}{3}$

3 Evaluate the pronumeral.

a $9^x = 27$ **b** $4^x = 32$ **c** $8^x = 4$ **d** $27^x = 81$
e $25^x = 125$ **f** $1000^x = 100$ **g** $128^x = 64$ **h** $32^x = 8$
i $8^x = \frac{1}{16}$ **j** $4^x = \frac{1}{8}$ **k** $27^x = \frac{1}{9}$ **l** $16^x = \frac{1}{8}$
m $(\frac{1}{4})^x = 8$ **n** $(\frac{1}{3})^x = 9$ **o** $(\frac{1}{9})^x = 27$ **p** $(0\cdot01)^x = 1000$

4 Solve:

a $2^{x-2} = 8$ **b** $3^{1-x} = 27$ **c** $2^{2x} = 32$
d $4^{2x-1} = 64$ **e** $4^{x-1} = \sqrt{2}$ **f** $(\sqrt{3})^x = 81$
g $2^{2-x} = \sqrt{8}$ **h** $5^{3x-2} = 125$ **i** $8^{2-x} = 4$
j $9^{1+2x} = 243$ **k** $8^{3x} = 32$ **l** $100^{3x-1} = 100\,000$
m $4^{x+1} = \dfrac{1}{8\sqrt{2}}$ **n** $8^{2x-1} = 4\sqrt{8}$ **o** $3^{2-x} = \dfrac{1}{3\sqrt{3}}$
p $9^{3x-5} = \dfrac{\sqrt{3}}{27}$ **q** $2^{2-x} = 4^{x-1}$ **r** $9^{3x} = 27^{x+1}$
s $4^{2x-1} = (\frac{1}{8})^x$ **t** $(4\sqrt{2})^{x-1} = (2\sqrt{8})^{1-x}$

Investigation 8:07 | Solving harder exponential equations by 'guess and check'

Please use the Assessment Grid on the following page to help you understand what is required for this Investigation.

Equations like $3^x = 5$ are more difficult than those in the previous exercise because the solution is obviously not an integer or simple fraction.

Later in this chapter, we will use logarithms to find the exact solution, but an approximate solution can be found by the 'guess and check' method.

Example

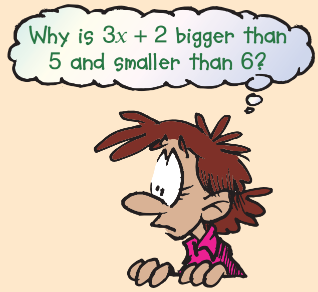
Find an approximate solution for $3^x = 5$.
Now $3^1 = 3$ and $3^2 = 9$.

As 5 is between 3 and 9, x must be between 1 and 2.

Now, trying $x = 1\cdot5$ gives $3^{1\cdot5} = 5\cdot196 \ldots$ which is too big.
Trying $x = 1\cdot4$ gives $3^{1\cdot4} = 4.655 \ldots$ which is too small.
$\therefore$ x must be between $1\cdot4$ and $1\cdot5$.

Now, trying $x = 1\cdot45$ gives $3^{1\cdot45} = 4\cdot918 \ldots$ which is too small.
$\therefore$ x must be between $1\cdot45$ and $1\cdot5$.

Repeating this procedure will quickly give the solution as $x = 1\cdot47$ correct to two decimal places.



Use the 'guess and check' method to solve:

1 $5^x = 500$

2 $3^x = 1690$

3 $7^{3x+2} = 19\,187$

Hint: If $3x + 2 > 5$ and $3x + 2 < 6$

then $x > 1$ and $x < 1\frac{1}{3}$.

$\therefore$ x is between 1 and $1\frac{1}{3}$.

4 $5^{2x-3} = 16\,755$

5 Suppose a new breed of rabbits triples its population every month. If there are three rabbits after one month the population after n weeks would be 3^n. Find the length of time, to the nearest day, for the population to reach:

- **a** 100 rabbits
- **b** 10000 rabbits

6 If a fungus growing over a pond doubles in size every day and takes 25 days to cover the entire pond, how many days did it take to cover half the pond?

Reflection:
Were your results easy to find and do they make sense?
Discuss the accuracy of your answers.
Can you describe any other real life applications for exponential functions?
Could you have improved your method in any way to make it faster or more efficient?

Assessment Grid for Investigation 8:07 | Solving harder exponential equations by 'guess and check'

The following is a sample assessment grid for this investigation. You should carefully read the criteria *before* beginning the investigation so that you know what is required.

		Assessment Criteria (C, D) for this investigation		Achieved ✓
Criterion C Communication in Mathematics	a	None of the following descriptors have been achieved.	0	
	b	There is a basic use of mathematical language and exponential notation. Lines of reasoning are insufficient.	1	
			2	
	c	There is satisfactory use of mathematical language and exponential notation. Explanations are clear but not always logical or complete.	3	
			4	
	d	There is a good use of mathematical language and exponential notation. Explanations and answers are complete and concise.	5	
			6	
Criterion D Reflection in Mathematics	a	None of the following descriptors have been achieved.	0	
	b	An attempt has been made to explain whether the results make sense, with connection to possible real-life applications.	1	
			2	
	c	There is a correct but brief explanation of whether results make sense and how they were found, including a good description in parts 5 and 6. Some consideration of the accuracy of the answers and the efficiency of the method is given.	3	
			4	
	d	There is a critical explanation of the results obtained including parts 5 and 6, with thorough consideration of their accuracy. The efficiency of the method is discussed with possible improvements and mention is made of other exponential applications.	5	
			6	

8:08 | Further Exponential Equations

In exercise 8:05, the solution of exponential equations of the form $a^x = b$ was investigated.

In all cases, b was able to be written as a power of a and comparison of the indices then allowed x to be found.

If b could not be written as a power of a, then we had to resort to the 'guess and check' method as outlined in Investigation 8:07.

Now, using the theory of logarithms, a quicker method can be found to solve these equations.

The method outlined below makes use of the log key on your calculator.

This key gives the logarithm of a number to the base 10.

worked examples

Solve the following:

1 $5^x = 12$ **2** $3^{x+2} = 7$ **3** $2^x = 5^{x-1}$

Solutions

The first step is to take logs of both sides, to base 10.

1
$$\begin{aligned}\log_{10} 5^x &= \log_{10} 12\\ x\log_{10} 5 &= \log_{10} 12\\ x &= \frac{\log_{10} 12}{\log_{10} 5}\\ &= 1\cdot54 \text{ (to 2 dec. pl.)}\end{aligned}$$

2
$$\begin{aligned}\log_{10} 3^{x+2} &= \log_{10} 7\\ (x+2)\log_{10} 3 &= \log_{10} 7\\ x+2 &= \frac{\log_{10} 7}{\log_{10} 3}\\ x+2 &= 1\cdot77 \text{ (to 2 dec. pl.)}\\ \therefore x &= -0\cdot23\end{aligned}$$

3
$$\begin{aligned}\log_{10} 2^x &= \log_{10} 5^{x-1}\\ x\log_{10} 2 &= (x-1)\log_{10} 5\\ &= x\log_{10} 5 - \log_{10} 5\\ \text{ie } x\log_{10} 5 - x\log_{10} 2 &= \log_{10} 5\\ x(\log_{10} 5 - \log_{10} 2) &= \log_{10} 5\\ \therefore x &= \frac{\log_{10} 5}{\log_{10} 5 - \log_{10} 2}\\ \therefore x &= 1\cdot76 \text{ (to 2 dec. pl.)}\end{aligned}$$

Exercise 8:08

1 Solve x (give answers correct to 3 dec. pl.).

a $10^x = 700$ **b** $10^x = 41\cdot6$ **c** $10^x = 49\ 168$

d $2^x = 7$ **e** $5^x = 100$ **f** $6^x = 2$

2 Solve for x (give answers correct to 3 dec. pl.).

a $10^{2x-3} = 1500$ **b** $5^{x+1} = 8$ **c** $2^{x-1} = 12$

d $3^{2x} = 15$ **e** $9^{2x-1} = 900$ **f** $5^{1-3x} = 27$

3 Determine the value of t in each of the following (correct to 3 dec. pl.).

a $1000 = 1{\cdot}8(10^{2t})$ **b** $1\,000\,000 = 100(2^{0{\cdot}8t})$ **c** $6(3^{2t-1}) + 3 = 27$

4 Determine the value of x in the following, giving answers correct to three decimal places.

a $3^x = 6^{x-1}$ **b** $2^x = 7^{x-3}$ **c** $3^{2x} = 7^x$

d $5^{x+1} = 8^{x-1}$ **e** $6^{x+2} = 10^{x-7}$ **f** $12^{2x} = 9^{x+7}$

Investigation 8:08 | Logarithmic scales and the history of calculating

The use of electronic calculators to do mathematical calculations only became common during the late 1970s.

Prior to then, many calculations were done using *logarithm tables*. Another common calculating device was a *slide-rule*. This consisted of a pair of logarithmic scales — one fixed and the other sliding.

Logarithms and logarithmic scales are still important, even though they are not as widely used as they once were.

1 Investigate the use of logarithms as a calculating tool. See if you can find a set of logarithm tables and find out how they were used.

2 Investigate the use of a slide-rule and try to construct one yourself using logarithmic graph paper.

Mathematical Terms 8

dependent variable

- A variable, the value of which depends on the value of another variable.
  eg for $y = x^2 + 1$
  y is the dependent variable because its value depends on the value of x.

exponent

- Another name for a power or index.
  eg for 2^3, the number 3 is an exponent.

exponential equation

- An equation where the unknown is part of an exponent.
  eg $2^x = 8$ or $3^{x-1} = 7$.

function

- A special rule or relationship that assigns to every input value a unique output value.

function notation

eg $f(x) = x^2 + 1$ tells us that the function f turns the input value x into the output value $x^2 + 1$.

independent variable

- A variable, the value of which does *not* depend on the value of any other variable.
  eg for $y = x^2 + 1$
  x is the independent variable.

inverse function

- Denoted by f^{-1}.
- An inverse function f^{-1} reverses the function f.
  eg if $f(x) = 2x + 1$
  then $f^{-1}(x) = \frac{x-1}{2}$.

logarithm

- An index.
- If $y = a^x$, then $\log_a y = x$.
  eg if $32 = 2^5$ then $\log_2 32 = 5$.

mathematical terms 8

Technology Applications — Mathematical terms 8 — Drag and Drops

- The nautilus shell is in the shape of a logarithmic spiral.

Diagnostic Test 8 | Functions and Logarithms

- These questions reflect the important skills introduced in this chapter.
- Errors made will indicate areas of weakness.
- Each weakness should be treated by going back to the section listed.

Section

1 **a** If $g(x) = 3x - 1$, find $g(3)$. **b** If $f(x) = x^2 + x$, find $f(-1)$. — 8:01

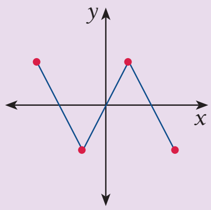
c If $H(a) = 3a + a^2$, find $H(\frac{1}{2})$.

2 If $f(p) = 2p - 5$, write an expression for: — 8:01

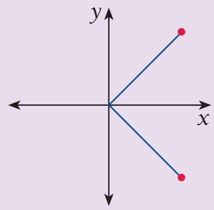
a $f(2p)$ **b** $f(a + 1)$ **c** $f(x^2)$

3 Which of the following represent graphs of functions? — 8:01

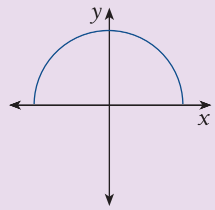
a



b

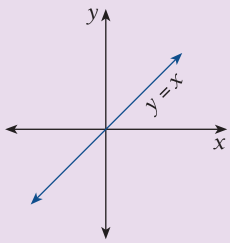


c



4 Write down the possible x and y values for the following. — 8:01

a



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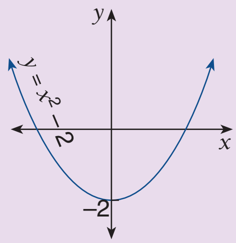
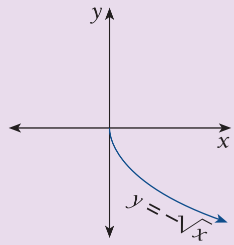


c



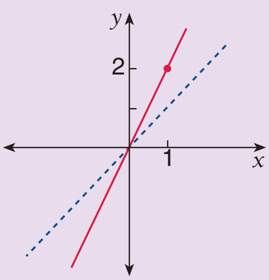
5 **a** If $y = 2x - 1$, find the inverse function. — 8:02

b If $f(x) = 3 - 2x$, find $f^{-1}(x)$. **c** If $g(x) = \frac{3x + 1}{2}$, find $g^{-1}(x)$.

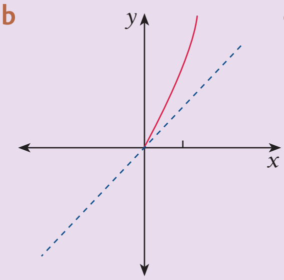
6 Which of the functions in question **4** would have an inverse function? — 8:02

7 Copy the following and add a sketch of the inverse function. — 8:02

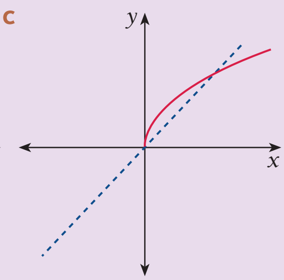
a



b



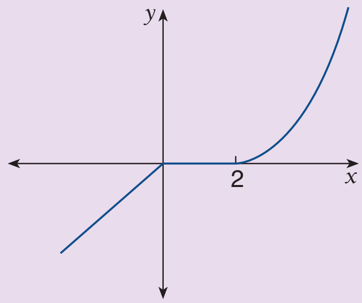
c



Section

8 Use the sketch of $y = H(x)$ to sketch: 8:03

a $y = H(x) + 1$ **b** $y = H(x - 1)$ **c** $y = H(x + 1)$ **d** $y = H(x) - 1$



9 Rewrite each expression in the form $x = a^y$. 8:04

a $\log_2 8 = 3$ **b** $\log_3 9 = 2$ **c** $\log_4 2 = \frac{1}{2}$ **d** $\log_5 1 = 0$

10 Rewrite each expression in the form $\log_a x = y$. 8:04

a $2^5 = 32$ **b** $4^3 = 64$ **c** $27^{\frac{1}{3}} = 3$ **d** $10^{-1} = 0{\cdot}1$

11 Evaluate: 8:04

a $\log_3 3$ **b** $\log_5 25$ **c** $\log_{10} 1000$ **d** $\log_2 \frac{1}{8}$

12 Find x. 8:04

a $\log_9 81 = x$ **b** $\log_x 4 = 2$ **c** $\log_2 x = 5$ **d** $\log_x 7 = 1$

13 Simplify: 8:06

a $\log_a 7 + \log_a 3$ **b** $\log_a 15 - \log_a 5$ **c** $\log_a 6 + \log_a 10$

d $\log_a 20 - \log_a 4$

14 Rewrite in the form $n \log_a 3$: 8:06

a $\log_a 3^4$ **b** $\log_a 9$ **c** $\log_a \sqrt{3}$ **d** $\log_a \frac{1}{3}$

15 Use the logarithm laws to write in expanded form: 8:06

a $\log_a xy^2$ **b** $\log_a \sqrt{xy}$ **c** $\log_a \frac{x}{\sqrt{y}}$

16 Use the logarithm laws to write in contracted form: 8:06

a $\log_a x - \log_a y - \log_a z$

b $2 \log_a x + 3 \log_a y + \frac{1}{2} \log_a z$

c $\log_a x + \frac{1}{2} \log_a (x - 1) - \frac{1}{2} \log_a (x + 1)$

17 Solve these equations. 8:07

a $2^x = 128$ **b** $2^x = \frac{1}{4}$ **c** $10^{1-x} = 1000$ **d** $9^{2x-1} = 3$

18 Solve for x (correct to 3 dec. pl.). 8:08

a $10^x = 20$ **b** $5^x = 2$ **c** $6^{3x-1} = 18$

assignment

8A

Chapter 8 | Revision Assignment

1 A function is defined by $f(x) = 2x + 1$.

a Find $f(-2)$.

b If $f(a) = -5$, find a.

c Find $f^{-1}(x)$ and hence find $f^{-1}(2)$.

2 Sketch the function $f(x) = x^2$ for the domain $-2 \leqslant x \leqslant 2$. For what values of x will $f(x) = 2$? Does this function have an inverse?

3 Solve:

a $5^x = 625$ b $9^x = 3^5$

c $4^x = 8$ d $3^{1-x} = \frac{1}{27}$

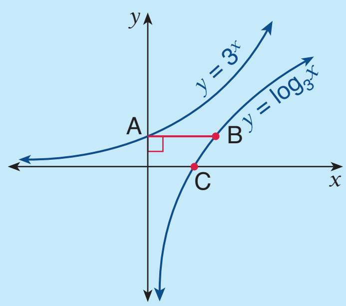
4 If $\log_x 5 = 0{\cdot}56$, find:

a $\log_x 25$ b $\log_x \sqrt{5}$

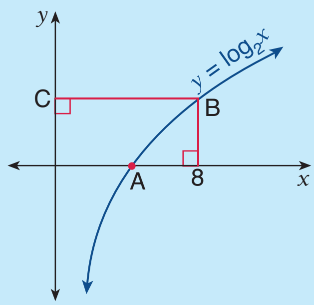
c $\log_x 5x$ d $\log_x 0{\cdot}2$

5 For each graph give the coordinates of A, B and C.

a



b



6 Solve:

a $5^x = 17$ b $6^{x+1} = 27$

c $3^{2x-1} = 147$ d $8^x = 5^{x+1}$

e $9^{x-1} = 4^{x+1}$ f $12^{2x+1} = 8^{3x}$

7 Simplify:

a $\log(x^2 - x - 2) - \log(x + 1)$

b $\log\sqrt{x^2 - 4x + 4} - \log(x - 2)$

8 Find the value of m if:

a $\log_a 4m - \log_a 3 = \log_a (m + 4)$

b $\log_a x + \log_a (x - 2) = \log_a 3$

9 Find a relationship between x and y, not involving logarithms.

a $\log x - \log y = \log (x + y)$

b $\log\left(\frac{x^2}{y}\right) = \log 2$

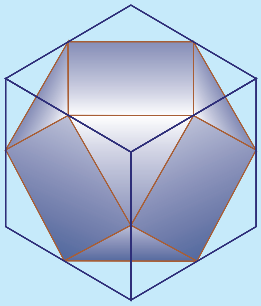
c $5 \log x + 3 \log y = \log 2$

• San Francisco suffers from earthquakes. Earthquake intensity is measured on a logarithmic scale called the Richter Scale.

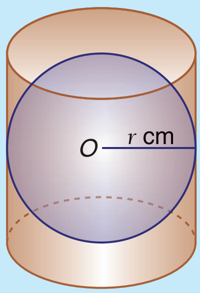
Chapter 8 | Working Mathematically

assignment 8B

1 A solid is formed from a cube by cutting off the corners in such a way that the vertices of the new solid will be at the midpoints of the edges of the original cube. How many edges will the new solid have?



2 The diagram shows a sphere of radius r cm which fits inside a cylinder. The cylinder has the same diameter and height as the sphere. Calculate (in terms of r):



a the volume of the sphere
b the volume of the cylinder
c the volume of the cylinder not occupied by the sphere
d the volume of a cone with a radius of r cm and a height of $2r$ cm.
e What relationship can you see between the answers to parts **a**, **b** and **d**?

3 John is given $10 each month for a year. Anna is given 5 cents for the first month, 10 cents for the second, 20 cents for the third and so on until the twelfth month. Who is given the most in the year and by how much is the amount bigger?

4 The table below has been prepared from information supplied by the Federal Office of Road Safety in a country with right-hand drive vehicles. It gives the number of car occupants in various seating positions and the number killed as a direct result of an impact to the side of the car. The figures exclude 38 occupants as there was insufficient information to determine their exact seating positions at the time of the crash.

a How many occupants were in:
  i the driver's seat?
  ii the front left side seat?
  iii a front seat?
  iv a passenger's side seat?
b What percentage of drivers were killed?
c What percentage of all the occupants who were killed were drivers?
d For each of the six seating positions, calculate the number of deaths as a percentage of the total number of people in that seat. Can you infer anything about the relative safety of the various seating positions from this information? Give reasons for your answer.

	Seating position						*All occupants*	
	Left side (passenger's side)		*Centre seat*		*Right side (driver's side)*			
	Dead	*Total*	*Dead*	*Total*	*Dead*	*Total*	*Dead*	*Total*
Front seat	114	187	0	1	199	316	313	504
Rear seat	29	63	2	19	20	44	51	126
All occupants	143	250	2	20	219	360	364	630

Chapter Review Questions

Technology Applications Drag and Drops
1 Logarithms
2 Function notation

Technology Applications Activities
Radioactive decay

9 Matrices

Chapter Contents

Learning Outcomes

Students will be able to:

- Arrange information into a rectangular array.
- Perform operations with matrices.
- Understand the significance of the identity and zero matrices.
- Calculate the determinant of a 2 × 2 matrix.
- Use the conditions for the existence of the inverse of a matrix.
- Find the inverse of a matrix.
- Solve systems of simultaneous equations using inverse matrices.

Areas of Interaction

Approaches to Learning (Knowledge Acquisition, Reflection, Organisation, Technology), Human Ingenuity

9:01 | Defining a Matrix, and Some Matrix Operations

A matrix is a rectangular array of numbers that usually represents some information.

These numbers (called elements) are arranged in rows and columns.

A common example of a matrix is a spreadsheet.

worked example

Michal delivers pizzas on Friday nights. He records his deliveries in the table as shown:

	Supreme	*Hawaiian*	*Mexican*	*Marghareta*
Delivery 1	2	0	3	0
Delivery 2	1	0	2	0
Delivery 3	0	2	1	0
Delivery 4	0	1	1	1
Delivery 5	3	3	2	2
Delivery 6	2	1	1	0
Delivery 7	1	1	3	0

This information can be arranged in a matrix:
A matrix is generally represented by a capital letter in bold type (as shown).
To refer to an element in the matrix, the row and column number is used.
For example, in row 5 column 3 is the element 2.

$$\mathbf{P} = \begin{bmatrix} 2 & 0 & 3 & 0 \\ 1 & 0 & 2 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 3 & 3 & 2 & 2 \\ 2 & 1 & 1 & 0 \\ 1 & 1 & 3 & 0 \end{bmatrix}$$

The order of a matrix is determined by the number of rows × the number of columns.

So matrix $\mathbf{P}$ is a 7×4 matrix.

A matrix that has the same number of rows as columns is called a square matrix.

continued →→→

Suppose on the next night Bob also makes 7 deliveries and again only delivers the same 4 types of pizza as shown in matrix $\boldsymbol{Q}$.

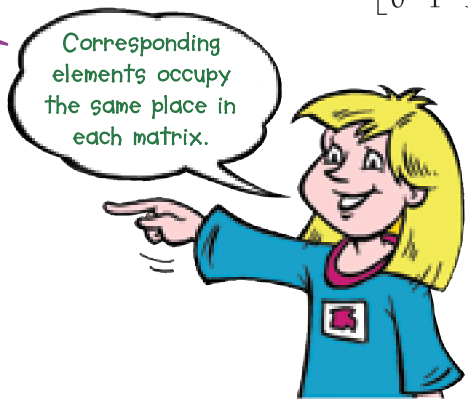
$$\boldsymbol{Q} = \begin{bmatrix} 3 & 2 & 0 & 1 \\ 2 & 2 & 0 & 1 \\ 1 & 1 & 0 & 1 \\ 3 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 3 & 1 \end{bmatrix}$$

To find out how many of each type of pizza he delivered over the 2 nights, we must add together the corresponding elements in each matrix.

This would then be called matrix $\boldsymbol{P} + \boldsymbol{Q}$.

$$\boldsymbol{P} + \boldsymbol{Q} = \begin{bmatrix} 2 & 0 & 3 & 0 \\ 1 & 0 & 2 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 3 & 3 & 2 & 2 \\ 2 & 1 & 1 & 0 \\ 1 & 1 & 3 & 0 \end{bmatrix} + \begin{bmatrix} 3 & 2 & 0 & 1 \\ 2 & 2 & 0 & 1 \\ 1 & 1 & 0 & 1 \\ 3 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 3 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 5 & 2 & 3 & 1 \\ 3 & 2 & 2 & 1 \\ 1 & 3 & 1 & 1 \\ 3 & 2 & 2 & 1 \\ 4 & 3 & 3 & 3 \\ 3 & 1 & 2 & 1 \\ 1 & 2 & 6 & 1 \end{bmatrix}$$



- It is only possible to add or subtract matrices that have the same order.

  This means that they must have the same number of rows and columns.

The same rule applies when subtracting matrices.

$$\boldsymbol{P} - \boldsymbol{Q} = \begin{bmatrix} 2 & 0 & 3 & 0 \\ 1 & 0 & 2 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 3 & 3 & 2 & 2 \\ 2 & 1 & 1 & 0 \\ 1 & 1 & 3 & 0 \end{bmatrix} - \begin{bmatrix} 3 & 2 & 0 & 1 \\ 2 & 2 & 0 & 1 \\ 1 & 1 & 0 & 1 \\ 3 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 3 & 1 \end{bmatrix} = \begin{bmatrix} -1 & -2 & 3 & -1 \\ -1 & -2 & 2 & -1 \\ -1 & 1 & 1 & -1 \\ -3 & 0 & 0 & 1 \\ 2 & 3 & 1 & 1 \\ 1 & 1 & 0 & -1 \\ 1 & 0 & 0 & -1 \end{bmatrix}$$

If Bob had the same deliveries three nights in a row then the total number of pizzas he delivered could be represented by the following:

$$\begin{bmatrix} 2 & 0 & 3 & 0 \\ 1 & 0 & 2 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 3 & 3 & 2 & 2 \\ 2 & 1 & 1 & 0 \\ 1 & 1 & 3 & 0 \end{bmatrix} + \begin{bmatrix} 2 & 0 & 3 & 0 \\ 1 & 0 & 2 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 3 & 3 & 2 & 2 \\ 2 & 1 & 1 & 0 \\ 1 & 1 & 3 & 0 \end{bmatrix} + \begin{bmatrix} 2 & 0 & 3 & 0 \\ 1 & 0 & 2 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 3 & 3 & 2 & 2 \\ 2 & 1 & 1 & 0 \\ 1 & 1 & 3 & 0 \end{bmatrix} = 3\boldsymbol{P} = 3\begin{bmatrix} 2 & 0 & 3 & 0 \\ 1 & 0 & 2 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 3 & 3 & 2 & 2 \\ 2 & 1 & 1 & 0 \\ 1 & 1 & 3 & 0 \end{bmatrix}$$

To evaluate the answer we could add together all the corresponding elements, or simply multiply the matrix $\boldsymbol{P}$ by 3.

$$\text{So that } 3\boldsymbol{P} = \begin{bmatrix} 6 & 0 & 9 & 0 \\ 3 & 0 & 6 & 0 \\ 0 & 6 & 3 & 0 \\ 0 & 3 & 3 & 3 \\ 9 & 9 & 6 & 6 \\ 6 & 3 & 3 & 0 \\ 3 & 3 & 9 & 0 \end{bmatrix}$$

When multiplying a matrix by a number, multiply all elements of the matrix by that number.

Exercise 9:01

1 Perform the following matrix operations.

a $\begin{bmatrix} 1 & 3 \\ 2 & 2 \\ 3 & 2 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ -1 & 2 \\ -3 & -1 \end{bmatrix}$

b $\begin{bmatrix} 4 & -1 \\ 3 & 5 \end{bmatrix} - \begin{bmatrix} -1 & 3 \\ 3 & -2 \end{bmatrix}$

c $\begin{bmatrix} 1 & 3 & 0 \\ -2 & 5 & -1 \end{bmatrix} + \begin{bmatrix} 3 & -1 & 4 \\ 5 & 0 & 0 \end{bmatrix}$

d $\begin{bmatrix} 4 & 4 & 3 \\ 1 & 6 & 0 \\ -1 & 2 & -1 \end{bmatrix} + \begin{bmatrix} -1 & 0 & -2 \\ 2 & 4 & 1 \\ 0 & 6 & 3 \end{bmatrix} - \begin{bmatrix} -3 & 0 & -2 \\ 0 & 0 & 3 \\ 5 & 0 & 4 \end{bmatrix}$

e $\begin{bmatrix} -2 & 0 \\ 2 & 3 \end{bmatrix} - 2\begin{bmatrix} 1 & -3 \\ -5 & 0 \end{bmatrix}$

f $\begin{bmatrix} 2 & 1 & 0 \\ 3 & 0 & -1 \\ -1 & 2 & 5 \\ 5 & -3 & 4 \end{bmatrix} + \begin{bmatrix} -1 & -2 & 3 \\ 2 & 2 & 3 \\ 1 & 0 & 0 \\ -3 & 4 & -2 \end{bmatrix}$

g $\begin{bmatrix} 3 \\ 1 \\ -2 \\ 0 \end{bmatrix} - \begin{bmatrix} -2 \\ 6 \\ 4 \\ 1 \end{bmatrix}$

h $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

i $3\left[\begin{bmatrix} -2 & 0 \\ 1 & 3 \end{bmatrix} - \begin{bmatrix} -1 & 1 \\ 4 & 0 \end{bmatrix}\right]$

j $\begin{bmatrix} 3 & 4 \\ -1 & 0 \end{bmatrix} - \begin{bmatrix} -2 & 0 \\ 3 & 1 \end{bmatrix} - \begin{bmatrix} 2 & 1 \\ -1 & 3 \end{bmatrix}$

2 If $\boldsymbol{A} = \begin{bmatrix} 2 & -1 \\ 3 & 1 \end{bmatrix}$, $\boldsymbol{B} = \begin{bmatrix} -3 & 0 \\ 5 & 2 \end{bmatrix}$ and $\boldsymbol{C} = \begin{bmatrix} 0 & 5 \\ -3 & -2 \end{bmatrix}$, calculate:

a $\boldsymbol{A} + \boldsymbol{B}$

b $\boldsymbol{C} - \boldsymbol{B}$

c $2\boldsymbol{A} - \boldsymbol{B}$

d $3\boldsymbol{C} + \boldsymbol{A}$

e $\boldsymbol{A} - \boldsymbol{B} - \boldsymbol{C}$

3 If $\boldsymbol{P} = \begin{bmatrix} 2 & -1 \\ 1 & 3 \\ 5 & 2 \end{bmatrix}$, $\boldsymbol{Q} = \begin{bmatrix} 2 & 6 & -1 \\ -3 & 3 & 0 \end{bmatrix}$, $\boldsymbol{R} = \begin{bmatrix} -1 & 0 \\ 5 & 0 \\ -2 & 3 \end{bmatrix}$ and $\boldsymbol{S} = \begin{bmatrix} -2 & -6 & 1 \\ 3 & -3 & 0 \end{bmatrix}$

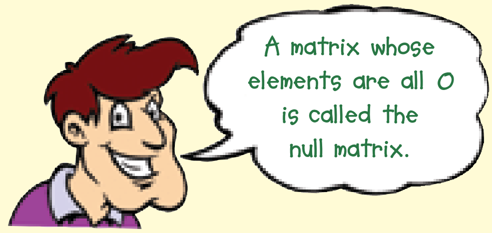
i calculate:

a $\boldsymbol{P} + \boldsymbol{R}$ **b** $\boldsymbol{Q} + \boldsymbol{S}$ **c** $\boldsymbol{S} - 2\boldsymbol{Q}$

d $\boldsymbol{R} - \boldsymbol{P}$ **e** $3\boldsymbol{P} - \boldsymbol{R}$

ii **a** Explain why it is not possible to add $\boldsymbol{P}$ and $\boldsymbol{Q}$.

b A matrix with all zeros is called a *null matrix*, which is denoted $\boldsymbol{0}$. Which of the answers for (i) gives the null matrix?



4 Find the value of the variables in the following:

a $\begin{bmatrix} x & -1 \\ 3 & y \end{bmatrix} + \begin{bmatrix} -2 & 4 \\ 5 & -2 \end{bmatrix} = \begin{bmatrix} 0 & 3 \\ 8 & -4 \end{bmatrix}$

b $\begin{bmatrix} 0 & b \\ -5 & d \end{bmatrix} - \begin{bmatrix} a & 4 \\ c & 3 \end{bmatrix} = \begin{bmatrix} -3 & 0 \\ -6 & -3 \end{bmatrix}$

c $\begin{bmatrix} p & 2 \\ 1 & q \\ 3 & -2 \end{bmatrix} - \begin{bmatrix} 5 & 4 \\ r & 0 \\ s & -1 \end{bmatrix} = \begin{bmatrix} 3 & t \\ 6 & -3 \\ 5 & -1 \end{bmatrix}$

d $\begin{bmatrix} k & 2 & -1 \\ 1 & l & 5 \end{bmatrix} + \begin{bmatrix} 4 & 3 & m \\ 3 & -1 & -2 \end{bmatrix} = \begin{bmatrix} 1 & 5 & -4 \\ 4 & 2 & n \end{bmatrix}$

e $2\begin{bmatrix} a & 3 \\ c & -1 \end{bmatrix} + \begin{bmatrix} 5 & b \\ \frac{1}{2} & d \end{bmatrix} = \begin{bmatrix} 9 & 6 \\ -1\frac{1}{2} & 1 \end{bmatrix}$

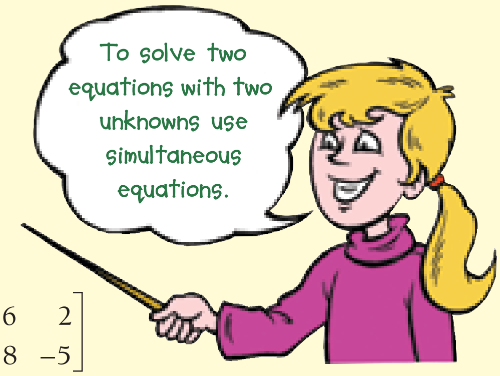
f $\begin{bmatrix} x & -3 \\ y & 2 \end{bmatrix} + \begin{bmatrix} x & 5 \\ 2 & z \end{bmatrix} = \begin{bmatrix} 5 & 2 \\ 1 & 2 \end{bmatrix}$

g $\begin{bmatrix} 2x & x \\ 5 & -1 \end{bmatrix} - \begin{bmatrix} y & 3y \\ 8 & -1 \end{bmatrix} = \begin{bmatrix} 4 & 7 \\ -3 & 0 \end{bmatrix}$

h $2\begin{bmatrix} a & b \\ c & d \end{bmatrix} + \begin{bmatrix} a & b \\ c & d \end{bmatrix} - 5\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} -6 & 2 \\ 8 & -5 \end{bmatrix}$

i $j\begin{bmatrix} 3 & 8 \\ -1 & 4 \end{bmatrix} - k\begin{bmatrix} 5 & -3 \\ 2 & 3 \end{bmatrix} = -1\begin{bmatrix} 13 & 2 \\ 3 & 10 \end{bmatrix}$

j $\begin{bmatrix} a & 6 \\ -3 & 2d \end{bmatrix} + 2\begin{bmatrix} 2a & b \\ c & -4d \end{bmatrix} = 3\begin{bmatrix} 5 & 4 \\ 0 & 4 \end{bmatrix}$



5 $\boldsymbol{A} = \begin{bmatrix} 1 & 5 & -2 \\ 3 & 3 & 0 \\ -2 & 0 & 3 \end{bmatrix}$ and $\boldsymbol{B} = \begin{bmatrix} 2 & 0 & -1 \\ 5 & -2 & 5 \\ -3 & 4 & 0 \end{bmatrix}$. If $2\boldsymbol{A} + \boldsymbol{B} - \boldsymbol{C} = \boldsymbol{0}$, find the matrix $\boldsymbol{C}$.

9:02 | Matrix Multiplication

Suppose we now look again at Bob's pizza delivery matrix $\boldsymbol{P}$.

$$\boldsymbol{P} = \begin{bmatrix} 2 & 0 & 3 & 0 \\ 1 & 0 & 2 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 3 & 3 & 2 & 2 \\ 2 & 1 & 1 & 0 \\ 1 & 1 & 3 & 0 \end{bmatrix}$$

Remember, the columns represent the types of pizza and the rows represent different deliveries.

We will now consider the cost of each pizza:

Supreme	\$12.50
Hawaiian	\$10.00
Mexican	\$10.50
Margharita	\$ 9.00

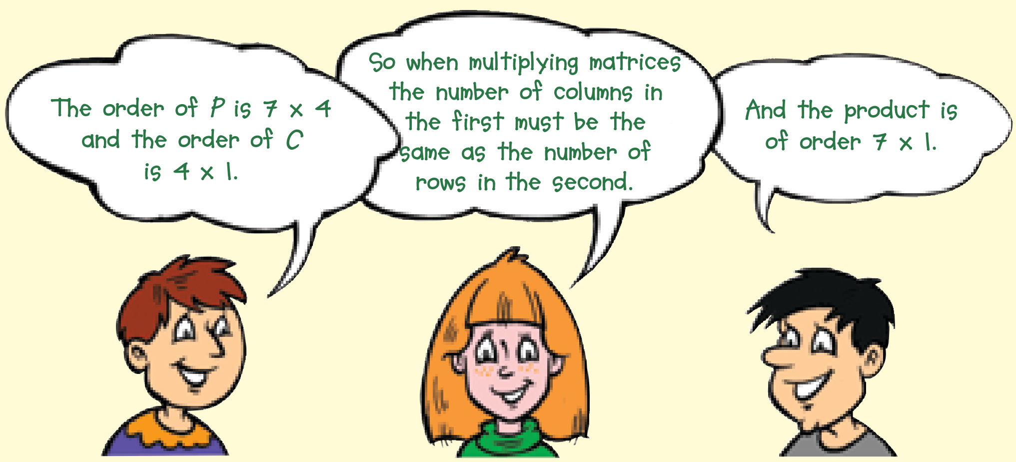
We will put the cost of each type of pizza in a new matrix $\boldsymbol{C}$: $\boldsymbol{C} = \begin{bmatrix} 12.50 \\ 10.00 \\ 10.50 \\ 9.00 \end{bmatrix}$

Now to figure out how much Bob collected in each delivery we must do the following:

$$\begin{bmatrix} 2 \times 12.50 + 0 \times 10.00 + 3 \times 10.50 + 0 \times 9.00 \\ 1 \times 12.50 + 0 \times 10.00 + 2 \times 10.50 + 0 \times 9.00 \\ 0 \times 12.50 + 2 \times 10.00 + 1 \times 10.50 + 0 \times 9.00 \\ 0 \times 12.50 + 1 \times 10.00 + 1 \times 10.50 + 1 \times 9.00 \\ 3 \times 12.50 + 3 \times 10.00 + 2 \times 10.50 + 2 \times 9.00 \\ 2 \times 12.50 + 1 \times 10.00 + 1 \times 10.50 + 0 \times 9.00 \\ 1 \times 12.50 + 1 \times 10.00 + 3 \times 10.50 + 0 \times 9.00 \end{bmatrix} = \begin{bmatrix} 56.50 \\ 33.50 \\ 30.50 \\ 29.50 \\ 106.50 \\ 15.50 \\ 54.00 \end{bmatrix}$$

This is known as the product of $\boldsymbol{P}$ and $\boldsymbol{C}$.

So if $\boldsymbol{P} = \begin{bmatrix} 2 & 0 & 3 & 0 \\ 1 & 0 & 2 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 3 & 3 & 2 & 2 \\ 2 & 1 & 1 & 0 \\ 1 & 1 & 3 & 0 \end{bmatrix}$ and $\boldsymbol{C} = \begin{bmatrix} 12.50 \\ 10.00 \\ 10.50 \\ 9.00 \end{bmatrix}$ then $\boldsymbol{P} \times \boldsymbol{C} = \begin{bmatrix} 2 & 0 & 3 & 0 \\ 1 & 0 & 2 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 3 & 3 & 2 & 2 \\ 2 & 1 & 1 & 0 \\ 1 & 1 & 3 & 0 \end{bmatrix} \times \begin{bmatrix} 12.50 \\ 10.00 \\ 10.50 \\ 9.00 \end{bmatrix}$.



$$= \begin{bmatrix} 56.50 \\ 33.50 \\ 30.50 \\ 29.50 \\ 106.50 \\ 15.50 \\ 54.00 \end{bmatrix}$$

You will notice that: Each element in the first row is multiplied by 12.50
Each element in the second row is multiplied by 10.00
Each element in the third row is multiplied by 10.50
Each element in the fourth row is multiplied by 9.00

$$\text{So if } \boldsymbol{P} \times \boldsymbol{C} = \begin{bmatrix} 2 & 0 & 3 & 0 \\ 1 & 0 & 2 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 3 & 3 & 2 & 2 \\ 2 & 1 & 1 & 0 \\ 1 & 1 & 3 & 0 \end{bmatrix} \times \begin{bmatrix} 12.50 \\ 10.00 \\ 10.50 \\ 9.00 \end{bmatrix}$$

> ■ This can be seen as multiplying down the columns as shown here and then adding along the rows.
>
> $$\begin{matrix} \times 12.50 & \times 10.00 & \times 10.50 & \times 9.00 \end{matrix}$$
> $$\begin{bmatrix} 2 & 0 & 3 & 0 \\ 1 & 0 & 2 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 3 & 3 & 2 & 2 \\ 2 & 1 & 1 & 0 \\ 1 & 1 & 3 & 0 \end{bmatrix}$$

If there is a second or third column in the second matrix, we simply repeat the process and the answers become the second or third row respectively in the answer.

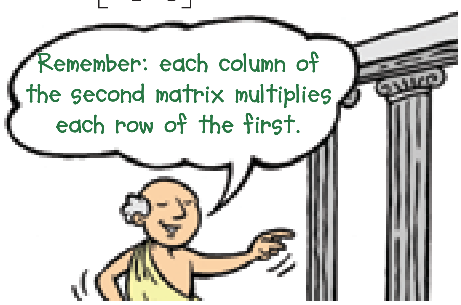
worked examples

Where possible, find the following products

1 $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \times \begin{bmatrix} -1 & 5 \\ 0 & 6 \end{bmatrix}$ **2** $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \times \begin{bmatrix} 0 & 3 \\ -2 & 2 \\ 1 & 3 \end{bmatrix}$ **3** $\begin{bmatrix} 0 & 3 \\ -2 & 2 \\ 1 & 3 \end{bmatrix} \times \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$

Solutions

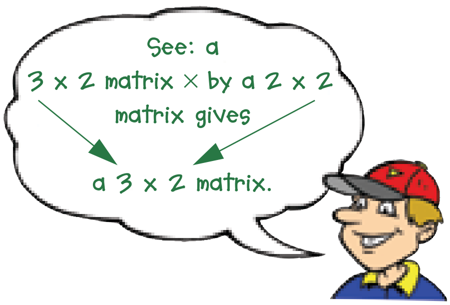
1
$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \times \begin{bmatrix} -1 & 5 \\ 0 & 6 \end{bmatrix} = \begin{bmatrix} 1 \times -1 + 2 \times 0 & 1 \times 5 + 2 \times 6 \\ 3 \times -1 + 4 \times 0 & 3 \times 5 + 4 \times 6 \end{bmatrix}$$
$$= \begin{bmatrix} -1 & 17 \\ -3 & 39 \end{bmatrix}$$



2
$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \times \begin{bmatrix} 0 & 3 \\ -2 & 2 \\ 1 & 3 \end{bmatrix}$$

Cannot be done since the first matrix is a 2×2 and the second is a 3×2. So the number of rows in the second is not the same as the number of columns in the first. *Therefore there is no solution.*

3
$$\begin{bmatrix} 0 & 3 \\ -2 & 2 \\ 1 & 3 \end{bmatrix} \times \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 0 \times 1 + 3 \times 3 & 0 \times 2 + 3 \times 4 \\ -2 \times 1 + 2 \times 3 & -2 \times 2 + 2 \times 4 \\ 1 \times 1 + 3 \times 3 & 1 \times 2 + 3 \times 4 \end{bmatrix}$$
$$= \begin{bmatrix} 9 & 12 \\ 4 & 4 \\ 10 & 14 \end{bmatrix}$$



Exercise 9:02

1 Calculate the following:

a $\begin{bmatrix} 3 & -1 \\ 2 & 0 \end{bmatrix} \times \begin{bmatrix} 4 & 1 \\ 5 & -2 \end{bmatrix}$

b $\begin{bmatrix} 2 & 0 \\ 1 & 3 \end{bmatrix} \times \begin{bmatrix} -2 & 3 \\ 1 & 5 \end{bmatrix}$

c $(2 \quad 3 \quad -4) \times \begin{bmatrix} -1 \\ 2 \\ 5 \end{bmatrix}$

d $\begin{bmatrix} 2 & 3 & -4 \\ -2 & 1 & 3 \end{bmatrix} \times \begin{bmatrix} -1 \\ 2 \\ 5 \end{bmatrix}$

e $\begin{bmatrix} 4 & 3 \\ 5 & 2 \end{bmatrix} \times \begin{bmatrix} 0 & 1 & -3 \\ -1 & 2 & 2 \end{bmatrix}$

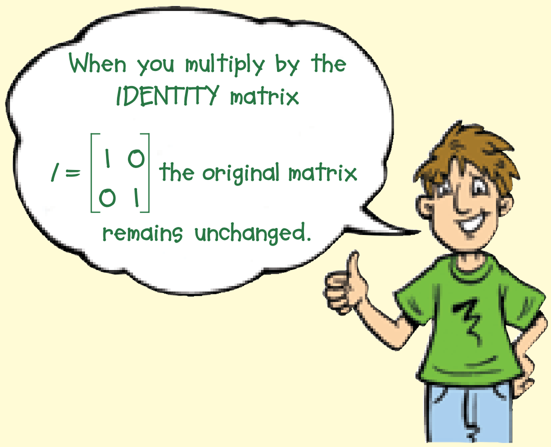
f $\begin{bmatrix} 2 & 1 \\ 5 & 0 \\ -1 & 3 \end{bmatrix} \times \begin{bmatrix} 2 & -2 & 0 \\ 3 & 5 & -1 \end{bmatrix}$

g $\begin{bmatrix} 2 & 1 \\ -2 & -1 \end{bmatrix} \times \begin{bmatrix} 1 & 3 \\ -2 & -6 \end{bmatrix}$

h $\begin{bmatrix} -2 & -1 \\ 3 & 2 \\ 1 & 4 \end{bmatrix} \times \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

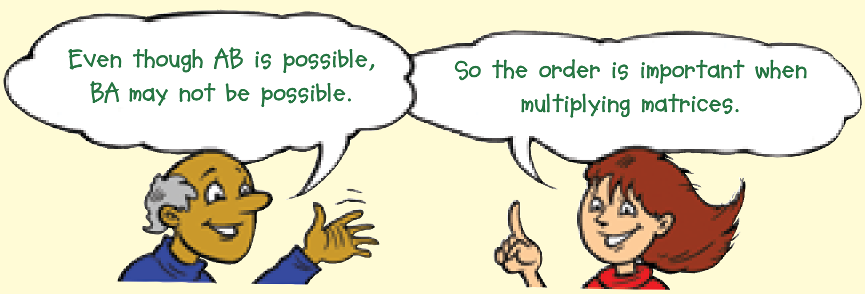
i $\begin{bmatrix} -2 & -1 \\ 3 & 2 \\ 1 & 4 \end{bmatrix} \times \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$

j $2\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \times \begin{bmatrix} -2 & -1 \\ 3 & 2 \\ 1 & 4 \end{bmatrix}$



2 Given that $A = \begin{bmatrix} 2 & 0 \\ -3 & 4 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 3 & -1 \\ -2 & 2 & 5 \end{bmatrix}$, $C = \begin{bmatrix} 3 \\ -2 \\ 0 \end{bmatrix}$ and $D = \begin{bmatrix} 1 & 0 \\ 1 & -2 \\ 2 & 3 \end{bmatrix}$,
calculate where possible:

a AB **b** BC **c** DA **d** BD **e** BA **f** A^2 **g** B^2



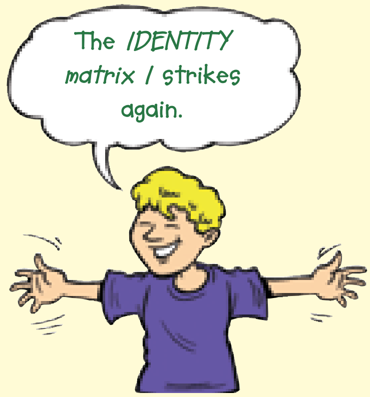
3 The matrix P has order 2×2, Q has order 3×2, R has order 2×4 and S has order 4×3. Where possible, give the order of the following:

a QP **b** PQ **c** QR **d** RQ **e** RS **f** SR **g** SQ
h QS **i** PR **j** RP

4 Find the value of a and b in the following:

a $\begin{bmatrix} 1 & 2 \\ 3 & -1 \end{bmatrix} \times \begin{bmatrix} a & 3a \\ 2b & b \end{bmatrix} = \begin{bmatrix} 9 & 7 \\ -1 & 7 \end{bmatrix}$

b $\begin{bmatrix} 1 & 2 \\ 3 & -1 \end{bmatrix} \times \begin{bmatrix} x & y \\ y & x \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 3 & -1 \end{bmatrix}$



5 If $X = \begin{bmatrix} 2 & 3 \\ -1 & 0 \end{bmatrix}$ show that $X^2 - 2X + 3I = 0$

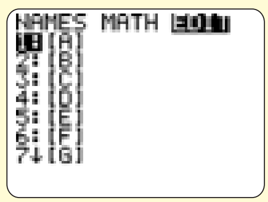
where I = the 2×2 identity matrix and 0 = the null matrix

9:03 | Matrix Operations and the GDC

The following directions are for the TI83, but there are similar steps when using other calculators.

Given two matrices: $A = \begin{bmatrix} 5 & 8 \\ 3 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & -1 \\ 3 & 5 \end{bmatrix}$ suppose we wish to find their product.

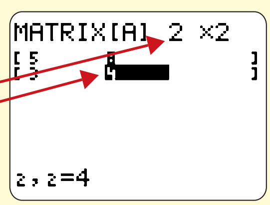
First we must enter the matrices into the calculator.
Using the MATRIX function we get the screen:

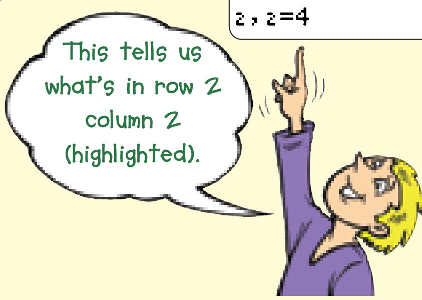


Using the edit function for matrix A we get the screen:

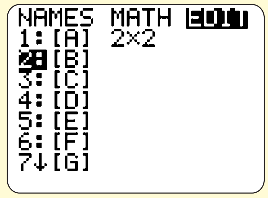
It is necessary to enter the order of the matrix

and the elements of the matrix.

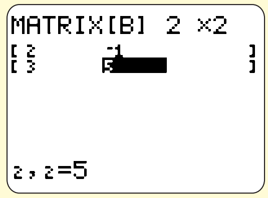




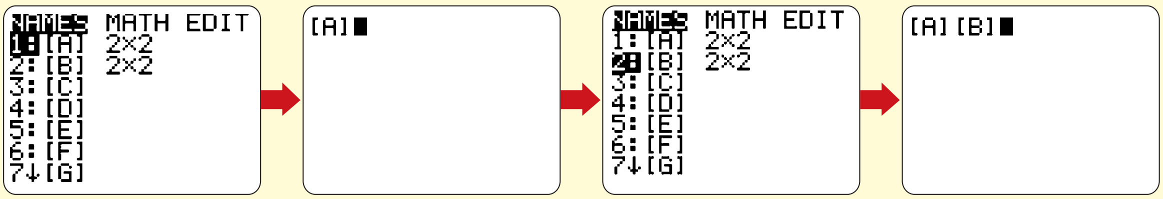
By quitting and then returning to the matrix function we choose to edit matrix ***B*** in the same way.



And enter the elements for matrix ***B***.

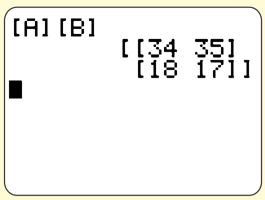


Quit and then choose the matrices to multiply by pressing enter.



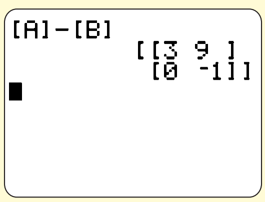
Now by pressing enter again, the calculator will calculate ***AB***:

$$\therefore \boldsymbol{AB} = \begin{bmatrix} 34 & 35 \\ 18 & 17 \end{bmatrix}$$



The same steps can be used when performing any operation simply by entering the operator between the two matrices. For example, to find ***A*** – ***B***

$$\therefore \boldsymbol{A} - \boldsymbol{B} = \begin{bmatrix} 3 & 9 \\ 0 & -1 \end{bmatrix}$$



Exercise 9:03

1 Given that $\mathbf{A} = \begin{bmatrix} 2 & -1 \\ 5 & 3 \end{bmatrix}$, $\mathbf{B} = \begin{bmatrix} 3 & 0 \\ 2 & 1 \\ -1 & -3 \end{bmatrix}$, $\mathbf{C} = \begin{bmatrix} 2 & 4 & 0 \\ -3 & 2 & 1 \end{bmatrix}$ and $\mathbf{D} = \begin{bmatrix} 4 \\ 3 \end{bmatrix}$

Perform, where possible, the following operations on your GDC.

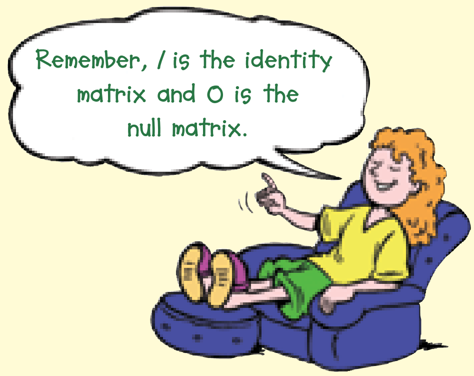
a $\mathbf{BA}$ b $\mathbf{CB}$ c $\mathbf{CB} - \mathbf{A}$ d $\mathbf{AD}$ e $\mathbf{AB}$ f $\mathbf{BD}$

2 a If $\mathbf{X} = \begin{bmatrix} 4 & -1 \\ -2 & 5 \end{bmatrix}$, use your GDC to calculate $\mathbf{X}^2$

b Use your GDC to calculate $\mathbf{X}^2 - 3\mathbf{X} + 2\mathbf{I}$ where $\mathbf{I}$ = the 2 × 2 identity matrix $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

3 a Given that $\mathbf{M} = \begin{bmatrix} 2 & 3 \\ 1 & 0 \end{bmatrix}$, use your GDC to calculate $\mathbf{M}^2$.

b Find the value of k such that $\mathbf{M}^2 - 2\mathbf{M} + k\mathbf{I} = 0$



investigation 9:03

Investigation 9:03 | Inverses and the GDC

Please use the Assessment Grid on page 246 to help you understand what is required for this Investigation.

1 Using your GDC complete the table below.
When calculating $\mathbf{A}^{-1}$, use the x^{-1} function on your calculator.

For example, if $\mathbf{A} = \begin{bmatrix} -3 & -4 \\ 5 & 6 \end{bmatrix}$ then $\mathbf{A}^{-1} = \begin{bmatrix} 3 & 2 \\ -2{\cdot}5 & -1{\cdot}5 \end{bmatrix}$

However by using *MATH* then *FRAC* functions the GDC will change the answer so the elements are fractions:

So that $\mathbf{A}^{-1} = \begin{bmatrix} 3 & 2 \\ -2{\cdot}5 & -1{\cdot}5 \end{bmatrix} = \begin{bmatrix} 3 & 2 \\ -\frac{5}{2} & -\frac{3}{2} \end{bmatrix}$

We can then take the fraction out of the brackets to write the inverse $A^{-1} = \frac{1}{2}\begin{bmatrix} 6 & 4 \\ -5 & -3 \end{bmatrix}$

	A	A^{-1}	$A\,A^{-1}$	$A\,A^{-1}$ = in the form $k\begin{bmatrix} p & q \\ r & s \end{bmatrix}$
a	$\begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix}$			
b	$\begin{bmatrix} 5 & 2 \\ 2 & 1 \end{bmatrix}$			
c	$\begin{bmatrix} 3 & 4 \\ 2 & 1 \end{bmatrix}$			
d	$\begin{bmatrix} -4 & -2 \\ 5 & 3 \end{bmatrix}$			
e	$\begin{bmatrix} -1 & 1 \\ 2 & 3 \end{bmatrix}$			
f	$\begin{bmatrix} 6 & 4 \\ 4 & 3 \end{bmatrix}$			
g	$\begin{bmatrix} -5 & 6 \\ 2 & -3 \end{bmatrix}$			

2 Describe any patterns you see and describe them with reference to the general 2×2 matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$

When looking for patterns try to see if the elements of the matrix have moved.
Also, look for how the fraction is obtained from the elements of the matrix in c, d, e, f and g.

3 Write a rule to show how to obtain the inverse of the general 2×2 matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$.

4 Describe how the patterns you found could be used to solve for x and y if $\begin{bmatrix} 2 & 1 \\ 3 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

Assessment Grid for Investigation 9:03 | Inverses and the GDC

The following is a sample assessment grid for this investigation. You should carefully read the criteria *before* beginning the investigation so that you know what is required.

		Assessment Criteria (B, C, D) for this investigation		Achieved ✓
Criterion B Investigating Patterns	a	None of the standards below has been reached.	0	
	b	Some help was needed to complete the tables and recognise patterns.	1	
			2	
	c	The tables have been completed independently. Patterns have been recognised and relationships suggested.	3	
			4	
	d	Tables have been completed and patterns recognised and described. Conclusions have been made that are consistent with the findings.	5	
			6	
	e	All of the above has been completed. In addition to this, patterns have been described in full using words and symbols and a logical response has been given to question 4.	7	
			8	
Criterion C Communication in Mathematics	a	None of the standards below has been reached.	0	
	b	There has been a basic use of mathematical language. Lines of reasoning are hard to follow.	1	
			2	
	c	The use of mathematical language is sufficient. Movement between the table and discussion has been achieved with some success. Lines of reasoning are clear but not always logical.	3	
			4	
	d	There is a good use of mathematical language and an effective movement between the table and discussion. Lines of reasoning are logical and complete.	5	
			6	
Criterion D Reflection in Mathematics	a	None of the standards below has been reached.	0	
	b	There has been an attempt to explain whether the results make sense in context.	1	
			2	
	c	The explanation of whether the findings make sense is correct but brief.	3	
			4	
	d	A full explanation of the findings has been given along with an explanation of how to apply the inverse to the problem in question 4.	5	
			6	

9:04 | The Determinant and Inverse of a 2 × 2 Matrix

The general 2 × 2 matrix is written $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$

The 2 × 2 identity matrix is written $\boldsymbol{I} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

The inverse of matrix $\boldsymbol{A}$ is written as $\boldsymbol{A}^{-1}$
When a matrix is multiplied by its inverse, the identity matrix results. So $\boldsymbol{A} \times \boldsymbol{A}^{-1} = \boldsymbol{I}$

- The inverse of matrix $\boldsymbol{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is given by $A^{-1} = \dfrac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$.

The expression $\boldsymbol{ad - bc}$ is called the **determinant** of the matrix.
This is because **when $\boldsymbol{ad - bc = 0}$ then the inverse does not exist**.

The determinant of matrix $\boldsymbol{A}$ is denoted as det$\boldsymbol{A}$ or $\begin{vmatrix} a & b \\ c & d \end{vmatrix}$.

worked examples

1 If $\boldsymbol{X} = \begin{bmatrix} 4 & 5 \\ 2 & 3 \end{bmatrix}$ find $\boldsymbol{X}^{-1}$, show that $\boldsymbol{X}\,\boldsymbol{X}^{-1} = \boldsymbol{I}$

2 If $\boldsymbol{B} = \begin{bmatrix} -1 & 3 \\ -2 & 5 \end{bmatrix}$, find $\boldsymbol{B}$ so that $\boldsymbol{AB} = \boldsymbol{I}$

3 Determine whether an inverse exists for each of the following matrices:

a $\begin{bmatrix} 2 & 5 \\ 3 & 4 \end{bmatrix}$

b $\boldsymbol{X} = \begin{bmatrix} 3 & 1 \\ 6 & 2 \end{bmatrix}$

4 Find the value of x if $\boldsymbol{G} = \begin{bmatrix} 3q & 2q \\ 6 & 5 \end{bmatrix}$ and det($\boldsymbol{G}$) = 64.

continued →→→

Solutions

1 $\boldsymbol{X} = \begin{bmatrix} 4 & 5 \\ 2 & 3 \end{bmatrix}$, $\therefore$

$$\boldsymbol{X}^{-1} = \frac{1}{12 - 10}\begin{bmatrix} 3 & -5 \\ -2 & 2 \end{bmatrix}$$

$$= \frac{1}{2}\begin{bmatrix} 3 & -5 \\ -2 & 4 \end{bmatrix}$$

$$\boldsymbol{X}\,\boldsymbol{X}^{-1} = \begin{bmatrix} 4 & 5 \\ 2 & 3 \end{bmatrix} \times \frac{1}{2}\begin{bmatrix} 3 & -5 \\ -2 & 4 \end{bmatrix}$$

$$= \frac{1}{2}\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \boldsymbol{I}$$

2 If $\boldsymbol{AB} = \boldsymbol{I}$ then

$$\boldsymbol{B} = \boldsymbol{A}^{-1} = \frac{1}{-5 + 6}\begin{bmatrix} 5 & -3 \\ 2 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 5 & -3 \\ 2 & -1 \end{bmatrix}$$

To check $\boldsymbol{AB} = \begin{bmatrix} -1 & 3 \\ -2 & 5 \end{bmatrix} \times \begin{bmatrix} 5 & -3 \\ 2 & -1 \end{bmatrix}$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \boldsymbol{I}$$

3 **a** $\begin{vmatrix} 2 & 5 \\ 3 & 4 \end{vmatrix} = 2 \times 4 - 5 \times 5$

$= 8 - 25$

$= -17$

$\therefore$ the inverse exists.

b Det $\boldsymbol{X} = 3 \times 2 - 6 \times 1$

$= 0$

$\therefore$ the inverse $(\boldsymbol{X}^{-1})$ does not exist.

4 $\det(\boldsymbol{G}) = 6$

$\therefore\ 15q - 12q = 6$

$\therefore\ 3q = 6$

$\therefore\ q = 2$

Using the GDC to find the determinant and the inverse of a matrix

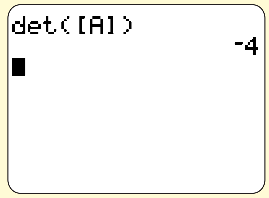
First enter the matrix into your calculator as described in the previous section.

Eg: $\boldsymbol{A} = \begin{bmatrix} 5 & 8 \\ 3 & 4 \end{bmatrix}$

In the matrix menu of the GDC choose *MATH* and the first option is *det(*

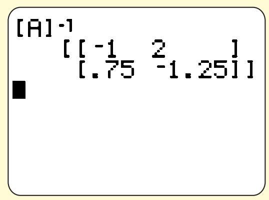
To find the determinant of a matrix that has already been entered, use this option:

$\therefore$ the determinant of matrix $\boldsymbol{A}$ is $\det(\boldsymbol{A}) = -4$



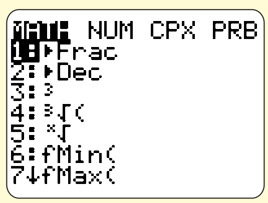
To find the inverse of matrix $\boldsymbol{A}$, use the $\boldsymbol{x}^{-1}$ function after the matrix has been entered.

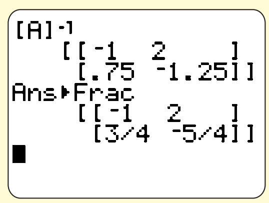
So the inverse of matrix $\boldsymbol{A}$ is $\boldsymbol{A}^{-1} = \begin{bmatrix} -1 & 2 \\ 0{\cdot}75 & -1{\cdot}25 \end{bmatrix}$



By using the *MATH* then *FRAC* functions, the GDC will change the answer so the elements are fractions:

So $\boldsymbol{A}^{-1} = \begin{bmatrix} -1 & 2 \\ \frac{3}{4} & -\frac{5}{4} \end{bmatrix}$





By taking $-\frac{1}{4}$ out as a common factor we get $\boldsymbol{A}^{-1} = -\frac{1}{4}\begin{bmatrix} 4 & -8 \\ -3 & 5 \end{bmatrix} = \frac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

Exercise 9:04

1 Use the GDC to find the determinant of the following matrices, and hence whether or not their inverse exists.

a $\begin{bmatrix} 2 & -1 \\ 4 & 3 \end{bmatrix}$ **b** $\begin{bmatrix} -1 & 2 \\ 3 & -6 \end{bmatrix}$ **c** $\begin{bmatrix} 3 & 9 \\ 2 & 6 \end{bmatrix}$

d $\begin{bmatrix} -1 & 3 \\ 4 & 0 \end{bmatrix}$ **e** $\begin{bmatrix} 3 & 1 \\ -2 & 3 \end{bmatrix}$

2 Find the inverse of the following matrices where possible.

a $\begin{bmatrix} -1 & -3 \\ 2 & 4 \end{bmatrix}$ **b** $\begin{bmatrix} 3 & -3 \\ -2 & 2 \end{bmatrix}$ **c** $\begin{bmatrix} -3 & -1 \\ 2 & 2 \end{bmatrix}$

d $\begin{bmatrix} 5 & 6 \\ 3 & 4 \end{bmatrix}$ **e** $\begin{bmatrix} 6 & 5 \\ 5 & 4 \end{bmatrix}$

3 Find the inverse of the following, writing your answer in the form $\frac{1}{k}\begin{bmatrix} p & q \\ r & s \end{bmatrix}$, where k = the determinant.

a $\begin{bmatrix} 3 & 4 \\ 2 & 2 \end{bmatrix}$ **b** $\begin{bmatrix} 5 & -4 \\ -4 & 3 \end{bmatrix}$ **c** $\begin{bmatrix} 1 & 2 \\ 0 & 4 \end{bmatrix}$ **d** $\begin{bmatrix} -3 & -2 \\ 6 & 2 \end{bmatrix}$ **e** $\begin{bmatrix} -7 & 5 \\ 4 & -4 \end{bmatrix}$

4 If $\boldsymbol{A} = \begin{bmatrix} 3 & -1 \\ 2 & 1 \end{bmatrix}$, $\boldsymbol{B} = \begin{bmatrix} 3 & 2 \\ 0 & -1 \end{bmatrix}$, $\boldsymbol{C} = \begin{bmatrix} -1 & 0 \\ 1 & 0 \end{bmatrix}$ and $\boldsymbol{D} = \begin{bmatrix} 1 & 2 \\ 0 & 4 \end{bmatrix}$, evaluate the following where possible, writing matrix answers in the form $\frac{1}{k}\begin{bmatrix} p & q \\ r & s \end{bmatrix}$, where k = the determinant.

a $\det(\boldsymbol{A})$ **b** $|\boldsymbol{B}|$ **c** $\det(\boldsymbol{AB})$ **d** $\det(\boldsymbol{C})$ **e** $\det(\boldsymbol{D})$ **f** $\det(\boldsymbol{CD})$

g $\det(\boldsymbol{AD})$ **h** $\det(\boldsymbol{BD})$ **i** $\boldsymbol{C}^{-1}$ **j** $\boldsymbol{DD}^{-1}$ **k** $\boldsymbol{D}^{-1}\boldsymbol{D}$

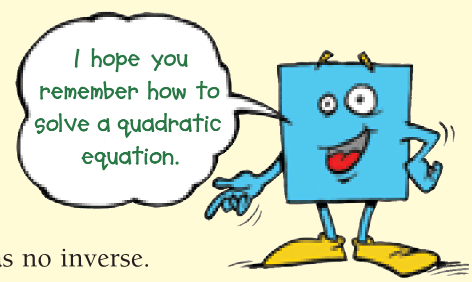
5 **a** What conclusion can be drawn from the answers **a** to **h** in the previous question?
b What conclusion can be drawn from the answers **j** and **k** in the previous question?

6 Use the information given in each of the following to find the possible values of the variables.

a $\begin{vmatrix} 2a & 5 \\ 3a & 4 \end{vmatrix} = -14$ **b** $\det\begin{bmatrix} x & 3 \\ -1 & 3x \end{bmatrix} = 15$

c $\boldsymbol{M} = \begin{bmatrix} p & 3 \\ 2p & p \end{bmatrix}$ and $\det(\boldsymbol{M}) = -8$

d $\begin{vmatrix} x-1 & 2 \\ 4 & x \end{vmatrix} = 4$ **e** The matrix $\begin{bmatrix} 2x+3 & 3 \\ x & x \end{bmatrix}$ has no inverse.



9:05 | Solving Matrix Equations

Just as with normal algebra, we can have matrix equations to solve. To do this we need to remember that:

- A matrix multiplied by its inverse gives the identity matrix ***I***.
- A matrix multiplied by its identity matrix remains unchanged.

worked examples

1 Find the values of x and y for which $\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 5 \\ 4 \end{bmatrix}$

2 Given that $\boldsymbol{A} = \begin{bmatrix} 1 & -3 \\ 2 & 0 \end{bmatrix}$ and $\boldsymbol{C} = \begin{bmatrix} 10 & -8 \\ 8 & 2 \end{bmatrix}$ find a matrix $\boldsymbol{B}$ such that $\boldsymbol{AB} = \boldsymbol{C}$

Solutions

1 In normal algebra we would divide both sides of the equation by $\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$
However, there is no division of matrices.
What we need to do is *PRE MULTIPLY* both sides of the equation by the inverse of the matrix $\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$
This means that the inverse comes *IN FRONT* of $\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$ and in front of $\begin{bmatrix} 5 \\ 4 \end{bmatrix}$
So to solve this equation we write:

$$\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}^{-1}\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}^{-1}\begin{bmatrix} 5 \\ 4 \end{bmatrix}$$

We already know that $\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}^{-1}\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} = \boldsymbol{I} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$$\therefore \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

Using GDC

$$\therefore \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

A matrix multiplied by ***I*** remains unchanged.

$\therefore x = 1$ and $y = 3$

2 $\boldsymbol{AB} = \boldsymbol{C}$ and we need to solve for B.
Again, normally we would divide both sides by ***A***, but this is not possible with matrices, so we *PRE MULTIPLY* by the inverse of ***A***.

$$\boldsymbol{AB} = \boldsymbol{C}$$
$$\boldsymbol{A}^{-1}\boldsymbol{AB} = \boldsymbol{A}^{-1}\boldsymbol{C}$$
$$\therefore \boldsymbol{IB} = \boldsymbol{A}^{-1}\boldsymbol{C}$$
$$\therefore \boldsymbol{B} = \begin{bmatrix} 4 & 1 \\ -2 & 3 \end{bmatrix}$$

We can do this by entering the matrices we know into the GDC and multiplying them.

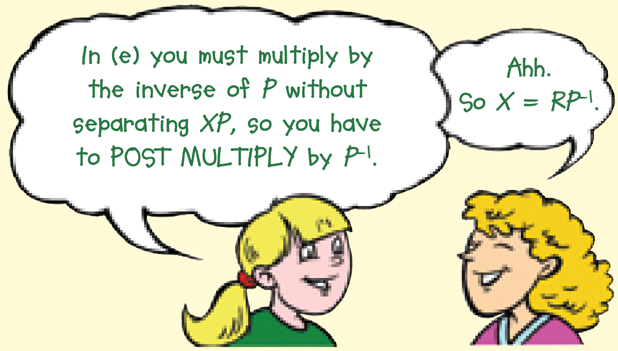
We already know that $\boldsymbol{A}^{-1}\boldsymbol{A} = \boldsymbol{I}$

Exercise 9:05

1 If $\boldsymbol{P} = \begin{bmatrix} 1 & 3 \\ 1 & 2 \end{bmatrix}$, $\boldsymbol{Q} = \begin{bmatrix} -2 & -1 \\ 4 & 3 \end{bmatrix}$, $\boldsymbol{R} = \begin{bmatrix} 1 & 3 \\ -2 & -5 \end{bmatrix}$ and $\boldsymbol{S} = \begin{bmatrix} 4 & 3 \\ -3 & -2 \end{bmatrix}$ find the matrix represented by $\boldsymbol{X}$ in the following matrix equations:

a $\boldsymbol{PX} = \boldsymbol{Q}$ **b** $\boldsymbol{QX} = \boldsymbol{P}$ **c** $\boldsymbol{SX} = \boldsymbol{R}$

d $\boldsymbol{RX} = \boldsymbol{S}$ **e** $\boldsymbol{XP} = \boldsymbol{R}$



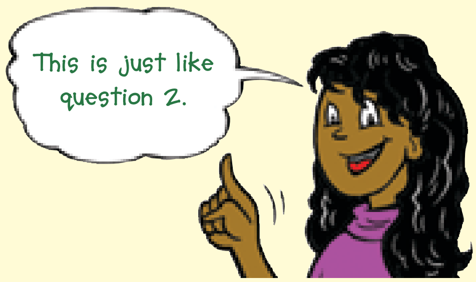
2 Find values for x and y such that

a $\begin{bmatrix} 5 & 4 \\ 3 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -2 \\ 0 \end{bmatrix}$ **b** $\begin{bmatrix} 5 & -3 \\ 3 & -5 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

3 **a** Show that $\begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2x + y \\ 3x + 2y \end{bmatrix}$

b The pair of simultaneous equations $\begin{aligned} 2x + y &= 1 \\ 3x + 2y &= 3 \end{aligned}$ can be written in matrix form as $\begin{bmatrix} 2x + y \\ 3x + 2y \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$

Use part (a) and matrix operations to solve this pair of simultaneous equations.



9:06 | Solving Simultaneous Equations Using Matrices

Matrices are very useful when solving simultaneous equations, as you found out in question 3 of the previous exercise.

worked examples

Use matrices and the GDC to solve the following pairs of simultaneous equations.

1 $\begin{aligned} 2x - 5y &= 1 \\ 3x - y &= -5 \end{aligned}$ **2** $\begin{aligned} y &= 4 - 2x \\ 26 + x + 1 &= 0 \end{aligned}$ **3** $\begin{aligned} 2x - 1 &= y \\ x - 2y + 7 &= 0 \end{aligned}$

Solutions

1 In matrix form $\begin{aligned} 2x - 5y &= 1 \\ 3x - y &= -5 \end{aligned}$ can be written $\begin{bmatrix} 2x - 5y \\ 3x - y \end{bmatrix} = \begin{bmatrix} 1 \\ -5 \end{bmatrix}$

$$\therefore \begin{bmatrix} 2x & -5 \\ 3 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ -5 \end{bmatrix}$$

Pre multiply both sides by the inverse $\therefore \begin{bmatrix} 2 & -5 \\ 3 & -1 \end{bmatrix}^{-1}\begin{bmatrix} 2x & -5 \\ 3 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 & -5 \\ 3 & -1 \end{bmatrix}^{-1}\begin{bmatrix} 1 \\ -5 \end{bmatrix}$

A matrix multiplied by its inverse gives ***I*** $\therefore \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -2 \\ -1 \end{bmatrix}$

A matrix multiplied by ***I*** remains unchanged $\therefore \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -2 \\ -1 \end{bmatrix}$

So that $x = -2$ and $y = -1$

2 The equations must first be rearranged so that $\begin{aligned} y &= 4 - 2x \\ 2y + x + 1 &= 0 \end{aligned}$ are rewritten $\begin{aligned} 2x + y &= 4 \\ x + 2y &= -1 \end{aligned}$.

They can now be written in matrix form: $\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 4 \\ -1 \end{bmatrix}$

$$\therefore \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}^{-1}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}^{-1}\begin{bmatrix} 4 \\ -1 \end{bmatrix}$$

$$\therefore \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 \\ -2 \end{bmatrix}$$

So that $x = 3$ and $y = -2$

continued →→→

3 The equations must first be rearranged so that $\begin{aligned} 2x - 1 &= y \\ x - 2y + 7 &= 0 \end{aligned}$ are rewritten $\begin{aligned} 2x - y &= 1 \\ x - 2y &= -7 \end{aligned}$

$$\therefore \begin{bmatrix} 2 & -1 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ -7 \end{bmatrix}$$

$$\therefore \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ 1 & -2 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ -7 \end{bmatrix}$$

$$\therefore \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 \\ 5 \end{bmatrix}$$

So that $x = 3$ and $y = 5$

Exercise 9:06

1 By writing the following in matrix form, find the solution to the following pairs of simultaneous equations.

a $\begin{aligned} 4x + 2y &= 0 \\ 3x - 2y &= 7 \end{aligned}$ **b** $\begin{aligned} x + 3y &= 5 \\ 2x + 14 &= 2y \end{aligned}$ **c** $\begin{aligned} 2x + y + 4 &= 0 \\ y - 3x &= 11 \end{aligned}$

d $\begin{aligned} 2x - 5y &= 10 \\ 3x + 2y + 4 &= 0 \end{aligned}$ **e** $\begin{aligned} 4y &= 5x - 5 \\ 3x + 2y - 14 &= 0 \end{aligned}$

2 By using matrix operations, find the point of intersection of the lines $3x + 2y = 4$ and $3y = 19 = 2x$.

3 Two lines have equations $6x - 3y - 3 = 0$ and $2x - y + 3 = 0$

a Write these equations as matrices in the form $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} p \\ q \end{bmatrix}$

b Find the value of $\begin{vmatrix} a & b \\ c & d \end{vmatrix}$

c Interpret the result for (c) geometrically with reference to the graphs of the lines.

4 By using matrices, show that the lines $6x - 3y = 15$ and $8x - 4y + 4 = 0$ are parallel (i.e. they have no point of intersection).

5 Three lines have equations $\begin{aligned} 2x + 3y - z &= 3 \\ x + 2y - 3z &= 0 \\ 3x - y + z &= -4 \end{aligned}$

a Write these equations as matrices in the form $\begin{bmatrix} a & b & c \\ d & e & f \\ g & f & i \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} p \\ q \\ r \end{bmatrix}$

b Use the GDC to find $\begin{vmatrix} q & b & c \\ d & e & f \\ g & f & i \end{vmatrix}$

c Hence, find the values of x, y and z that satisfy all three equations simultaneously.

Mathematical Terms 9

determinant
- The determinant of the 22 matrix

  $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is $ad - bc$
- If the determinant is zero, the matrix has no inverse.
- The determinant of matrix $\boldsymbol{A}$ is written $\det(\boldsymbol{A})$ or $|\boldsymbol{A}|$

element
- A term in a matrix.

order of a matrix
- The number of rows × the number of columns.

identity matrix
- When a matrix is multiplied by the identity matrix it remains unchanged.
  For example,
  2×2 identity matrix: $\boldsymbol{I} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

inverse of a matrix
- When a matrix is multiplied by its inverse, the identity matrix ($\boldsymbol{I}$) results.

matrix
- A rectangular array of data like a spreadsheet.

null matrix
- A matrix in which all the elements are zero.

square matrix
- The number of rows = the number of columns.

Mathematical terms 9

Diagnostic Test 9: | Matrices

- Each section of the test has similar items that test a certain type of example.
- Failure in more than one item will identify an area of weakness.
- Each weakness should be treated by going back to the section listed.

	Section
1 Alberto the cobbler makes shoes. He makes four types of shoes. **a** His sales over a four-week period are shown below. Express this information as a matrix. Week 1: 5 pairs of sandals, 0 pairs of dress shoes, 2 pairs of sports shoes and 2 pairs of casual shoes Week 2: 3 pairs of sandals, 4 pairs of dress shoes, 1 pair of sports shoes and 2 pairs of casual shoes Week 3: 7 pairs of sandals, 8 pairs of dress shoes, 7 pairs of sports shoes and 5 pairs of casual shoes Week 4: 1 pair of sandals, 3 pairs of dress shoes, 5 pairs of sports shoes and 0 pairs of casual shoes	9:01

	Section
b His brother, Franco, also makes shoes. His sales for the same four-week period are shown in the matrix $\begin{bmatrix} 0 & 1 & 3 & 6 \\ 5 & 2 & 0 & 3 \\ 4 & 3 & 5 & 2 \\ 4 & 4 & 7 & 1 \end{bmatrix}$ Add this to the matrix in (a) to show how many shoes Alberto and Franco sold altogether over the four-week period.	9:01
c Their profit on each type of shoe is: sandal \$2.50, dress shoe \$3.00, sports shoe \$3.50 and casual shoes \$2.75. Write a matrix multiplication statement to calculate how much profit they make over the four weeks.	9:02, 9:03
d Use your GDC to calculate their profit for each of the four weeks.	
2 Given that $\boldsymbol{P} = \begin{bmatrix} 3 & 0 & 4 \\ -2 & 1 & 2 \end{bmatrix}$, $\boldsymbol{Q} = \begin{bmatrix} 5 & 0 \\ -4 & 2 \\ 1 & 3 \end{bmatrix}$, $\boldsymbol{R} = \begin{bmatrix} 3 & 2 & -1 \\ 5 & 0 & -1 \end{bmatrix}$, $\boldsymbol{S} = \begin{bmatrix} 5 & -2 \\ 3 & 0 \end{bmatrix}$ Calculate where possible: **a** ***PQ*** **b** ***QR*** **c** **PR** **d** ***RS*** **e** ***SR*** **f** $\boldsymbol{P}^2$ **g** $\boldsymbol{S}^2$	9:02, 9:03
3 Find (i) the determinant of the following and (ii) the inverse if it exists. **a** $\begin{bmatrix} 3 & 2 \\ 1 & 1 \end{bmatrix}$ **b** $\begin{bmatrix} 3 & 6 \\ 2 & 4 \end{bmatrix}$ **c** $\begin{bmatrix} 5 & 6 \\ 3 & 4 \end{bmatrix}$ **d** $\begin{bmatrix} 3 & 2 \\ 5 & 3 \end{bmatrix}$ **e** $\begin{bmatrix} 3 & 4 \\ 6 & 8 \end{bmatrix}$	9:04
4 Use the given information in the following to calculate the unknown. **a** $\begin{vmatrix} x & 2x \\ -3 & -5 \end{vmatrix} = 5$ **b** $\begin{vmatrix} 3a & 1 \\ 4 & a \end{vmatrix} = 8$ **c** $\det\begin{bmatrix} 5y & -2y \\ 6 & -3 \end{bmatrix} = -3$ **d** $\begin{bmatrix} 3z & 2 \\ 4 & 1 \end{bmatrix}$ has no inverse	9:04
5 If $\boldsymbol{A} = \begin{bmatrix} 2 & 1 \\ -3 & -2 \end{bmatrix}$, $\boldsymbol{B} = \begin{bmatrix} 5 & 2 \\ -3 & -1 \end{bmatrix}$ and $\boldsymbol{C} = \begin{bmatrix} 6 & 5 \\ 4 & 3 \end{bmatrix}$ use the GDC to find the matrix $\boldsymbol{X}$ in the following matrix equations. **a** $\boldsymbol{AX} = \boldsymbol{B}$ **b** $\boldsymbol{XA} = \boldsymbol{B}$ **c** $\boldsymbol{BC} = \boldsymbol{X}$ **d** $\boldsymbol{BX} = \boldsymbol{C}$ **e** $\boldsymbol{CX} = \boldsymbol{I}$	9:05
6 Use matrices to solve the following pairs of simultaneous equations. **a** $2x + y + 1 = 0$, $x + 3y = 7$ **b** $3x - 2y = 0$, $x - y - 10 = 0$	9:06

Chapter 9 | Revision Assignment

assignment 9A

1 Let $\boldsymbol{A} = \begin{bmatrix} 1 & 2 \\ 3 & -1 \end{bmatrix}$ and $\boldsymbol{B} = \begin{bmatrix} 3 & 0 \\ -2 & 1 \end{bmatrix}$.

Find

a $\boldsymbol{A} + \boldsymbol{B}$;

b $-3\boldsymbol{A}$;

c $\boldsymbol{AB}$.

2 Matrices $\boldsymbol{A}$, $\boldsymbol{B}$ and $\boldsymbol{C}$ are defined by

$$\boldsymbol{A} = \begin{bmatrix} 5 & 1 \\ 7 & 2 \end{bmatrix} \quad \boldsymbol{B} = \begin{bmatrix} 2 & 4 \\ -3 & 15 \end{bmatrix} \quad \boldsymbol{C} = \begin{bmatrix} 9 & -7 \\ 8 & 2 \end{bmatrix}.$$

Let $\boldsymbol{X}$ be an unknown 2×2 matrix satisfying the equation

$$\boldsymbol{AX} + \boldsymbol{B} = \boldsymbol{C}.$$

This equation may be solved for $\boldsymbol{X}$ by rewriting it in the form

$$\boldsymbol{X} = \boldsymbol{A}^{-1}\,\boldsymbol{D}.$$

where $\boldsymbol{D}$ is a 2×2 matrix.

a Write down $\boldsymbol{A}^{-1}$.

b Find $\boldsymbol{D}$.

c Find $\boldsymbol{X}$.

3 $\boldsymbol{A}$ and $\boldsymbol{B}$ are 2×2 matrices, where

$\boldsymbol{A} = \begin{bmatrix} 5 & 2 \\ 2 & 0 \end{bmatrix}$ and $\boldsymbol{BA} = \begin{bmatrix} 11 & 2 \\ 44 & 8 \end{bmatrix}$. Find $\boldsymbol{B}$.

4 Let $\boldsymbol{M} = \begin{bmatrix} 2 & 1 \\ 2 & -1 \end{bmatrix}$.

a Write down the determinant of $\boldsymbol{M}$.

b Write down $\boldsymbol{M}^{-1}$.

c **Hence** solve $\boldsymbol{M} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 4 \\ 8 \end{bmatrix}$.

5 Let $\boldsymbol{A} = \begin{bmatrix} 3 & 2 \\ k & 4 \end{bmatrix}$ and $\boldsymbol{B} = \begin{bmatrix} 2 & 2 \\ 1 & 3 \end{bmatrix}$.

Find, in terms of k,

a $2\boldsymbol{A} - \boldsymbol{B}$;

b $\det(2\boldsymbol{A} - \boldsymbol{B})$.

6 Jacques can buy six CDs and three DVDs for \$163.17 or he can buy nine CDs and two DVDs for \$200.53.

a Express the above information using two equations relating the price of CDs and the price of DVDs.

b Find the price of one DVD.

c If Jacques has \$180 to spend, find the exact amount of change he will receive if he buys nine CDs.

assignment 9B

Chapter 9 | Working Mathematically

1 Let $A = \begin{bmatrix} a & b \\ c & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 0 \\ d & e \end{bmatrix}$. Giving your answers in terms of a, b, c, d and e,

a write down $A + B$;

b find AB.

2 Let $A = \begin{bmatrix} 0 & 2 \\ 2 & 0 \end{bmatrix}$.

a Find

i A^{-1};

ii A^2.

Let $B = \begin{bmatrix} p & 2 \\ 0 & q \end{bmatrix}$.

b Given that $2A + B = \begin{bmatrix} 2 & 6 \\ 4 & 3 \end{bmatrix}$, find the value of p and of q.

c Hence find $A^{-1}B$.

d Let X be a 2×2 matrix such that $AX = B$. Find X.

3 a Let $\begin{bmatrix} b & 3 \\ 7 & 8 \end{bmatrix} + \begin{bmatrix} 9 & 5 \\ -2 & 7 \end{bmatrix} = \begin{bmatrix} 4 & 8 \\ a & 15 \end{bmatrix}$.

i Write down the value of a.

ii Find the value of b.

b Let $3\begin{bmatrix} -4 & 8 \\ 2 & 1 \end{bmatrix} - 5\begin{bmatrix} 2 & 0 \\ q & -4 \end{bmatrix} = \begin{bmatrix} -22 & 24 \\ 9 & 23 \end{bmatrix}$.

Find the value of q.

4 The matrices A, B, X are given by

$$A = \begin{bmatrix} 3 & 1 \\ -5 & 6 \end{bmatrix}, B = \begin{bmatrix} 4 & 8 \\ 0 & -3 \end{bmatrix}, X = \begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

Given that $AX + X = B$, find the **exact** values of a, b, c and d.

5 If $A = \begin{bmatrix} 2p & 3 \\ -4p & p \end{bmatrix}$ and $\det A = 14$, find the possible values of p.

6 Mal is shopping for a school trip. He buys 50 tins of beans and 20 packets of cereal. The total cost is 260 Australian dollars (AUD).

a Write down an equation showing this information, taking b to be the cost of one tin of beans and c to be the cost of one packet of cereal in AUD.

Stephen thinks that Mal has not bought enough so he buys 12 more tins of beans and 6 more packets of cereal. He pays 66 AUD.

b Write down another equation to represent this information.

c Find the cost of one tin of beans.

Technology Applications Drag and Drops

10

Surface Area and Volume

Chapter Contents

Learning Outcomes

Students will be able to:

- Calculate the surface areas of pyramids, cones and spheres.
- Calculate the volume of pyramids, cones and spheres.
- Apply the surface area and volumes of these shapes to the solution of problems.

Areas of Interaction

Approaches to Learning (Knowledge Acquisition, Problem Solving, Communication, Logical Thinking, Information and Communication, Technology Skills, Reflection), Human Ingenuity, Environments

10:01 | Review of Surface Area

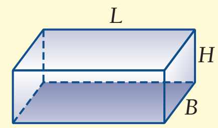
In Book 4, the surface areas of prisms, cylinders and composite solids were calculated by adding the areas of the faces (or surfaces).

The following formula may be needed.

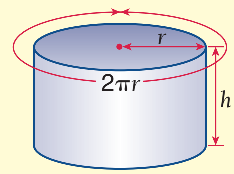
Area formulae

1 square: $A = s^2$
2 rectangle: $A = LB$
3 triangle: $A = \frac{1}{2}bh$
4 trapezium: $A = \frac{1}{2}h(a + b)$
5 parallelogram: $A = bh$
6 rhombus: $A = \frac{1}{2}xy$
7 circle: $A = \pi r^2$

8 Surface area of a rectangular prism:
$A = 2LB + 2LH + 2BH$

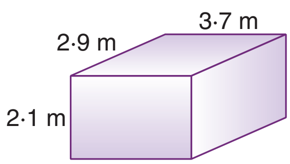


9 Surface area of a cylinder:
$A = 2\pi rh + 2\pi r^2$

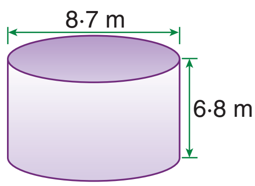


worked examples

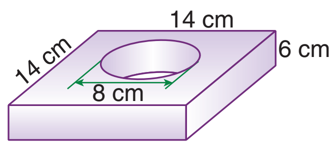
1 Find the surface area of this prism.



2 Find the surface area of this cylinder.



3 Find the surface area of this solid.

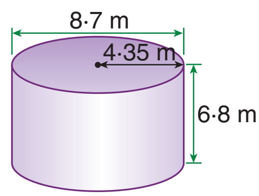


The hole is 4 cm deep

Solutions

1 Surface area $= 2LB + 2LH + 2BH$
$= 2 \times 2{\cdot}9 \times 3{\cdot}7 + 2 \times 2{\cdot}9 \times 2{\cdot}1 + 2 \times 3{\cdot}7 \times 2{\cdot}1$
$= 49{\cdot}18\ \text{m}^2$

2 For a cylinder:
Surface area = curved surface area + area of circles
Surface area $= 2\pi rh + 2\pi r^2$
$= (2\pi \times 4{\cdot}35 \times 6{\cdot}8) + (2 \times \pi \times 4{\cdot}35^2)$
$= 304{\cdot}75\ \text{m}^2$ (correct to 2 dec. pl.)

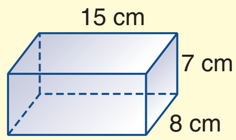


3 Surface area $= 2 \times (14 \times 14) - \pi \times 4^2 + 4 \times (14 \times 6) + 2\pi \times 4 \times 4 + \pi \times 4^2$
$= 728 + 32\pi$
$= 829\ \text{cm}^2$ (to nearest cm^2)

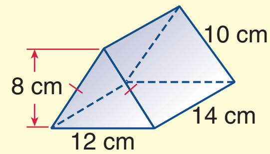
Exercise 10:01

1 Find the surface area of the following prisms.

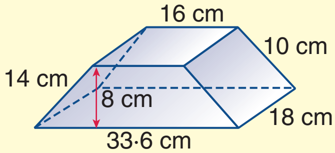
a



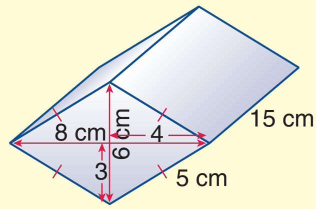
b



c

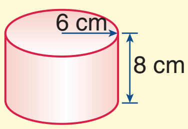


d

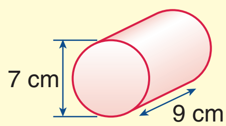


2 Find the surface area of the following solids.

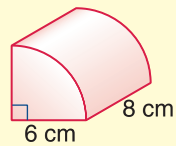
a



b

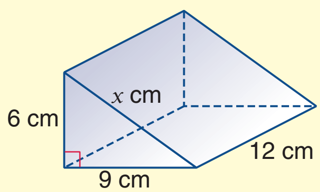


c

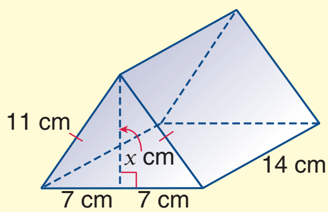


3 In each of the following questions use Pythagoras' theorem to calculate the unknown length *x*, correct to two decimal places, and then calculate the surface area.

a

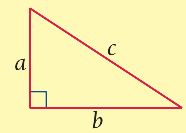


b



Caution!

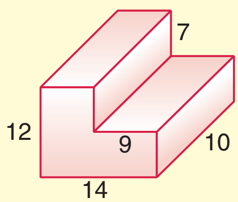
$c^2 = a^2 + b^2$



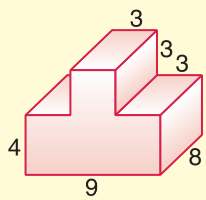
Right-angled triangles

4 Find the surface area of the following solids. All measurements are in metres.

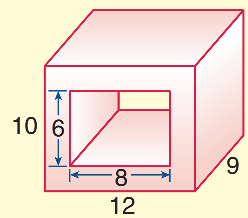
a



b

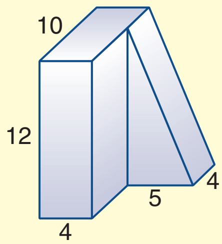


c

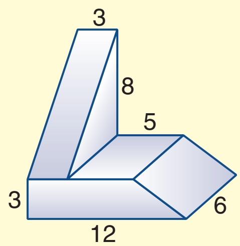


5 Find the surface area of the following solids. All measurements are in centimetres.

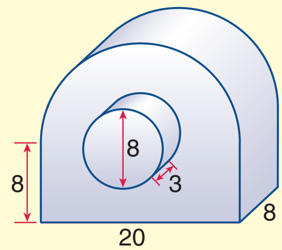
a



b



c

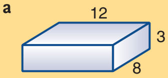


Foundation Worksheet 10:01

Surface area review

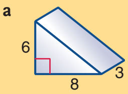
1 Find the surface area of the following rectangular prisms.

a



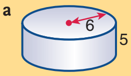
2 Find the surface area of the following triangular prisms.

a



3 Find the surface area of the following cylinders.

a



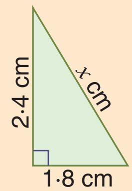
10:02 | Surface Area of a Pyramid

prep quiz 10:02

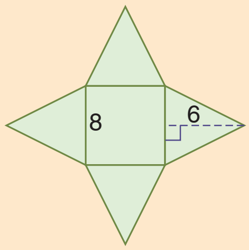
1 Are the triangular faces of a square pyramid congruent?

2 Are the triangular faces of a rectangular pyramid congruent?

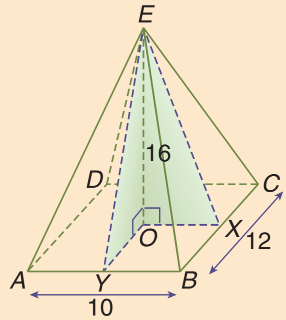
3 Find x



4 The net of a square pyramid is shown. Find the area of the net.



ABCDE is a rectangular pyramid.



Find:

5 OX

6 EX

7 OY

8 EY

Find the area of:

9 ΔEBC

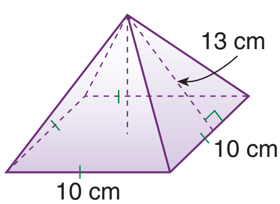
10 ΔABE

To calculate the surface area of a pyramid with a polygonal base, we add the area of the base and the area of the triangular faces.

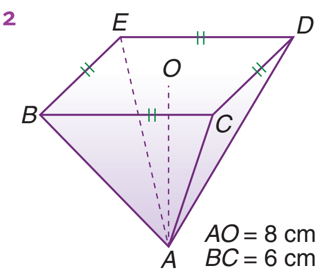
worked examples

Calculate the surface area of the following square and rectangular pyramids.

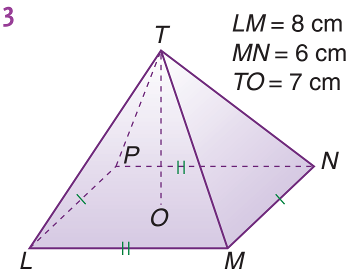
1



2



3



Solutions

1 Surface area = (area of square) + 4 × (area of a triangular face)

$$= (10 \times 10) + 4 \times \left(\frac{10 \times 13}{2}\right)$$

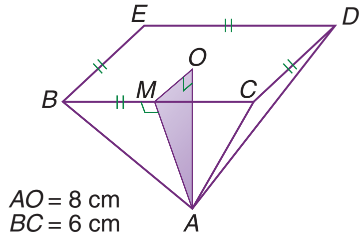
$$= 100 + 260$$

$$= 360 \text{ cm}^2$$

In right square pyramids, all the triangular faces are congruent. This simplifies the calculation of the surface area.

2 As the perpendicular height of the triangular face is not given, this must be calculated.

In the diagram, AM is the perpendicular height of the face.



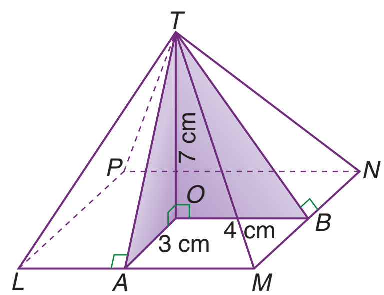
In ΔAOM

$$\begin{aligned} AM^2 &= AO^2 + OM^2 \quad \text{(Pythagoras' theorem)} \\ &= 8^2 + 3^2 \quad (\textit{Note: } OM = \tfrac{1}{2}CD) \\ &= 64 + 9 \\ &= 73 \\ AM &= \sqrt{73} \end{aligned}$$

Surface area = (area of square) + 4 × (area of a triangular face)

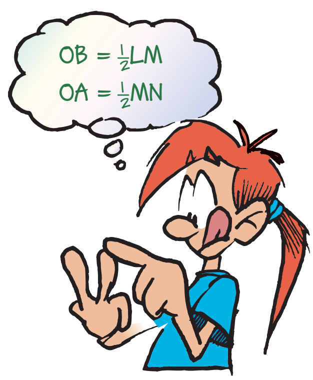
$$\begin{aligned} &= (6 \times 6) + 4 \times \left(\frac{6 \times \sqrt{73}}{2}\right) \\ &= 36 + 12\sqrt{73} \\ &= 138{\cdot}5 \text{ cm}^2 \text{ (correct to 1 dec. pl.)} \end{aligned}$$

3 The perpendicular heights TA and TB must be calculated, as these are not given.



In ΔTOA,

$$\begin{aligned} TA^2 &= AO^2 + OT^2 \\ &= 3^2 + 7^2 \\ &= 58 \\ \therefore\ TA &= \sqrt{58} \text{ cm} \end{aligned}$$



In ΔTOB,

$$\begin{aligned} TB^2 &= BO^2 + OT^2 \\ &= 4^2 + 7^2 \\ &= 65 \\ \therefore\ TB &= \sqrt{65} \text{ cm} \end{aligned}$$

Now,

Surface area = (area of rect. $LMNP$) + 2 × (area of ΔTMN) + 2 × (area of ΔTLM)

$$\begin{aligned} &= (LM \times MN) + 2 \times \left(\frac{MN \times TB}{2}\right) + 2 \times \left(\frac{LM \times TA}{2}\right) \\ &= 8 \times 6 + \frac{2 \times 6 \times \sqrt{65}}{2} + \frac{2 \times 8 \times \sqrt{58}}{2} \\ &= 48 + 6\sqrt{65} + 8\sqrt{58} \\ &= 157{\cdot}3 \text{ cm}^2 \text{ (correct to 1 dec. pl.)} \end{aligned}$$

In right rectangular pyramids, the opposite triangular faces are congruent.

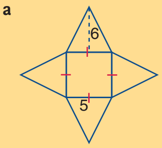
Exercise 10:02

Foundation Worksheet 10:02

Surface area of a pyramid

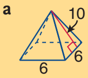
1 Calculate the area of each net.

a



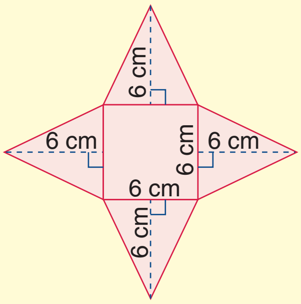
2 Find the surface area of each square or rectangular pyramid.

a

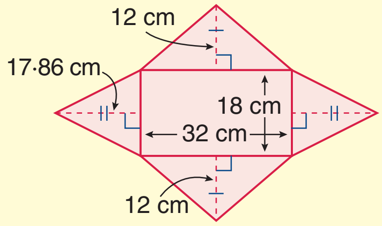


1 The following diagrams represent the nets of pyramids. Calculate the area of each net.

a

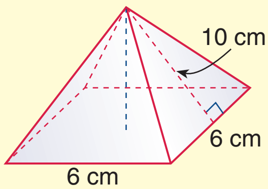


b

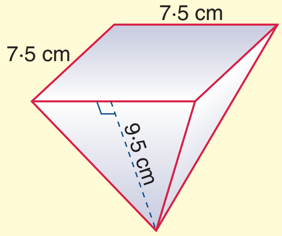


2 Calculate the surface area of the following square and rectangular pyramids.

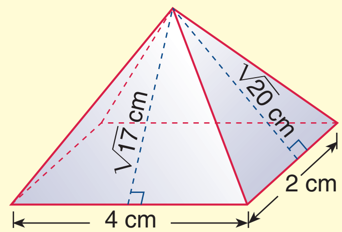
a



b

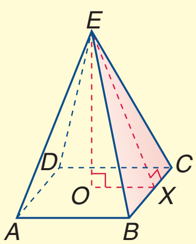


c



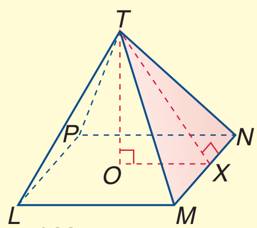
3 Use Pythagoras' theorem to calculate the perpendicular height of each face and then calculate the surface area of each pyramid. Give the answers in surd form.

a



AB = 10 cm
EO = 12 cm

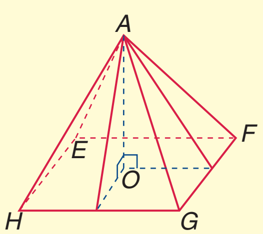
b



LM = 10 cm
TO = 10 cm

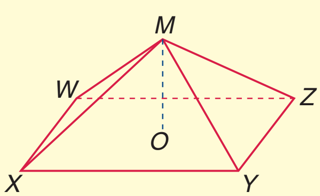
4 Calculate the surface area of the following pyramids. Give all answers correct to one decimal place where necessary.

a



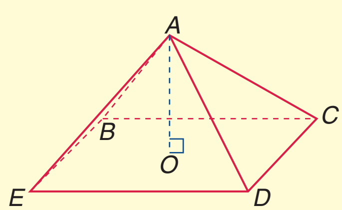
EFGH is a rectangle.
AO = 8 cm, *HG* = 12 cm,
GF = 9 cm.

b



WXYZ is a square.
MO = 3 cm, *XY* = 11 cm.

c



BCDE is a rectangle.
AO = 4 cm, *ED* = 10 cm,
DC = 6 cm.

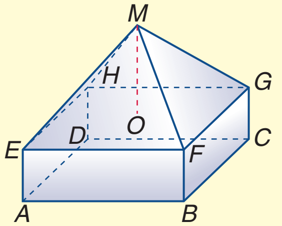
5 Find the surface area of:

a a square pyramid, base edge 6 cm, height 5 cm

b a rectangular pyramid, base 7 cm by 5 cm, height 10 cm

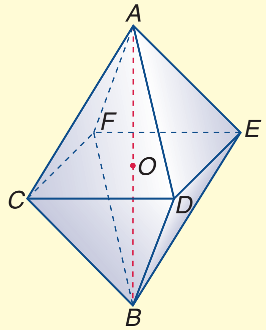
6 Find the surface area of the following solids. Give all answers correct to three significant figures.

a



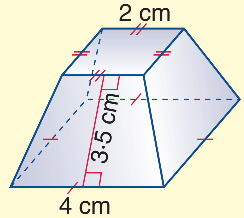
ABCD is a square.
AB = 5 m, *MO* = 3 m,
CG = 2 m

b

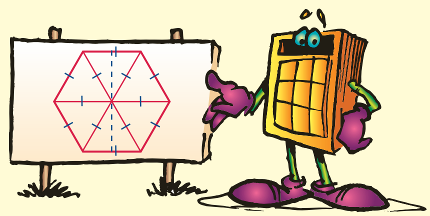


AO = *OB* = 9 cm
CD = *DE* = *FE* = *FC* = 6 cm

c



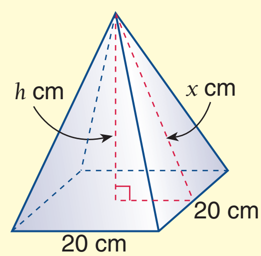
7 Find the surface area of a pyramid that has a regular hexagonal base of edge 6 cm and a height of 8 cm.



8 A square pyramid has to have a surface area of 2000 cm². If the base edge is 20 cm, calculate:

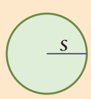
a the perpendicular height, x cm, of one of the triangular faces

b the perpendicular height, h cm, of the pyramid



10:03 | Surface Area of a Cone

prep quiz 10:03



1 Area = ?

2 Circumference = ?

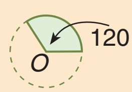
3 Simplify: $\dfrac{2\pi r}{2\pi s}$

4 Simplify: $\dfrac{r}{s} \times \pi s^2$

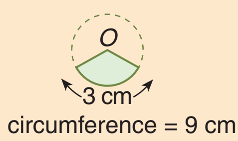
What fraction of a circle is each of the following sectors?

5

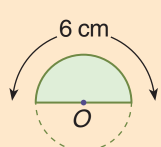
6



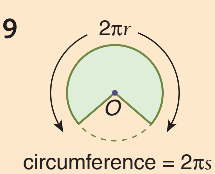
7



8



9



10 Evaluate πrs if $r = 3{\cdot}5$ and $s = 6{\cdot}5$. Answer correct to 1 dec. pl.

investigation 10:03

Investigation 10:03 | The surface area of a cone

The surface area of a cone comprises two parts: a circle and a curved surface. The curved surface is formed from a sector of a circle.

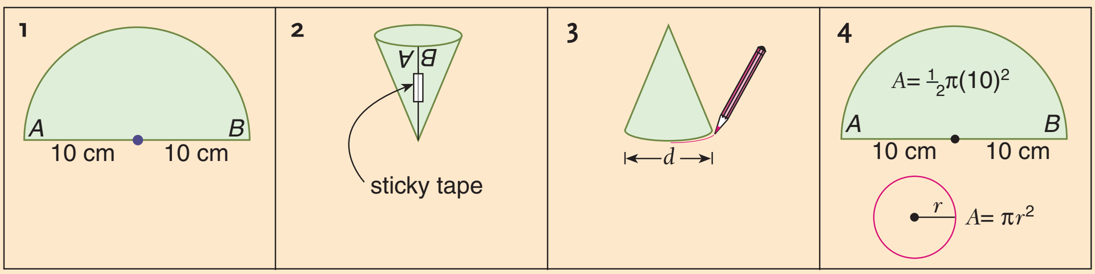
- This investigation involves the making of two cones and the calculation of their surface area.

  Step 1
  Draw a semicircle of radius 10 cm.

  Step 2
  Make a cone by joining opposite sides of the semicircle, as shown below.

  Step 3
  Put the cone face down and trace the circular base. Measure the diameter of this base.

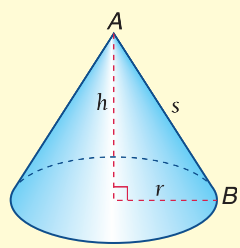
  Step 4
  Calculate the area of the original semicircle plus the area of the circular base. This would be the total surface area of the cone if it were closed.

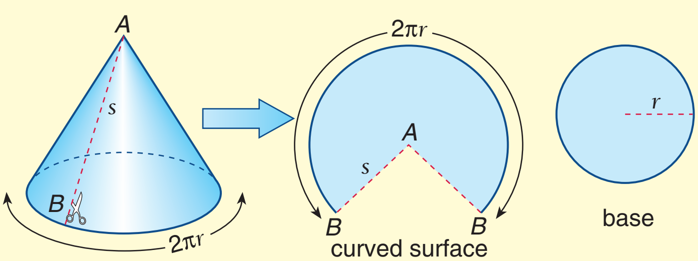


- Repeat the steps above, making the original sector a quarter circle of radius 10 cm. What is the surface area of a closed cone of these dimensions?

A cone may be thought of as a pyramid with a circular base. Consider a cone of slant height s and base radius r.

- Imagine what would happen if we cut along a straight line joining the vertex to a point on the base.
- By cutting along this line, which is called the **slant height**, we produce the net of the curved surface. The net of the curved surface is a sector of a circle, radius s.





To calculate the area of a sector, we must find what fraction it is of the complete circle. Normally this is done by looking at the sector angle and comparing it to 360°, but it can also be done by comparing the length of the sector's arc to the circumference of the circle. Hence, if the sector's arc length is half the circumference of the circle, then the sector's area is half the area of the circle. (See Prep Quiz 10:03.)

$$\therefore \text{ Area of sector} = \frac{\text{length of sector's arc}}{\text{circumference of circle}} \times \text{area of circle}$$
$$= \frac{2\pi r}{2\pi s} \times \pi s^2$$
$$= \pi rs$$

Now, since the area of the sector = area of the curved surface,

curved surface area $= \pi rs$

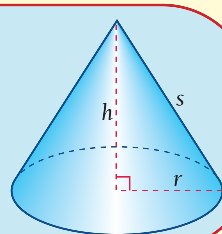
Surface area of a cone:

surface area $= \pi rs + \pi r^2$

where r **= radius of the cone and**

s **= slant height of the cone**

Note: $s = \sqrt{h^2 + r^2}$



worked examples

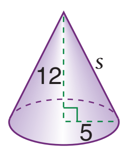
1 Find the surface area of a cone with a radius of 5 cm and a slant height of 8 cm.

2 Find the surface area of a cone with a radius of 5 cm and a height of 12 cm.

Solutions

1 Surface area $= \pi rs + \pi r^2$
$= \pi \times 5 \times 8 + \pi \times 5^2$
$= 40\pi + 25\pi$
$= 65\pi \text{ cm}^2$
$= 204{\cdot}2 \text{ cm}^2$
(correct to 1 dec. pl.)

2



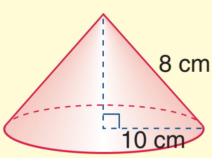
First the slant height must be calculated.
Now, $s^2 = 5^2 + 12^2$
(Pythag. theorem)
$= 169$
$\therefore s = 13$
surface area $= \pi rs + \pi r^2$
$= 65\pi + 25\pi$
$= 90\pi \text{ cm}^2$
$= 282{\cdot}7 \text{ cm}^2$
(correct to 1 dec. pl.)

■ The height of a cone is the perpendicular height.

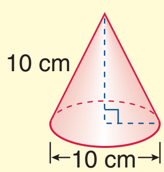
Exercise 10:03

1 Find the curved surface area of the following cones, giving answers in terms of π.

a



b

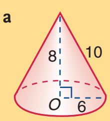


Foundation Worksheet 10:03

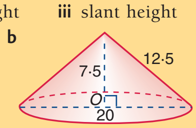
Surface area of a cone

1 For each cone shown, find the:
i radius **ii** height **iii** slant height

a



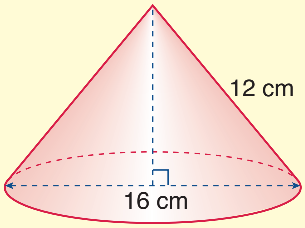
b



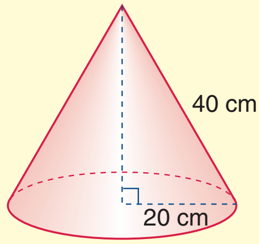
2 For each of the cones in question **1** find:
i the curved surface area **ii** the surface area

3 Use Pythagoras' theorem to find the slant height if:
a radius = 3 cm; height = 4 cm
b diameter = 16 cm; height = 15 cm

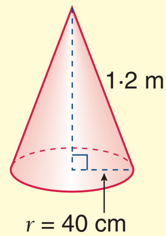
c



d



e



2 Find the surface area of each of the cones in question **1** giving all answers in terms of π.

3 Calculate the surface area of the following cones, giving all answers in terms of π.

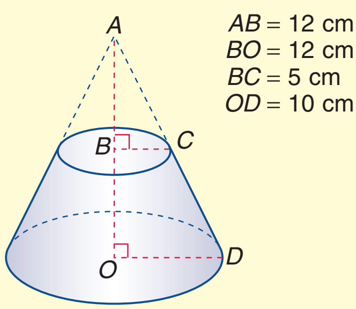
- **a** radius 8 cm and height 6 cm
- **b** radius 1·6 m and height 1·2 m
- **c** diameter 16 cm and height 15 cm
- **d** diameter 1 m and height 1·2 m

4 In each of the following, find the surface area of the cone correct to four significant figures.

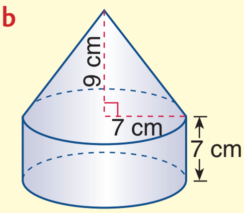
- **a** radius 16 cm and height 20 cm
- **b** radius 5 cm and slant height 12 cm
- **c** radius 12·5 cm and height 4·5 cm
- **d** diameter 1·2 m and height 60 cm
- **e** diameter 3·0 m and slant height 3·5 m

5 Find the surface area of the following solids. Give all answers correct to one decimal place.

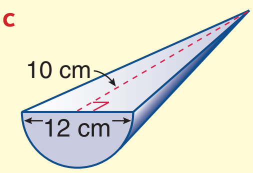
a



b



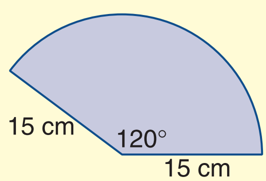
c



Note: This is half a cone

6 A cone is to be formed by joining the radii of the sector shown. In the cone that is formed, find:

- **a** the slant height
- **b** the radius
- **c** the perpendicular height



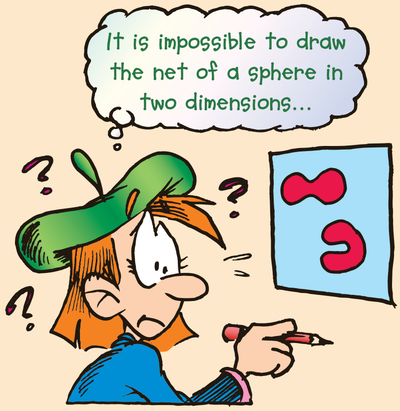
7 **a** A cone with a radius of 5 cm has a surface area of 200π cm^2. What is the perpendicular height of the cone?

b A cone cannot have a surface area greater than 1000π cm^2. What is the largest radius, correct to one decimal place, that will achieve this if the slant height is 20 cm?

10:04 | Surface Area of a Sphere

Investigation 10:04 | The surface area of a sphere

Carry out the experiment outlined below to demonstrate that the reasoning is correct.



Step 1
Cut a solid rubber ball or an orange into two halves.
The faces of the two hemispheres are identical circles.

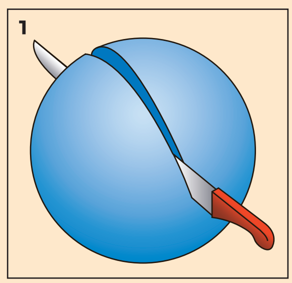
Step 2
Push a long nail through the centre of a hemisphere, as shown.

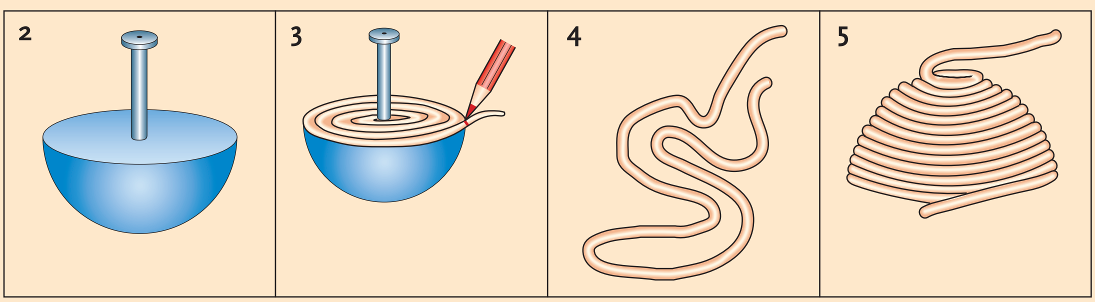
Step 3
Using thick cord, cover the circular face of one of the hemispheres as shown, carefully working from the inside out.
Mark the length of cord needed. Call this length A.
Length A covers the area of the circle, ie πr^2.

Step 4
Put a second mark on the cord at a point that is double the length A.
The length of the cord to the second mark is $2A$.
$2A$ covers the area of two identical circles, ie $2\pi r^2$.

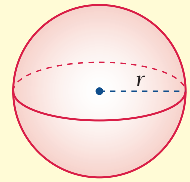
Step 5
Turn the other hemisphere over and use the cord of length $2A$ to cover the outside of the hemisphere.
It should fit nicely.
It seems that $2A$ covers half of the sphere.
It would take $4A$ to cover the whole sphere.
ie the surface area of a sphere $= 4A$

Surface area $= 4\pi r^2$





The proof of the formula of the surface area of a sphere is beyond the scope of this course. However, the formula is given below.



The surface area of a sphere is given by the formula:

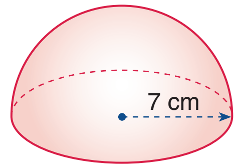
$$SA = 4\pi r^2$$

where SA is the surface area and r is the radius.

worked examples

1 Find the surface area of a sphere of diameter 12 cm.

2 Find the surface area of the hemisphere shown here.



Solutions

1 Diameter = 12 cm

$\therefore$ Radius = 6 cm

Now, $S = 4\pi r^2$

$= 4 \times \pi \times 6^2$

$= 144\pi \text{ cm}^2$

$= 452 \text{ cm}^2$

(correct to 3 sig. figs)

2 S = area of curved surface + area of circle

$= 2\pi r^2 + \pi r^2$

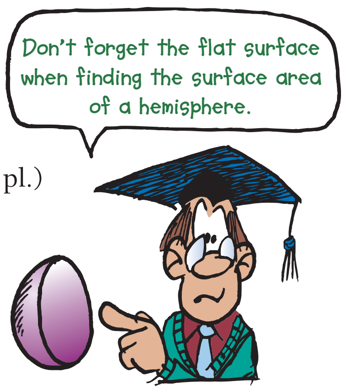
$= 3\pi r^2$

$= 3\pi \times 7^2$

$= 147\pi \text{ cm}^2$

$= 461{\cdot}8 \text{ cm}^2$

(correct to 1 dec. pl.)



Exercise 10:04

1 Find the surface area of a sphere with:

a radius = 5 cm **b** radius = 7·6 cm **c** radius = 3·2 m

d diameter = 18 cm **e** diameter = 1·6 m **f** diameter = 8000 km

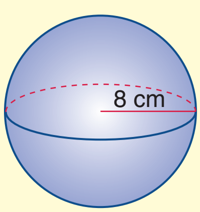
Give all answers as exact answers (multiples of π) and also as approximations correct to three significant figures.

2 Calculate the exact surface area of a hemisphere with:

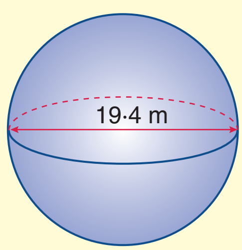
a a radius of 12 cm **b** a diameter of 12 cm

3 Calculate the surface area of each solid correct to two decimal places.

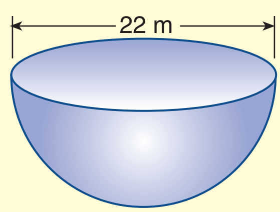
a



b



c

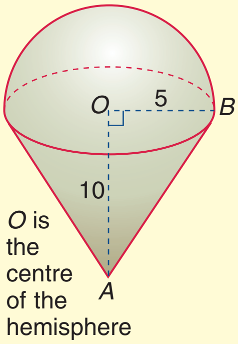


4 A cylinder is 3 m long and has a radius of 80 cm. A hemispherical cap is placed on each end of the cylinder. Calculate the surface area of the solid correct to four significant figures.

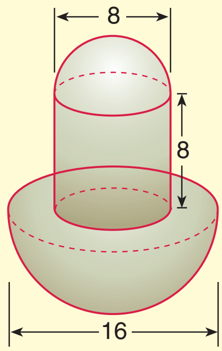
5 A sphere is to have a surface area of 200 cm^2. Find its radius correct to one decimal place.

6 Calculate the surface area of each of the following solids, correct to three significant figures. (All measurements are in centimetres unless stated otherwise.)

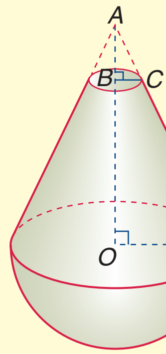
a



b



c



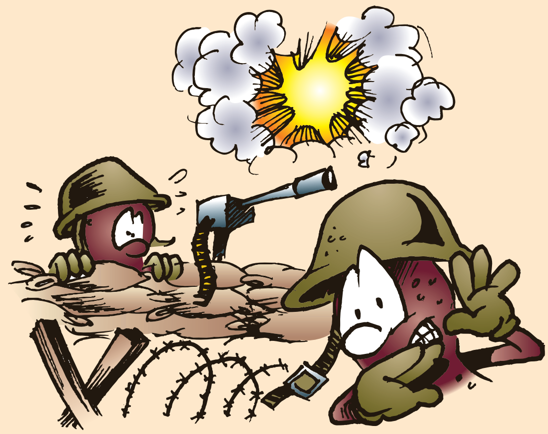
- The dimetrodon was a dinosaur that had a large sail to absorb and dissipate heat efficiently. Dimetrodons of different sizes had a slightly different shape. Larger specimens had proportionally larger sails. These sails varied according to the volume rather than the surface area or the length of the creature.

  Being cold blooded, they needed to remove or absorb heat according to their body mass (or volume).

fun spot 10:04

Fun Spot 10:04 | How did the raisins win the war against the nuts?

Work out the answer to each question and put the letter for that part in the box that is above the correct answer.



Simplify where possible:

B $8x + x$ **C** $8x \div x$
D $8x \times x$ **E** $8x - x$
H $xy + yx$ **E** $2x - 2$
E $(x + 1)^2 - 1$ **E** $3(2x - 4) - 6x$
T $3x \times 3x$ **T** $-2x - 5x$

Solve:

G $\frac{x}{3} = 3$ **H** $\frac{x}{3} = \frac{1}{3}$ **E** $3x = \frac{1}{3}$ **H** $\frac{3}{4}x = \frac{3}{8}$
I $x^2 = 9$ **I** $0{\cdot}3x = 3$ **J** $\sqrt{x} = 9$ **L** $x = \frac{x}{2}$

For the formula $A = \pi rs$:

L what is the subject?
M how many variables are used?
N what is the value of π correct to four significant figures?

Find the value of $b^2 - 4ac$ if:

N $a = 2, b = 10, c = 5$ **N** $a = 3, b = -2, c = 4$
O $a = 1, b = -1, c = 4$ **S** $a = 6, b = -3, c = -1$

Simplify:

S $(a^{-1})^{-2}$ **S** $a^{-1} \times a^1$ **E** $(2a^2)^3$ **T** $\sqrt{x^{16}}$
T $\frac{a}{4} + \frac{a}{5}$ **E** $\frac{a^2}{5} \times \frac{5}{a}$ **T** $3a \div \frac{a}{4}$ **U** $a - \frac{3a}{16}$
Y $6a + 3a \div 3$

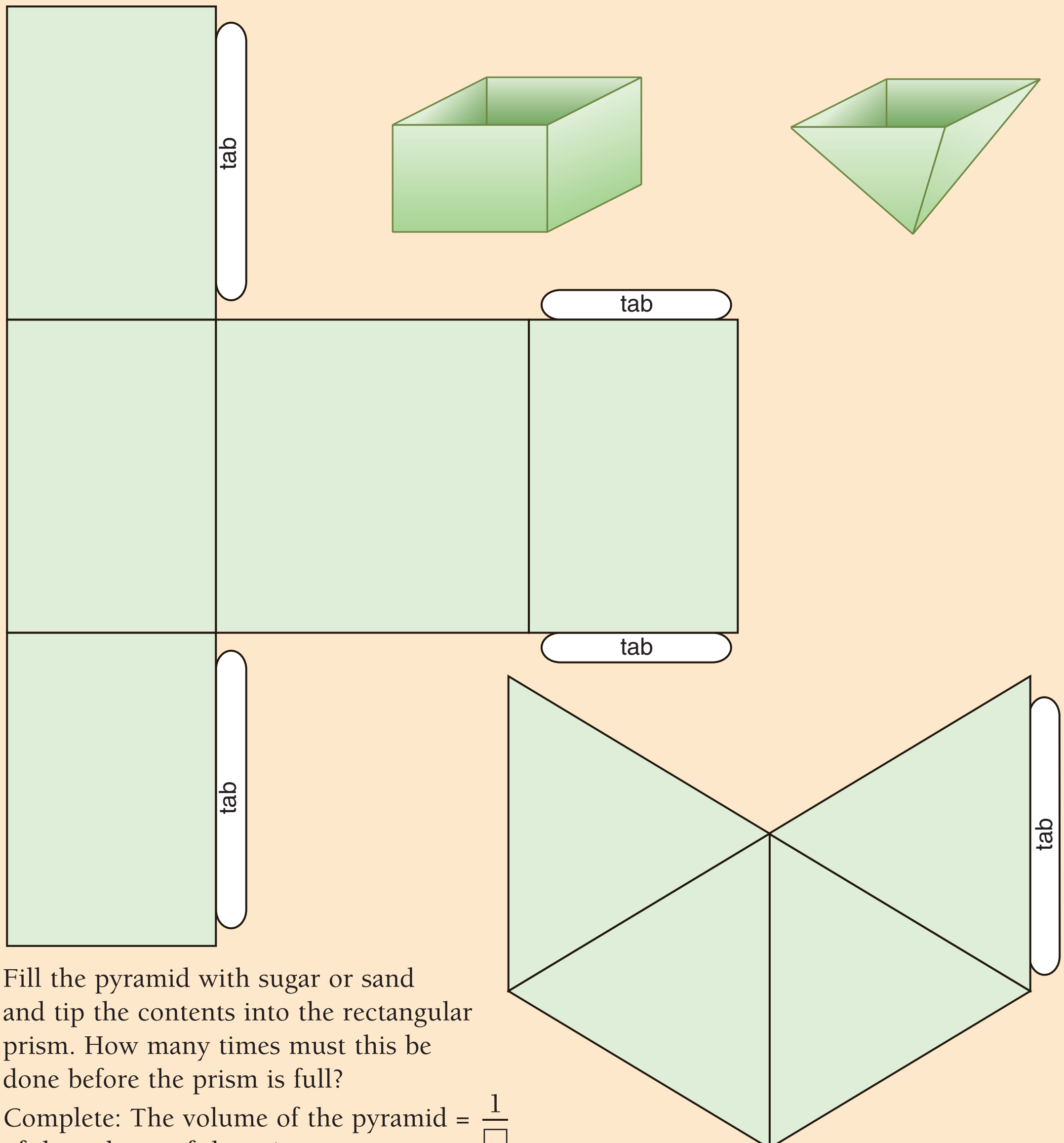
x^8	$x = \frac{1}{2}$	$7x$	$7a$	a^2	$\frac{13a}{16}$	$9x$	$x = 3$	a	8	$9x^2$	−12	$8x^2$	12	$x = 1$	$x = \frac{1}{9}$	3

$\frac{9a}{20}$	−15	$x = 10$	60	$-7x$	$2x - 2$	−44	1	$8a^6$	33	$2xy$	$x^2 + 2x$	$x = 0$	A	$x = \pm 3$	3·142	$x = 9$

10:05 | Volume of a Pyramid

Investigation 10:05 | The volume of a pyramid

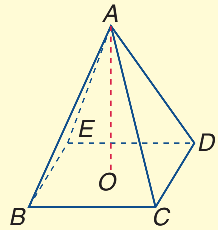
- Photocopy the nets below onto light cardboard.
- Use these nets (with tabs) to construct an open rectangular prism of length 4·2 cm, breadth 4·2 cm and height 2·8 cm and an open pyramid of length 4·2 cm, breadth 4·2 cm and height 2·8 cm.
  (*Note:* to produce this pyramid, each triangle must have a height of 3·5 cm.)



- Fill the pyramid with sugar or sand and tip the contents into the rectangular prism. How many times must this be done before the prism is full?
- Complete: The volume of the pyramid = $\frac{1}{\square}$ of the volume of the prism.

A pyramid is named according to the shape of its base.

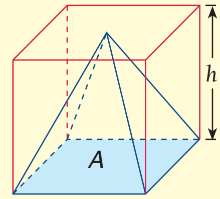
- *BCDE* is the **base** of the pyramid.
- *A* is the **apex** of the pyramid.
- *AO* is the **height** of the pyramid. It is sometimes called the altitude.
- Investigation 10:05 above demonstrates that the volume of a pyramid is one-third of the volume of a prism with the same base area and height.



The volume of all pyramids is given by the formula:

$$V = \frac{1}{3}Ah$$

where V = volume, *A* = area of base and *h* is the height of the pyramid.



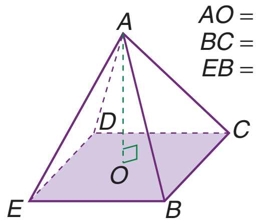
worked examples

1 Find the volume of a rectangular pyramid that has a base 6·2 cm long and 4·5 cm wide and a height of 9·3 cm.

2 A pyramid has a hexagonal base with an area of 12·6 cm². If the height of the pyramid is 7·1 cm, calculate its volume.

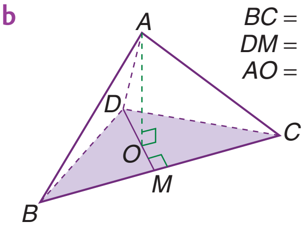
3 Calculate the volume of the pyramids pictured.

a

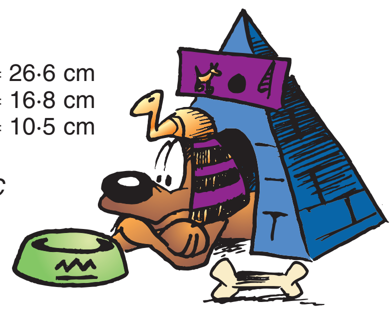


$AO = 12$ cm
$BC = 6{\cdot}5$ cm
$EB = 6{\cdot}5$ cm

b



$BC = 26{\cdot}6$ cm
$DM = 16{\cdot}8$ cm
$AO = 10{\cdot}5$ cm



Solutions

1
$V = \frac{1}{3}Ah$
$A = 6{\cdot}2 \times 4{\cdot}5 \text{ cm}^2$
$h = 9{\cdot}3 \text{ cm}$
$\therefore V = \frac{1}{3} \times 6{\cdot}2 \times 4{\cdot}5 \times 9{\cdot}3 \text{ cm}^3$
$= 86{\cdot}49 \text{ cm}^3$

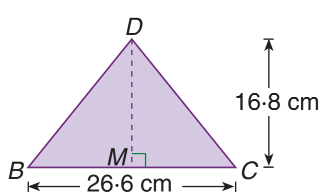
2
$V = \frac{1}{3}Ah$
$A = 12{\cdot}6 \text{ cm}^2$
$h = 7{\cdot}1 \text{ cm}$
$\therefore V = \frac{1}{3} \times 12{\cdot}6 \times 7{\cdot}1$
$= 29{\cdot}82 \text{ cm}^3$

3 a
$V = \frac{1}{3}Ah$
$A = 6{\cdot}5 \times 6{\cdot}5 \text{ cm}^2$
$h = 12 \text{ cm}$
$\therefore V = \frac{1}{3} \times 6{\cdot}5 \times 6{\cdot}5 \times 12$
$= 169 \text{ cm}^3$

3 b
$V = \frac{1}{3}Ah$
$A = \qquad \text{cm}^2$
$h = 10{\cdot}5 \text{ cm}$
$\therefore V = \frac{1}{3} \times \frac{26{\cdot}6 \times 16{\cdot}8}{2} \times 10{\cdot}5$
$= 782{\cdot}04 \text{ cm}^3$

The base is triangular. It's drawn below.

$\text{Area of base} = \frac{BC \times DM}{2}$
$= \frac{26{\cdot}6 \times 16{\cdot}8}{2} \text{ cm}^2$

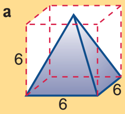


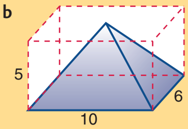
Exercise 10:05

Foundation Worksheet 10:05

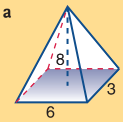
Volume of a pyramid

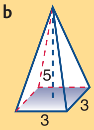
1 In each of the following, calculate the volume of the prism and then divide by 3 to find the volume of the pyramid.

a 

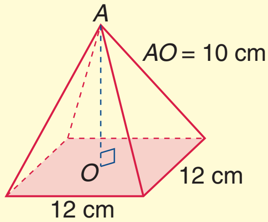
b 

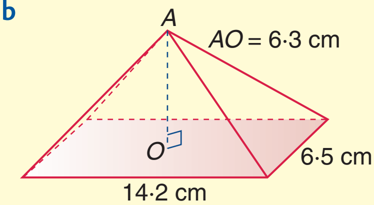
2 For each of the following pyramids, find the value of A and h and then use the formula $V = \frac{Ah}{3}$ to find the volume.

a 

b 

1 Calculate the volume of the square and rectangular pyramids drawn below.

a 

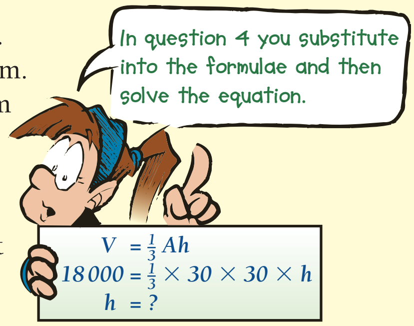
b 

2 **a** Calculate the volume correct to two decimal places of a square pyramid with a base edge of 3·25 m and a height of 6·3 m.

b A rectangular pyramid is 16·2 cm high. Its base is 5·8 cm long and 7·5 cm wide. Find its volume.

3 **a** A pyramid has a triangular base with an area of 4·32 m². Find the volume of the pyramid if it has a height of 2·5 m.

b Find the volume of a hexagonal pyramid of height 15 cm if the hexagonal base has an area of 6·2 cm².



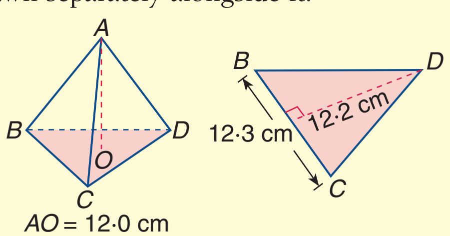
4 **a** A square pyramid has a volume of 18 000 cm³. If the side length of the square is 30 cm, what is its height?

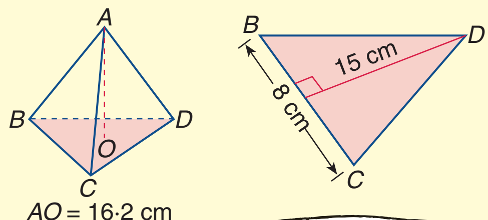
b A pyramid has a 12 cm square as its base. How high must the pyramid be if it is to have a volume of 500 cm³?

c A square pyramid with a height of 120 cm has a volume of 64 000 cm³. What is the area of its base and the length of the side of the square?

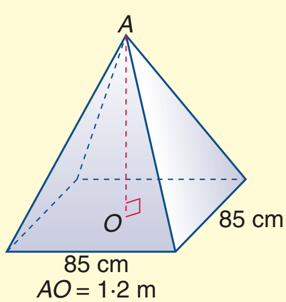
d A rectangle that is twice as long as it is wide forms the base of a rectangular pyramid that is 60 cm high. If the volume of the pyramid is 4000 cm³, what are the dimensions of the rectangle?

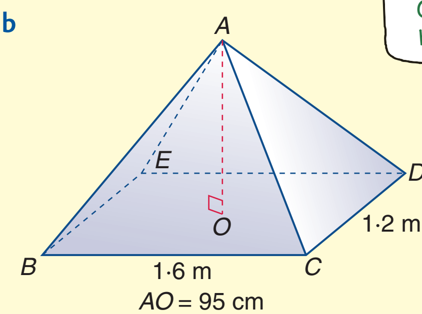
5 Calculate the volume of the triangular pyramids below. The base of each pyramid has been drawn separately alongside it.

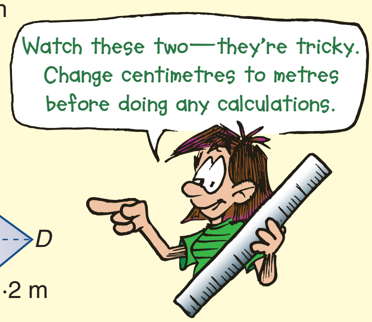
a 

b 

6 Calculate the volume of the pyramids drawn below.

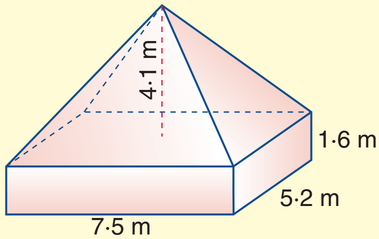
a 

b 

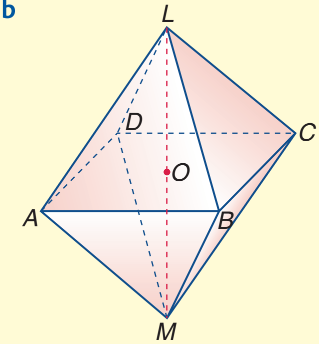


7 Calculate the volume of the following solids.

a

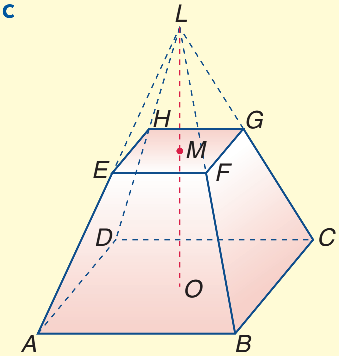


b



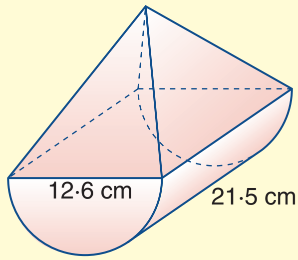
ABCD is a square.
$AB = 12$ cm
$LO = MO = 10$ cm

c



ABCD and *EFGH* are squares.
$AB = 20$ cm, $EF = 10$ cm
$MO = 15$ cm, $LM = 15$ cm

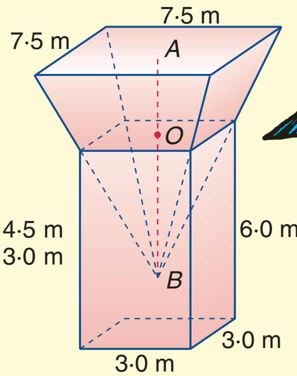
d

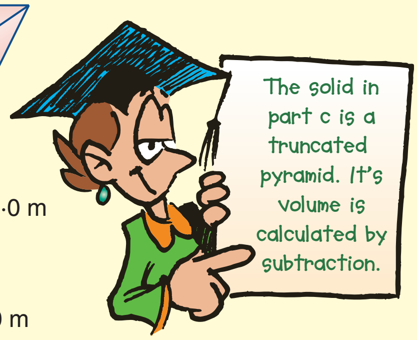


The height of the pyramid is 8·5 cm. (Answer correct to 3 sig. figs.)

e

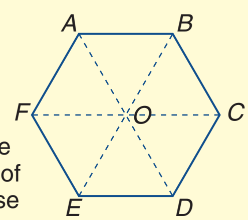
$AO = 4{\cdot}5$ m
$BO = 3{\cdot}0$ m





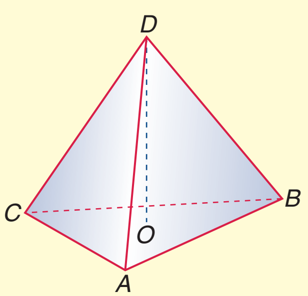
8 **a** Calculate the volume of a square pyramid if it has a base area of 64 cm^2 and the distance from the apex to a corner of the base is 15 cm.

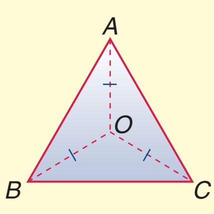
b Calculate the volume of a pyramid that has a height of 8 cm. The base of the pyramid is a regular hexagon with a side length of 6 cm. The base is shown in the diagram.

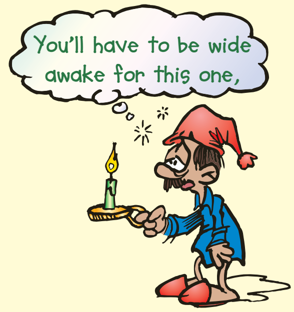


O is the centre of the base

9 A tetrahedron is a triangular pyramid in which each face is an equilateral triangle. Calculate the volume of a tetrahedron that has all its edges 6 cm in length. *Hint:* You will need to know some geometry and trigonometry. A diagram of the pyramid and its base are drawn below.



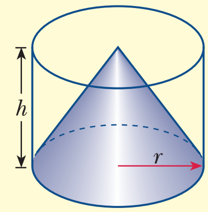




10:06 | Volume of a Cone

Just as the cylinder could be thought of as a 'circular prism', so the cone can be thought of as a 'circular pyramid'.

The volume of a cone is one-third of the volume of a cylinder with the same base area and height.



The volume of a cone is given by the formula:

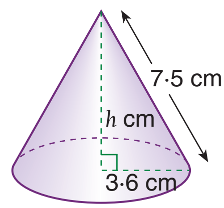
$$V = \frac{1}{3}\pi r^2 h$$

where *r* is the radius of the circular base and *h* is the height.

worked examples

1 Find the volume of a cone with a radius of 6·2 cm and a height of 5·8 cm. Give your answer correct to three significant figures.

2 Use Pythagoras' theorem to calculate the height, *h*, correct to three decimal places and then use this value to calculate the volume correct to three significant figures.



Solutions

1

$V = \frac{1}{3}\pi r^2 h$

$r = 6{\cdot}2$ cm

$h = 5{\cdot}8$ cm

$\therefore V = \frac{1}{3} \times \pi \times (6{\cdot}2)^2 \times 5{\cdot}8$

$= 233$ cm^3 (correct to 3 sig. figs)

2 By Pythagoras' theorem

$h^2 + 3{\cdot}6^2 = 7{\cdot}5^2$

$h^2 = 7{\cdot}5^2 - 3{\cdot}6^2$

$h = \sqrt{7{\cdot}5^2 - 3{\cdot}6^2}$

$= 6{\cdot}580$ (correct to 3 dec. pl.)

Now $V = \frac{1}{3}\pi r^2 h$

$r = 3{\cdot}6$

$h = 6{\cdot}580$

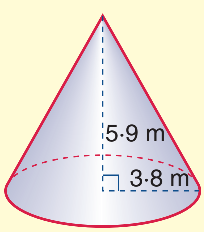
$\therefore V = \frac{1}{3} \times \pi \times 3{\cdot}6^2 \times 6{\cdot}580$

$= 89{\cdot}3$ cm^3 (correct to 3 sig. figs)

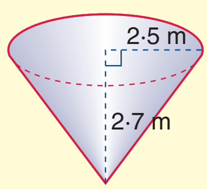
Exercise 10:06

1 Find the volume of the following cones correct to two decimal places.

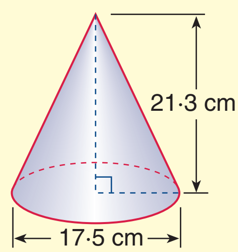
a



b



c



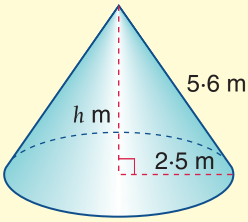
2 **a** A cone has a base radius of 5·2 cm and a height of 7·8 cm. Calculate its volume correct to two significant figures.

b The diameter of the base of a cone is 12·6 cm. If the cone has a height of 15·3 cm, find its volume correct to three significant figures.

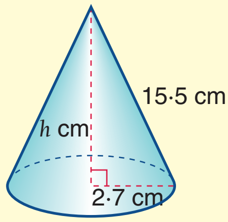
c A cone has a base diameter of 2·4 m and a height of 45 cm. Calculate its volume to the nearest tenth of a cubic metre.

3 Use Pythagoras' theorem to calculate h, correct to three decimal places, and then use this value to calculate the volume correct to three significant figures.

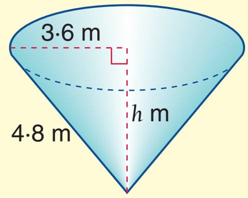
a



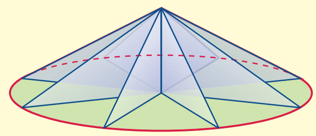
b



c



4 A right-angled triangle with sides of 5, 12 and 13 cm is rotated to form a cone. What is the volume of the cone if it is rotated about:



a the 12 cm side? **b** the 5 cm side?

5 **a** A cone has a radius of 10 cm. It has a volume of 1000 cm^3. Calculate the height of the cone correct to one decimal place.

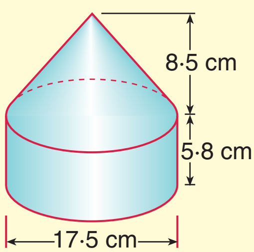
b A cone with a height of 20 cm has a volume of 1500 cm^3. Calculate the radius of the cone correct to one decimal place.

c It is noticed that the height of a cone is twice its radius and that the cone's volume is exactly 1000 cm^3. Calculate the dimensions of the cone correct to one decimal place.

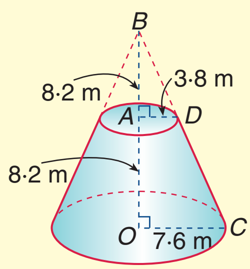
d A conical flask has a radius of 10 cm and a height of 10 cm. The contents of the cone are emptied into a cylinder with a radius of 5 cm. How high must the cylinder be to hold the contents of the cone?

6 Calculate the volume of the following solids. Give all answers correct to three significant figures.

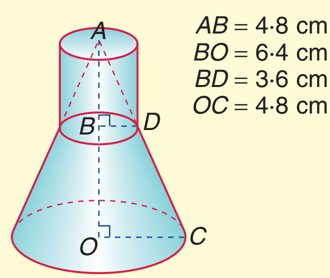
a



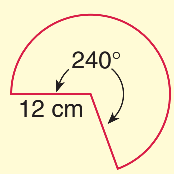
b



c



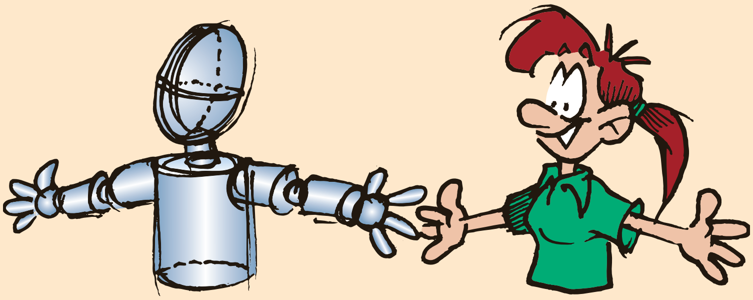
7 The sector shown is formed into a cone by joining its two radii. Calculate the volume of the cone correct to the nearest whole number.



10:07 | Volume of a Sphere

Investigation 10:07 | Estimating your surface area and volume

- Describe how you could estimate your surface area using a suitable roll of material.



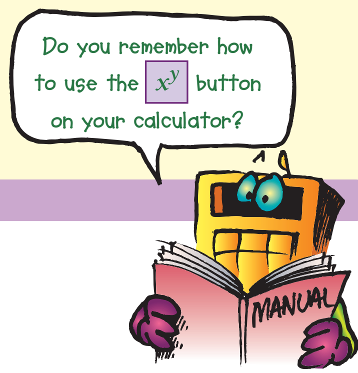
- *The volume of a solid can be established by placing it in a container of water. The amount of water that it displaces is equivalent to the volume of the solid.*
  Describe how you could use this principle, which was discovered by Archimedes, to calculate your volume.

As with the pyramid and cone, the formula for the volume of a sphere is hard to prove. Here the formula is just stated without proof.

The volume of a sphere is given by the formula:

$$V = \frac{4}{3}\pi r^3$$

where r is the radius of the sphere.



worked examples

1. Find the volume of a sphere that has a radius of 5·20 cm. Give your answer correct to three significant figures.
2. If the diameter of a sphere is 3·6 m, calculate the volume of the sphere correct to one decimal place.
3. If the earth is considered to be a sphere of radius 6378 km, find its volume correct to four significant figures.

Solutions

1
$$V = \frac{4}{3}\pi r^3$$
$$= \frac{4}{3} \times \pi \times (5{\cdot}20)^3$$
$$= 589 \text{ cm}^3 \text{ (correct to 3 sig. figs)}$$

3
$$V = \frac{4}{3}\pi r^3$$
$$= \frac{4}{3} \times \pi \times 6378^3$$
$$= 1{\cdot}087 \times 10^{12} \text{ km}^3$$
(correct to 4 sig. figs)

2 Diameter = 3·6 m

$$\therefore \text{Radius} = \frac{3{\cdot}6}{2} \text{ m}$$
$$= 1{\cdot}8 \text{ m}$$
$$\therefore V = \frac{4}{3}\pi r^3$$
$$= \frac{4}{3} \times \pi \times (1{\cdot}8)^3$$
$$= 24{\cdot}4 \text{ m}^3$$
(correct to 1 dec. pl.)

Exercise 10:07

1 Find the exact volumes of the following spheres:

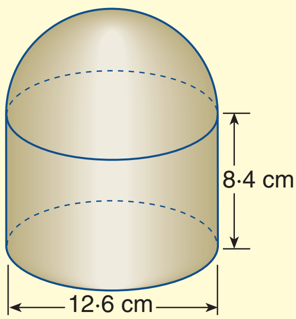
a radius = 3 cm
b diameter = 4 m
c radius = 3·6 m
d diameter = 4·8 m

2 Find the volumes of the following solids. Give the answers correct to three significant figures.

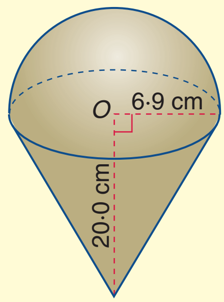
a sphere, radius 1·2 m
b sphere, diameter = 25·6 cm
c hemisphere, radius 3·15 cm
d hemisphere, diameter = 2·40 m

3 Calculate the volume of the following solids, correct to three significant figures.

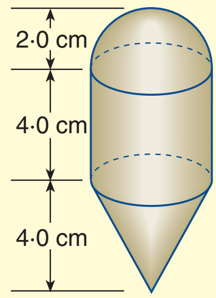
a



b



c



4 Calculate the volume of a spherical shell that has an inner radius of 5 cm and an outer radius of 6 cm. Give the answer in terms of π.

5 **a** A sphere has a volume of 2000 cm^3. Calculate its radius correct to one decimal place.

b What is the smallest radius a sphere can have if it has to have a volume greater than 5000 cm^3? Round your answer up to the nearest whole number.

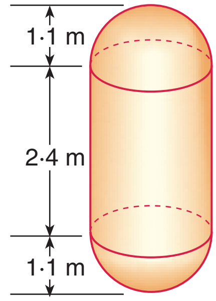
c A sphere has to have a volume of 1 m^3. What is the radius of the sphere? Round your answer up to the nearest centimetre.

- Why is gas stored in spherical tanks? Estimate the volume of gas held in one of these tanks.

10:08 | Practical Applications of Surface Area and Volume

worked example

A buoy consists of a cylinder with two hemispherical ends, as shown in the diagram. Calculate the volume and surface area of this buoy, correct to one decimal place.



Solution

Since the two hemispheres have the same radius, they will form a sphere if joined.

$$\text{Volume of buoy} = \begin{pmatrix}\text{volume of}\\ \text{sphere}\end{pmatrix} + \begin{pmatrix}\text{volume of}\\ \text{cylinder}\end{pmatrix}$$

$$= \tfrac{4}{3}\pi r^3 + \pi r^2 h$$

$$= \tfrac{4}{3}\pi(1{\cdot}1)^3 + \pi \times 1{\cdot}1^2 \times 2{\cdot}4$$

$$= 14{\cdot}7 \text{ m}^3 \text{ (correct to 1 dec. pl.)}$$

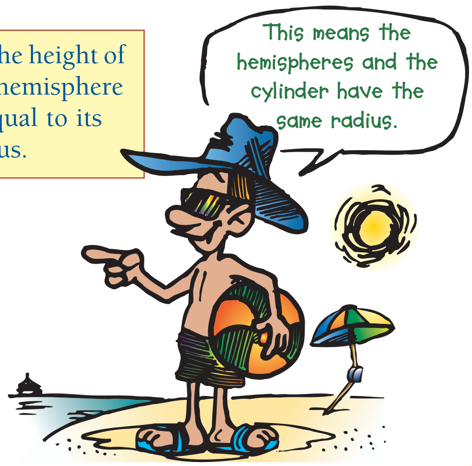
$$\text{Surface area} = \text{surface area of sphere} + \text{curved surface area of cylinder}$$

$$= 4\pi r^2 + 2\pi rh$$

$$= 4\pi \times 1{\cdot}1^2 + 2 \times \pi \times 1{\cdot}1 \times 2{\cdot}4$$

$$= 31{\cdot}8 \text{ m}^2 \text{ (correct to 1 dec. pl.)}$$

■ The height of the hemisphere is equal to its radius.



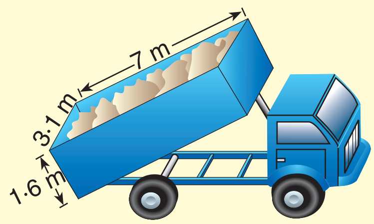
Exercise 10:08

1 A swimming pool is rectangular in shape and has uniform depth. It is 12 m long, 3·6 m wide and 1·6 m deep. Calculate:

- **a** the cost of tiling it at $75/m^2
- **b** the amount of water in litres that needs to be added to raise the level of water from 1·2 m to 1·4 m

■ 1 m^3 = kL

2 The tipper of a trick is a rectangular prism in shape. It is 7 m long, 3·1 m wide and 1·6 m high.



- **a** Calculate the volume of the tipper.
- **b** If the truck carries sand and 1 m^3 of sand weighs 1·6 tonnes, find the weight of sand carried when the truck is three-quarters full.
- **c** Calculate the area of sheet metal in the tipper.

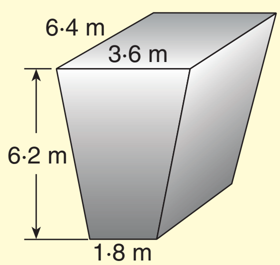
3 A large cylindrical reservoir is used to store water. The reservoir is 32 m in diameter and has a height of 9 m.

a Calculate the volume of the reservoir to the nearest cubic metre.
b Calculate its capacity to the nearest kilolitre below its maximum capacity.
c In one day, the water level drops 1·5 m. How many kilolitres of water does this represent?
d Calculate the outside surface area of the reservoir correct to the nearest square metre. Assume it has no top.

4 Assuming that the earth is a sphere of radius 6400 km, find (correct to 2 sig. figs):

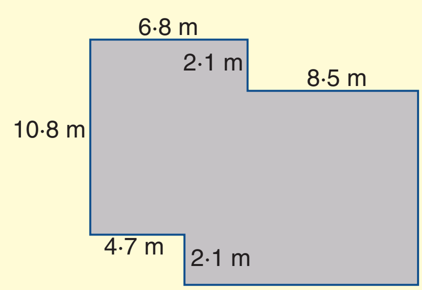
a the volume of the earth in cubic metres
b the mass of the earth if the average density is 5·4 tonnes/m^3
c the area of the earth's surface covered by water if 70% of the earth is covered by water.

5 A bridge is to be supported by concrete supports. Calculate the volume of concrete needed for each support.

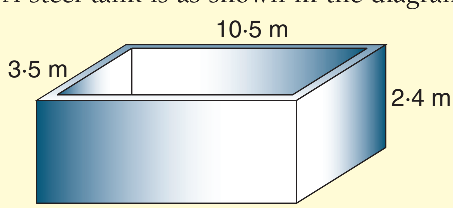


The supports are trapezoidal prisms, as pictured in the diagram.
Give the answer to the nearest cubic metre.

6 A house is to be built on a concrete slab which is 20 cm thick. The cross-sectional shape of the slab is shown in the diagram. Calculate the volume of concrete needed for the slab.

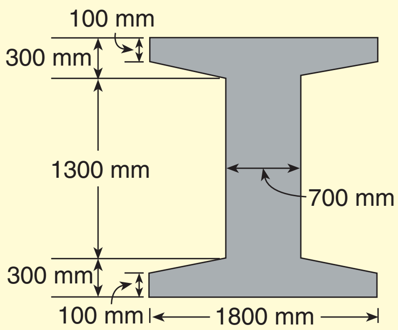


7 A steel tank is as shown in the diagram.



Given that the dimensions are external dimensions and that the steel plate is 2 cm thick, calculate the mass of the tank if the density of the steel is 7·8 g/cm^3. Give the answer correct to one decimal place.

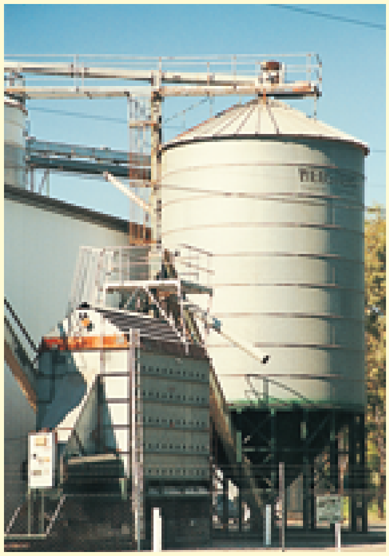
8



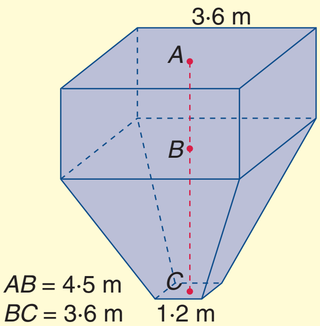
Calculate the volume of a concrete beam that has the cross-section shown in the diagram. The beam is 10 m long.

Calculate the mass of this beam if 1 m^3 of concrete weighs 2·5 tonnes.

9 The large tank in the photo consists of two cones and a cylinder. If the diameter of the cylinder is 5·2 m and the heights of the bottom cone, cylinder and top cone are 2·8 m, 8·5 m and 1·8 m respectively, calculate the volume of the tank correct to one decimal place.



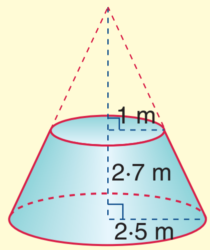
10



A storage bin has been made from a square prism and a square pyramid. The top 1·8 m of the pyramid has been removed. Calculate the volume of the bin.

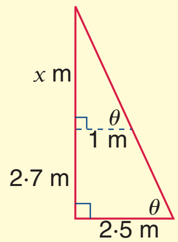
11 A glasshouse is in the shape of a square pyramid. Calculate the area of the four triangular faces to the nearest square metre if the side of the square is 20 m and the height of the pyramid is 17 m.

12 **a** The solid shown is known as a frustrum.
It is formed by removing the top part of the cone.

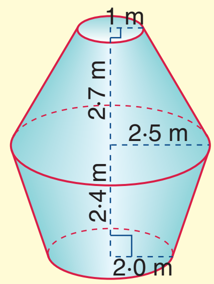


i By comparing the values of tan θ in two different triangles, find the value of x.

ii Find the volume of the frustrum.



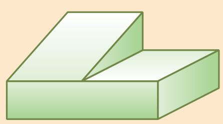
b A storage bin for mixing cement is formed from two truncated cones (frustrums). Calculate the volume of this bin.



Mathematical Terms 10

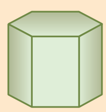
composite solid

- A solid that is formed by joining simple solids.



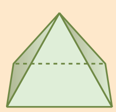
prism

- A solid that has two identical ends joined by rectangular faces.



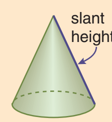
pyramid

- A solid that has a base from which triangular faces rise to meet at a point.



slant height (of a cone)

- The distance from a point on the circumference of the circular base to the apex of the cone.



surface area

- The sum of the areas of the faces or surfaces of a three-dimensional figure (or solid).

volume

- The amount of space (cubic units) inside a three-dimensional shape.

Mathematical terms 10

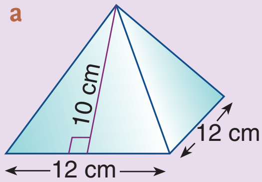
Diagnostic Test 10 | Surface Area and Volume

diagnostic test 10

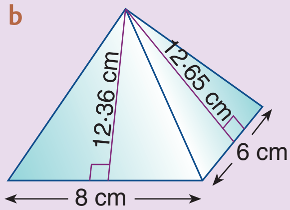
- Each part of this test has similar items that test a certain question type.
- Errors made will indicate areas of weakness.
- Each weakness should be treated by going back to the section listed.

Question	Section
1 Calculate the surface area of the following pyramids.	10:02

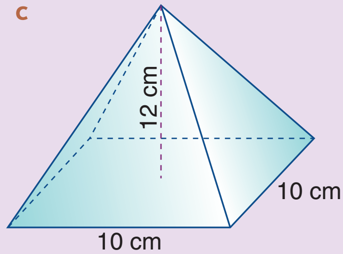
a



b

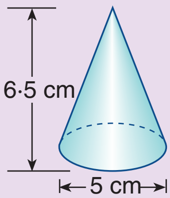


c

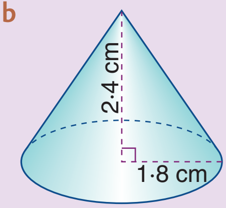


Question	Section
2 Calculate the surface area of the following cones. Give answers correct to one decimal place.	10:03

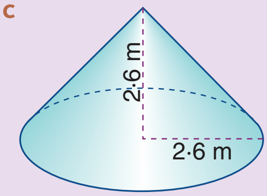
a



b

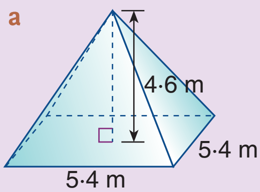


c

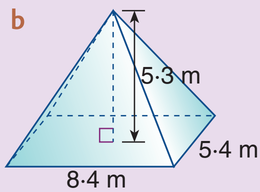


Question	Section
3 Calculate the surface area of: **a** a sphere of radius 5 cm, correct to 2 dec. pl. **b** a sphere of diameter 16·6 cm, correct to 2 dec. pl. **c** a hemisphere of radius 3 cm, correct to 2 dec. pl.	10:04
4 Calculate the volume of the following solids.	10:05

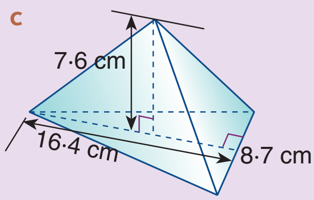
a



b

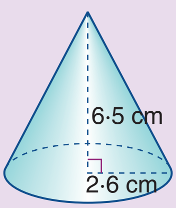


c

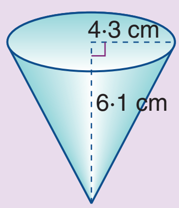


Question	Section
5 Calculate the volume of the following solids correct to one decimal place.	10:06

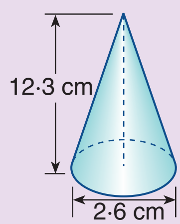
a



b



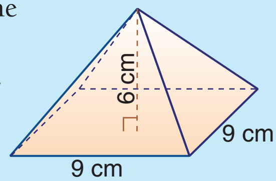
c



Question	Section
6 Calculate the volume of the following solids correct to one decimal place: **a** a sphere of radius 5 cm **b** a sphere of diameter 8·6 cm **c** a hemisphere of diameter 15 cm	10:07

Chapter 10 | Revision Assignment

1 Calculate the volume and surface area of the pyramid shown.

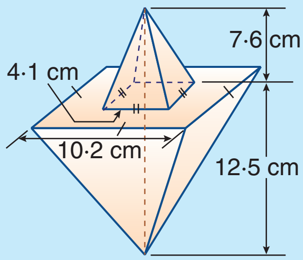


2 A cone has a diameter of 16 cm and a height of 15 cm. Calculate the surface area of the cone in terms of π.

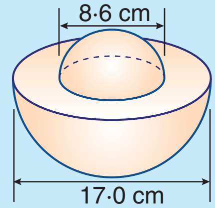
3 Calculate the surface area of a hemisphere that has a diameter of 16 cm. Give your answer correct to two significant figures.

4 A spherical shaped tank is to hold 100 m^3. What radius to the nearest millimetre will give a volume closest to 100 m^3?

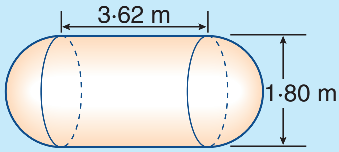
5 Calculate the volume of the solid pictured. Give the answer correct to three significant figures.



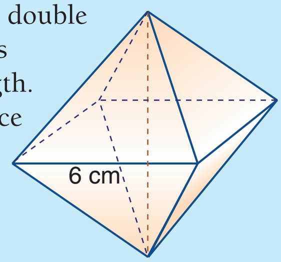
6 Calculate the surface area of the solid shown. Give the answer correct to three significant figures.



7 A tank for holding liquid chemicals consists of a cylinder with two hemispherical ends as shown in the diagram. Calculate its volume (correct to 3 sig. figs.)



8 An octahedron is a double pyramid with all its edges equal in length. Calculate the surface area and volume of an octahedron with all its edges 6 cm in length.



• Engineers solve many surface area and volume problems in the design and construction of buildings.

Chapter 10 | Working Mathematically

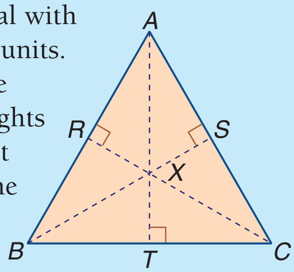
assignment 10B

1 The value of a library is depreciated at a rate of 15% pa. If the library is presently valued at \$800 000, what will its value be after 4 years?

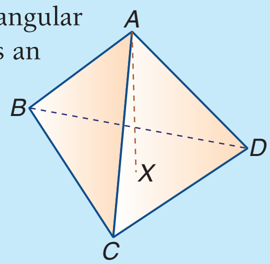
2 ΔABC is equilateral with a side of length a units. AT, BS and RC are perpendicular heights of the triangle that meet at X. Find the lengths:

a BX

b XT

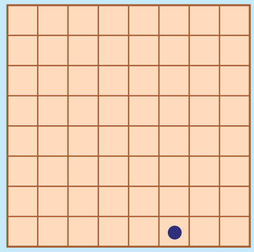


3 $ABCD$ is a right triangular pyramid. Its base is an equilateral triangle of side $\sqrt{72}$ units and the edges AB, AC and AD are all 6 units long. Find the height, AX and the volume of the pyramid.

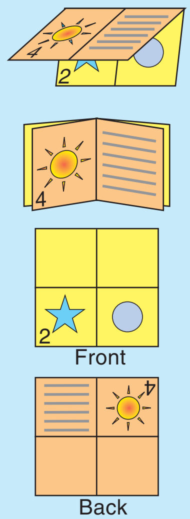


4 Three positive whole numbers are multiplied in pairs. The answers obtained are 756, 1890 and 4410. What are the numbers?

5 The dot indicates the position of one chess queen on a chess board. How many more queens can you place on the board so that none of the queens threatens another?



6 One large sheet of paper was ruled up and folded. It was then cut along the fold shown in the diagram on the top to form an 8-page booklet. On the diagrams to the right, put the page number on each quarter (as has been done for 2 and 4).



Chapter Review Questions

Technology Applications Animations

1 The box

2 Greatest volume

Technology Applications Drag and Drops

1 Volumes of pyramids

2 Volumes of cylinders, cones and spheres

- The glass pyramids outside the Louvre, Paris.

11

Similarity

Chapter Contents

Learning Outcomes

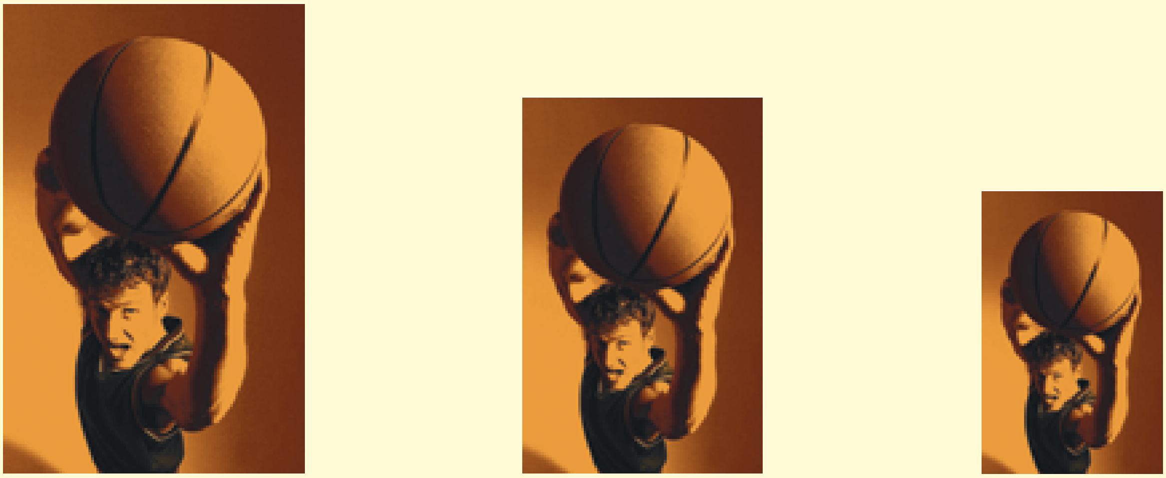
Students will be able to:

- Match similar figures.
- Identify similar triangles.
- Find unknown sides in similar triangles
- Prove triangles similar.
- Compare the areas of similar shapes.
- Compare the volumes of similar solids.

Areas of Interaction

Approaches to Learning (Knowledge Acquisition, Problem Solving, Communication, Logical Thinking, Reflection), Human Ingenuity, Environments

11:01 | Review of Similarity



From earlier studies of similar figures, you should recall the following.

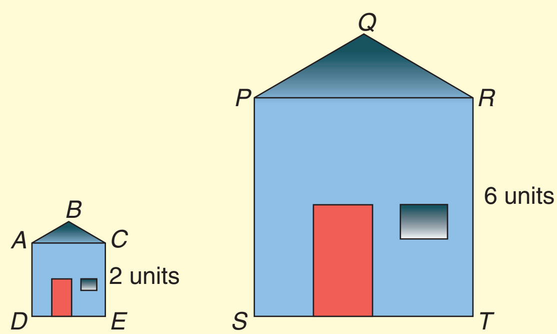
- In mathematics, the word *similar* does *not* mean 'almost equal' or 'nearly the same' but actually means 'the same shape'.

> - **Two figures are similar when one figure can be enlarged and superimposed on the other so that they coincide exactly.**
> - **Similar figures have the same shape, but different size.**

We can superimpose one figure onto a similar figure by using enlargements, translations, rotations and/or reflections.

- We use similar figures in house plans, when enlarging photographs, when using overhead projectors, when making models, in scale drawings, and when using maps.
- If two figures are similar, one can be thought of as the *enlargement* of the other. The **enlargement factor** can be found by dividing any side in the second figure by the corresponding side of the first.

  If the enlargement factor is 3, then each length in the second figure is 3 times the corresponding length in the first figure.

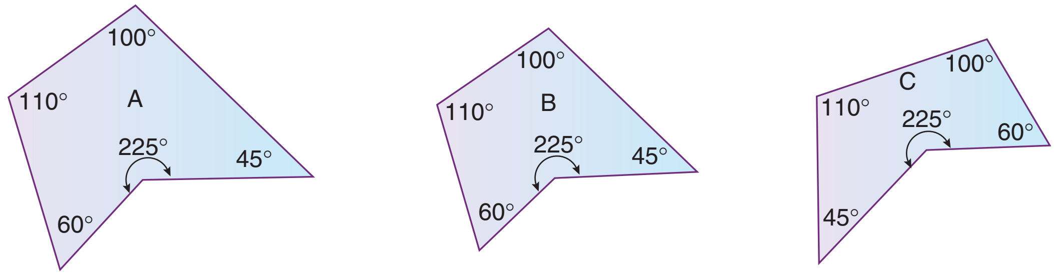


eg Figure 2 is an enlargement of Figure 1. The enlargement factor is 3.

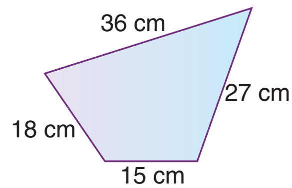
> **If two figures are similar (have the same shape) then:**
> - **matching angles are equal, and**
> - **the ratios of matching sides are equal.**

worked examples

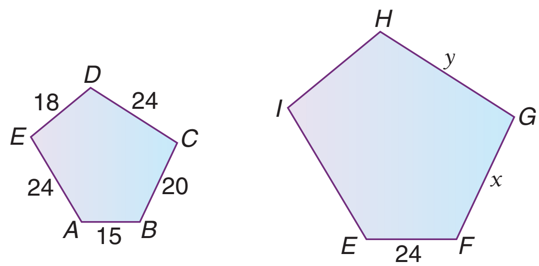
1 Which of the figures below cannot be similar to the other two? Give a reason for your answer.



2 The figure shown is to be enlarged using an enlargement factor of 1·5. What will be the lengths of the sides in the enlarged figure?



3 The two figures shown are similar. Calculate the enlargement factor and use it to find the value of the pronumerals.



Solutions

1 Figure C is not similar to the others. Its angles are the same size as those of figures A and B, but they are not in matching positions. For figures to be similar, matching angles must be equal.

2 The sides of the enlarged figure will be:

$1{\cdot}5 \times 18$ cm $= 27$ cm $1{\cdot}5 \times 36$ cm $= 54$ cm

$1{\cdot}5 \times 27$ cm $= 40{\cdot}5$ cm $1{\cdot}5 \times 15$ cm $= 22{\cdot}5$ cm

3 Enlargement factor = ratio of matching sides

$$= \frac{EF}{AB}$$

$$= \frac{24}{15}$$

$$= 1{\cdot}6$$

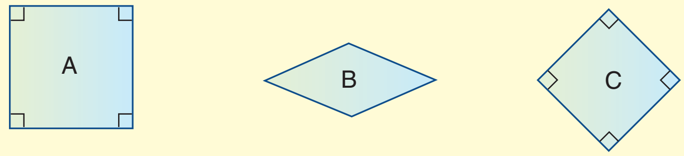
$\therefore\ x = 1{\cdot}6 \times 20$
$= 32$

$y = 1{\cdot}6 \times 24$
$= 38{\cdot}4$

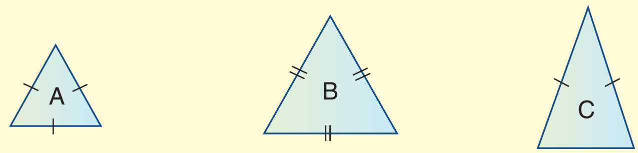
Exercise 11:01

1 Which of the figures cannot be similar to the other two? Give a reason for your answer.

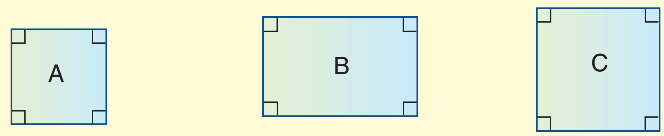
a



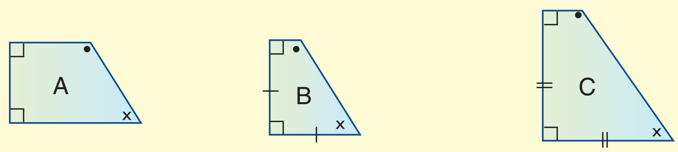
b



c



d



2 **a** A square with sides of 6 cm is enlarged using an enlargement factor of 2·4. What will be the length of the side of the enlarged square?

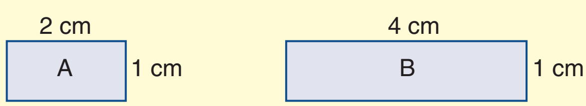
b A rectangle is 24 cm long and 16 cm wide. A rectangle similar to this rectangle is produced by reducing it using a reduction factor of $\frac{3}{4}$. What are the dimensions of the reduced rectangle?

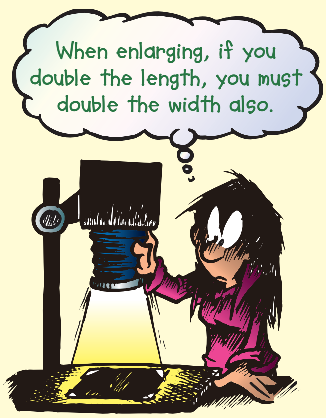
c Explain why any two squares or any two equilateral triangles are similar?

3 **a** A photograph is 8 cm long and 4 cm wide. If the photograph is to be enlarged so that the length is 16 cm, what will the width be?

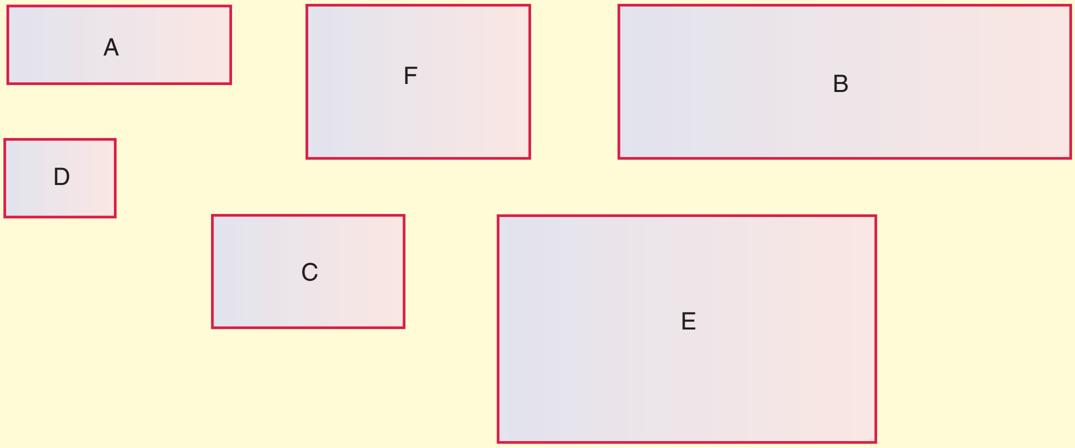
b A photograph is enlarged. If the length is tripled, what happens to its width?

c Could rectangle A be enlarged to give rectangle B?





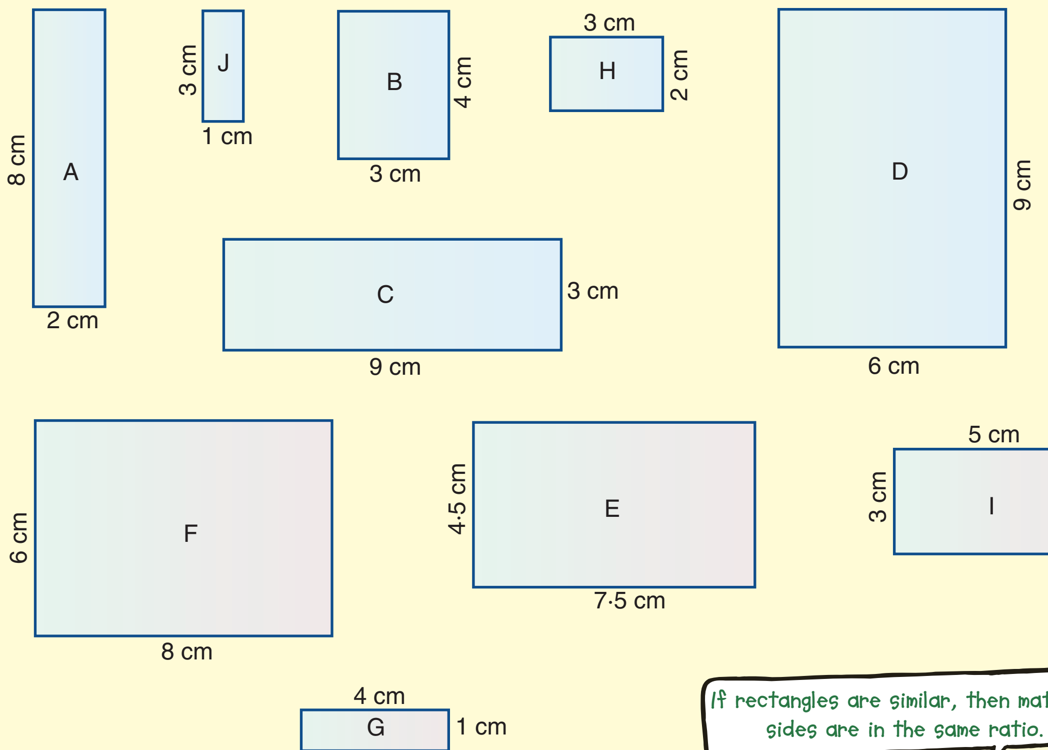
4 **a** Measure the lengths and widths of each of the rectangles below.

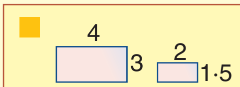


b By comparing the ratios of matching sides, select the pairs of rectangles that are similar.

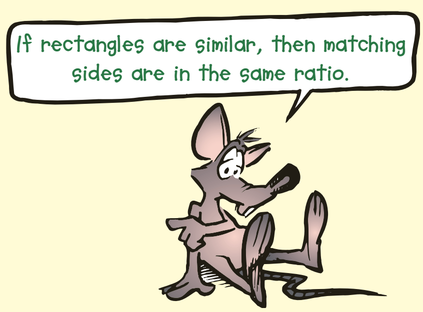
5 Select the rectangle that is similar to:

a A **b** B **c** C **d** D **e** E



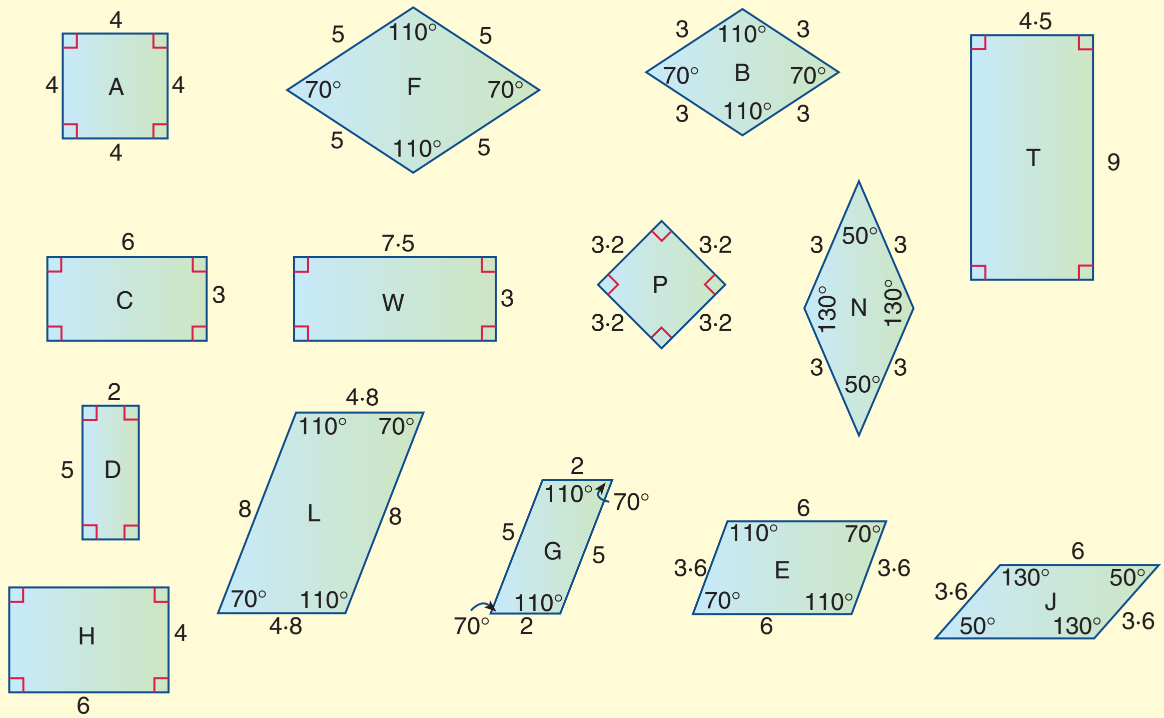


$$\frac{4}{2} = \frac{3}{1 \cdot 5}$$



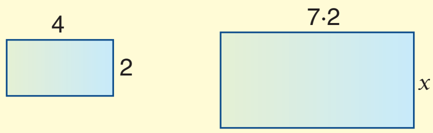
6 After checking that matching angles are equal and that matching sides are in the same ratio, find the figure that is similar to figure:

a A **b** B **c** C **d** D **e** E

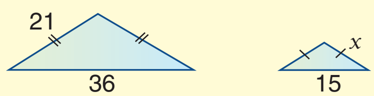


7 Each pair of figures shown is similar. Calculate the enlargement or reduction factor and use it to find the value of the pronumerals. (All measurements are in centimetres.)

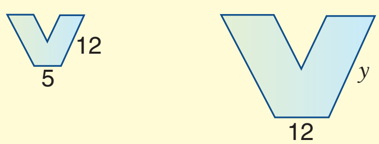
a



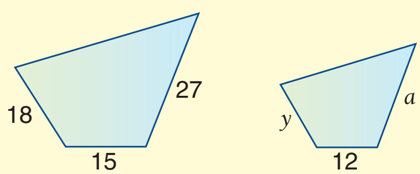
b



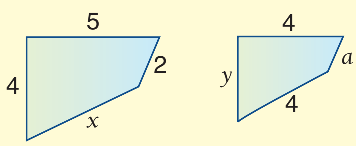
c



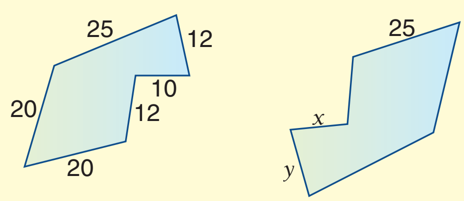
d



e



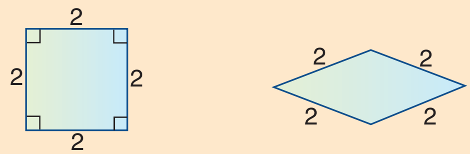
f



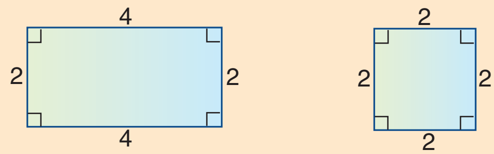
11:02 | Similar Triangles

prep quiz 11:02

1 Are these two figures similar?



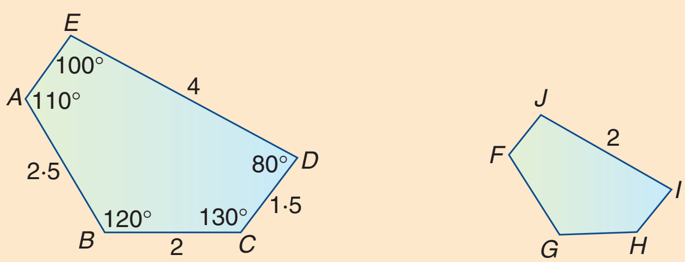
2 Are these two figures similar?



True or false?

3 If two figures have matching angles equal, then they are similar.

4 If the ratio of matching sides in two figures are equal, then the figures are similar.



The figures above are similar.

5 What side matches the side *BC*?

6 What angle matches $\angle E$?

7 What is the size of $\angle G$?

8 What is the value of the ratio $\frac{JI}{ED}$?

9 What is the length of *HI*?

10 What is the length of *FG*?

The Prep Quiz should have reminded you that:

- if two figures are similar, then matching angles are equal and the ratios of matching sides are also equal.
- conversely, if two figures have their matching angles equal and the ratios of matching sides equal, then they are similar.

Because triangles are just special figures, one would expect that similar triangles would have matching angles equal and matching sides in the same ratio.

The following exercises investigate the conditions that are necessary for two triangles to be similar.

11:02A | Matching angles

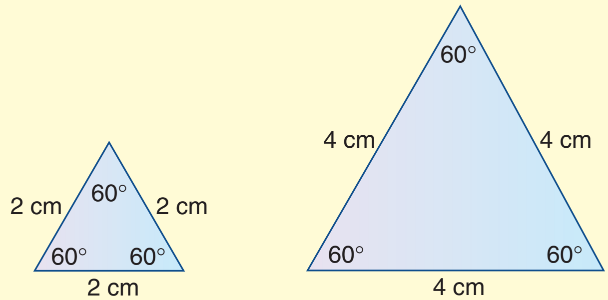
Similar triangles have matching angles that are equal.

Exercise 11:02A

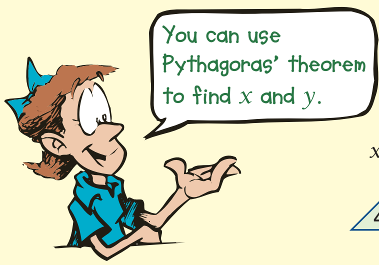
1 The diagrams show two equilateral triangles which have their matching angles equal.

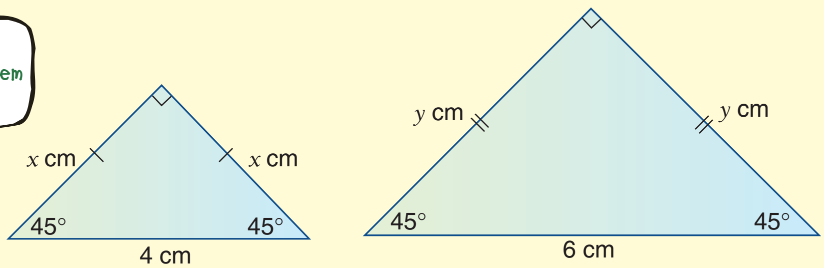
a Are the ratios of the matching sides equal?

b Are the triangles similar?



2 The diagrams show two right-angled triangles with their matching angles equal.



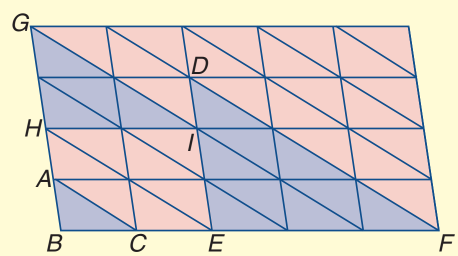


a By measurement or calculation, find the values of x and y.

b Calculate the ratios of the matching sides. Are they equal?

c Are the triangles similar?

3 The three triangles shown are drawn on a grid of parallel lines.



a Consider ΔABC and ΔDEF.

- **i** Are the matching angles equal?
- **ii** Name the sides in ΔDEF that match the sides AB, BC and AC.
- **iii** Are the ratios of the matching sides equal?

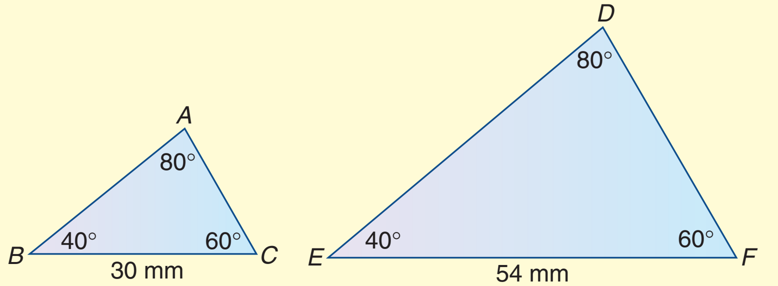
b Consider ΔGHI and ΔDEF.

- **i** Name the pairs of matching angles. Are the matching angles equal?
- **ii** What is the value of each of the following ratios: $\frac{DE}{GH}$, $\frac{EF}{HI}$, $\frac{ED}{IG}$?
- **iii** Are the triangles similar?

c Consider ΔABC and ΔGHI.

- **i** Are the matching angles equal?
- **ii** Are the ratios of the matching sides equal?
- **iii** Are the triangles similar?

4 The triangles shown have their matching angles equal.



The lengths of the sides have been measured to the nearest millimetre and recorded in the table.

Side	*AB*	*BC*	*AC*	*DE*	*EF*	*DF*
Length	26	30	20	47	54	35

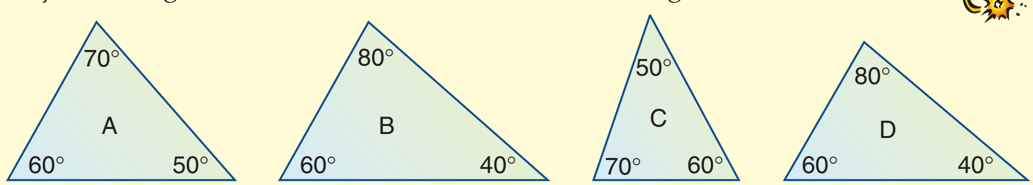
Use the results in the table to calculate the value of the ratio of each pair of matching sides correct to one decimal place.

Questions **1** to **4** illustrate the following result.

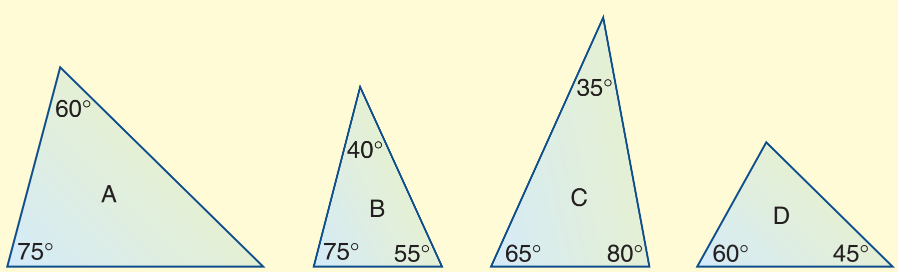
> **Two triangles are similar if matching angles are equal.**
> **This means that the ratios of matching sides are equal.**

5 Identify the triangles that are similar in each of the following.

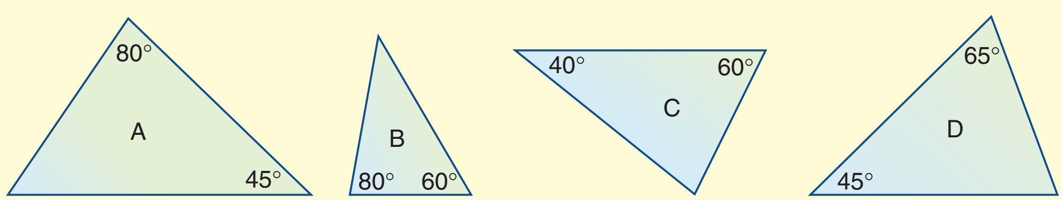
a



b

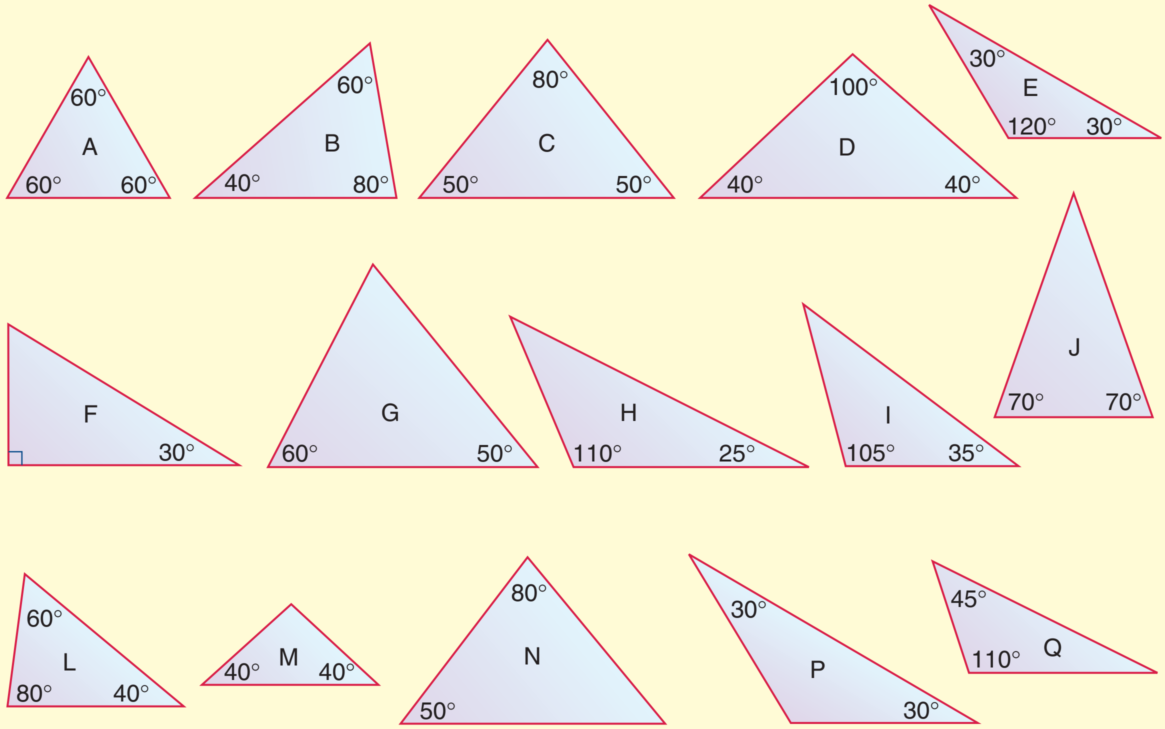


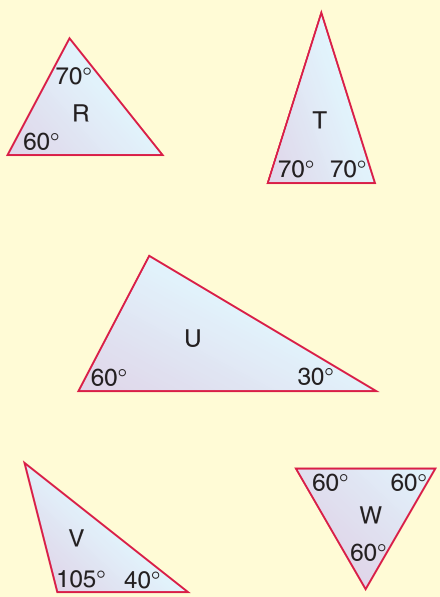
c

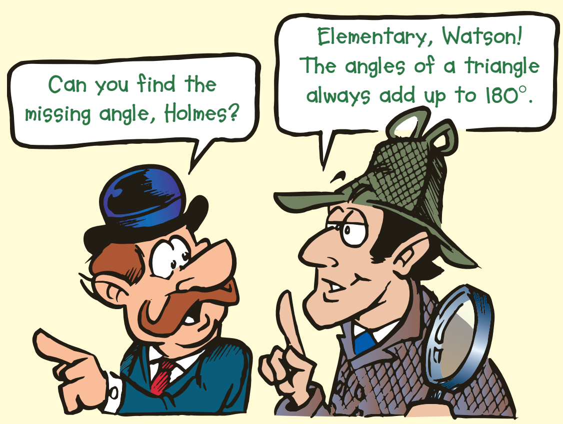


6 Below are shown ten pairs of similar triangles. Each of the triangles A to J is similar to one of the triangles L to W. Select the triangle that is similar to triangle:

a A	**b** B	**c** C	**d** D	**e** E
f F	**g** G	**h** H	**i** I	**j** J



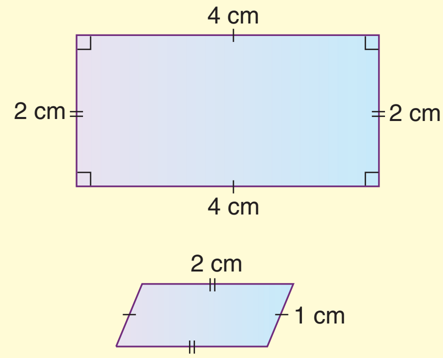




11:02B | Ratios of matching sides

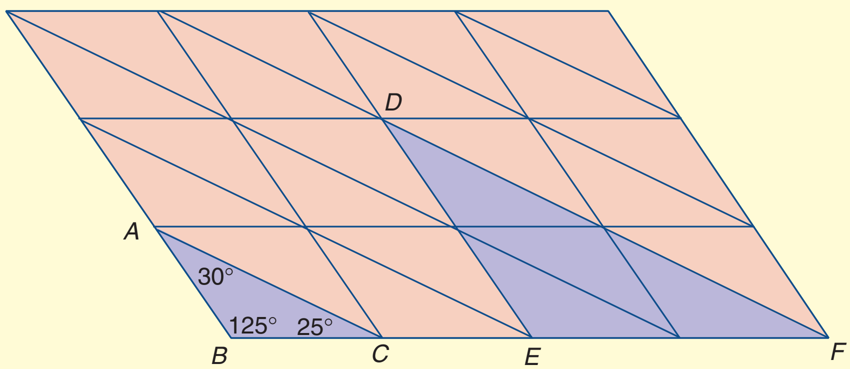
The rectangle and parallelogram shown here are a reminder that if the ratios of matching sides are equal the figures are not necessarily similar. The matching angles must also be equal.

In this exercise, we investigate two triangles that have matching sides in the same ratio.



Exercise 11:02B

1 Triangle ABC has angles of 25°, 30° and 125°, as shown. Copies of $\triangle ABC$ have been used to make the grid which has three sets of parallel lines.



a Are the matching sides of ΔABC and ΔDEF in the same ratio?

b Why does $\angle D = \angle A$, $\angle E = \angle B$ and $\angle F = \angle C$?

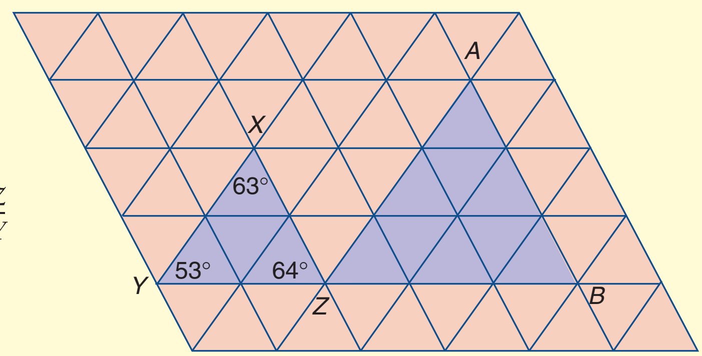
2 Triangles XYZ and AZB have been drawn on a grid based on parallelograms.

a What is the value of the following ratios?

i $\dfrac{ZB}{YZ}$ **ii** $\dfrac{AB}{XZ}$ **iii** $\dfrac{AZ}{XY}$

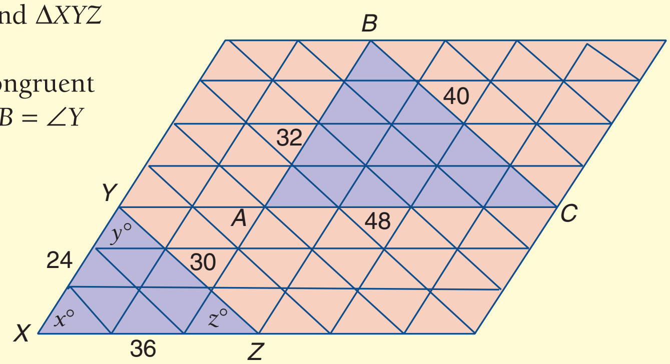
b Why are all the small triangles congruent?

c Are the matching angles of ΔXYZ and ΔAZB equal?



3 **a** Are the matching sides of ΔABC and ΔXYZ in the same ratio?

b State why each small triangle is congruent and hence show that $\angle A = \angle X$, $\angle B = \angle Y$ and $\angle C = \angle Z$.

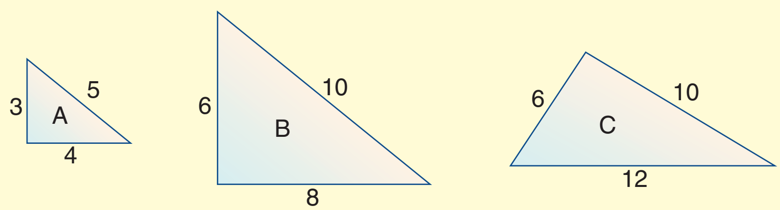


Questions **1** to **3** illustrate the following result.

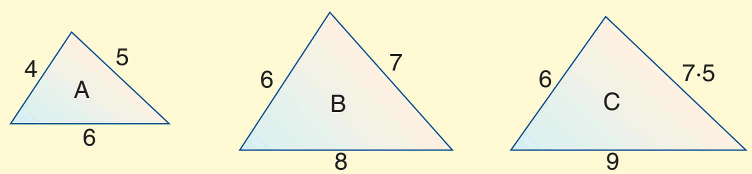
Two triangles are similar if the ratios of matching sides are equal. This means that matching angles are equal.

4 Identify the triangles that are similar in each of the following.

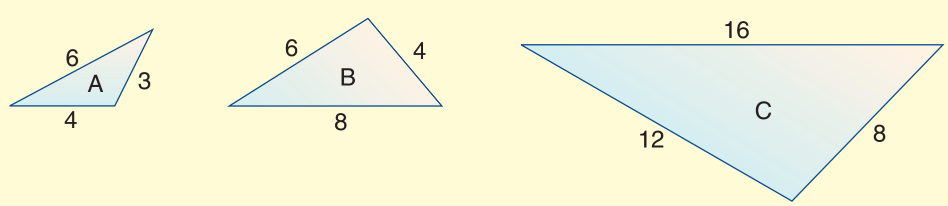
a



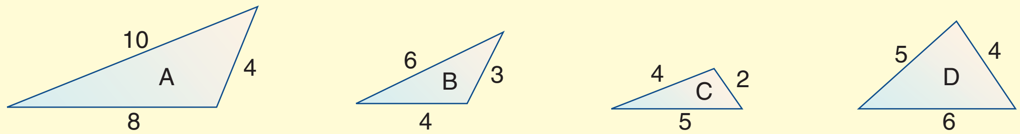
b

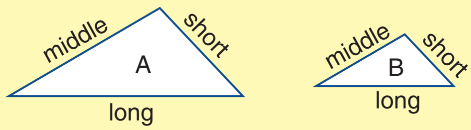


c



d





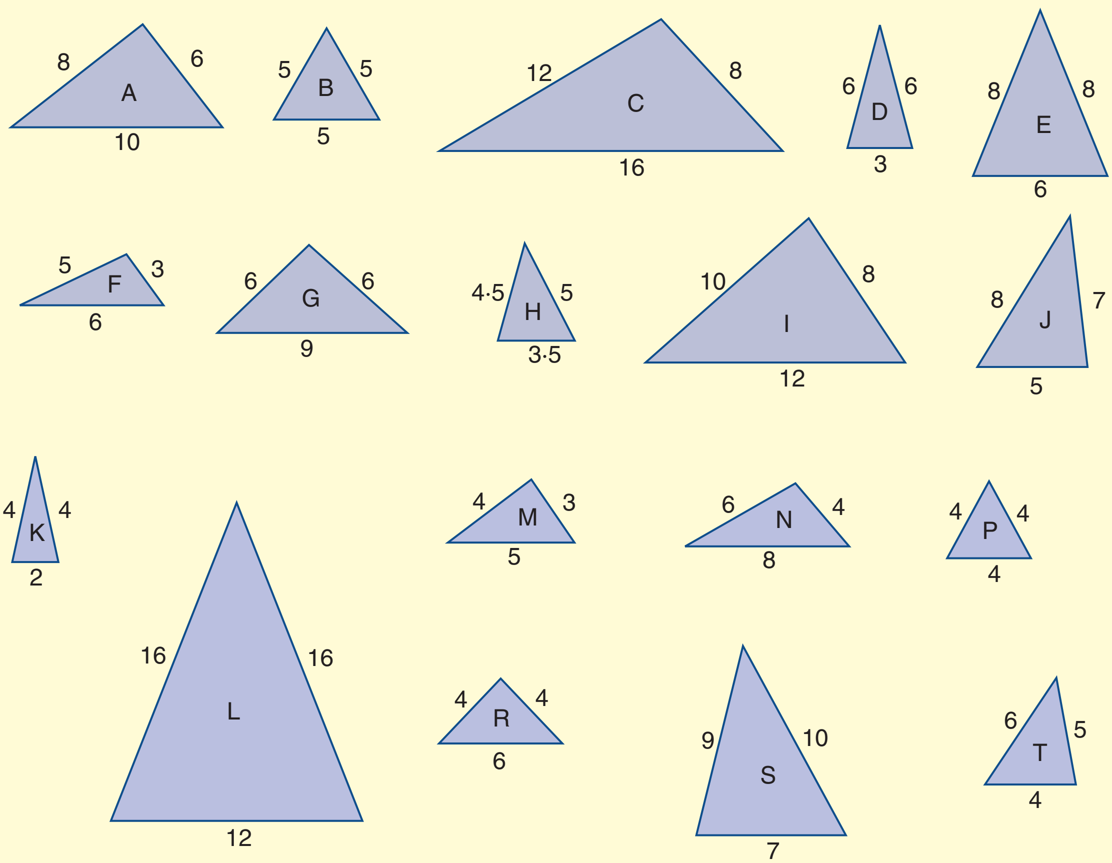
Triangle A is similar to triangle B if:

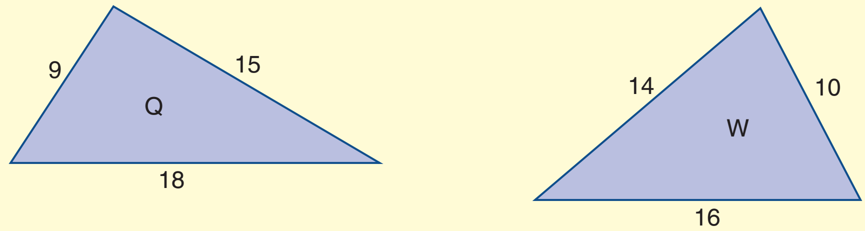
$$\frac{\text{long side of A}}{\text{long side of B}} = \frac{\text{middle side of A}}{\text{middle side of B}} = \frac{\text{short side of A}}{\text{short side of B}}$$

5 Below are shown ten pairs of similar triangles. Each of the triangles A to J is similar to one of the triangles K to W. By checking the ratios of corresponding sides, find the triangle that is similar to:

a A	**b** B	**c** C	**d** D	**e** E
f F	**g** G	**h** H	**i** I	**j** J

(All measurements are in cm.)

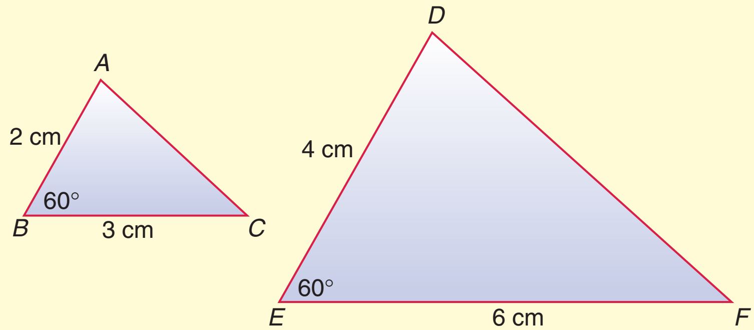




6 In ΔABC and ΔDEF,

$$\frac{DE}{AB} = \frac{EF}{BC} = 2$$

and $\angle B = \angle E$.

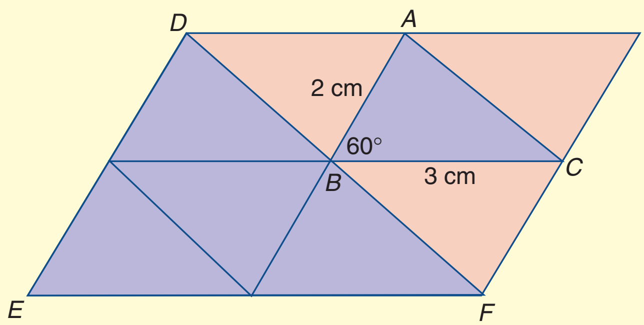


a What else would need to be known before we could say that the triangles are similar?

b In the diagram, a grid based on parallelograms formed from ΔABC is shown. ΔDEF is shown on the grid.

i Why does $\angle A = \angle D$ and $\angle C = \angle F$?

ii Does $\frac{DF}{AC} = 2$?

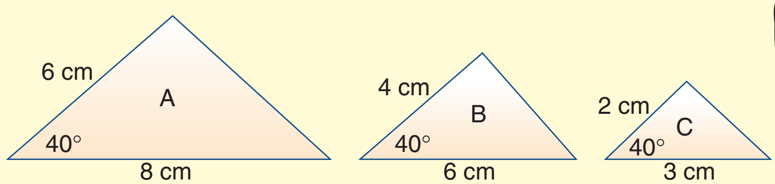


Question **6** illustrates the third condition for two triangles to be similar.

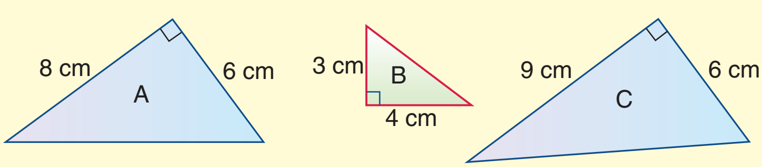
Two triangles are similar if an angle of one triangle is equal to an angle of the other and the lengths of the corresponding sides that form the angles are in the same ratio.

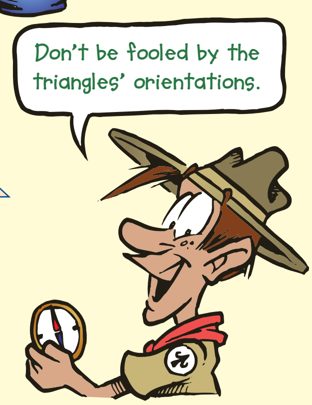
7 Identify the pair of similar triangles in each of the following.

a

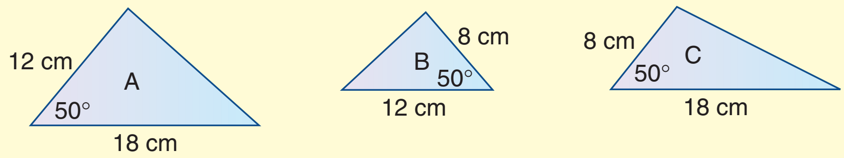


b

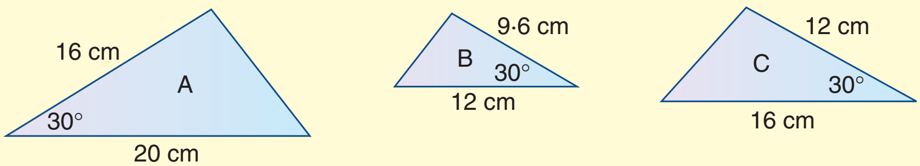




c

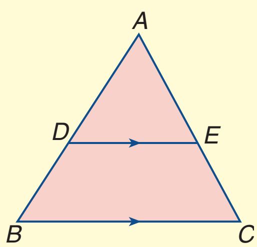


d

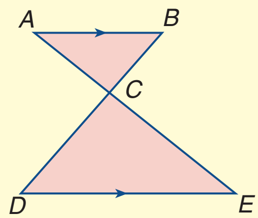


8 Find two similar triangles in each of the following. State which condition could be used to show that the triangles are similar.

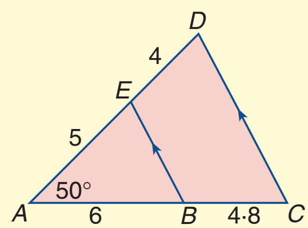
a



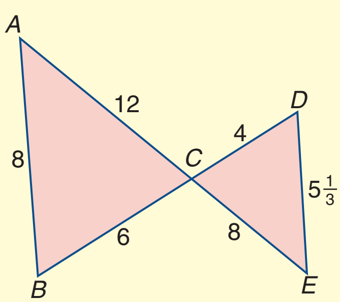
b



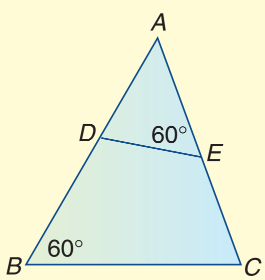
c



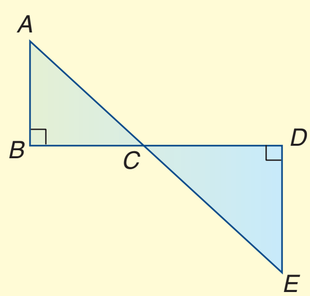
d



e

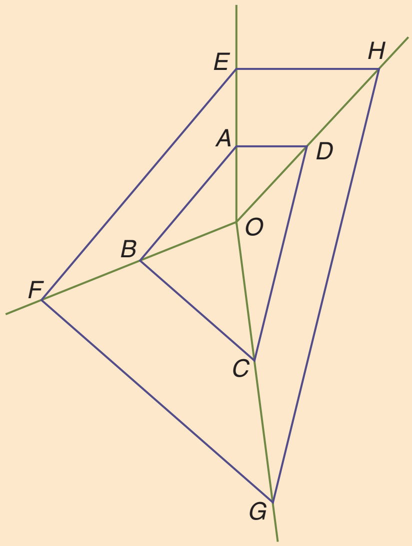


f



Fun Spot 11:02 | Drawing enlargements

- The following questions illustrate a simple way of constructing similar figures by the enlargement method.



ABCD is similar to *EFGH*.

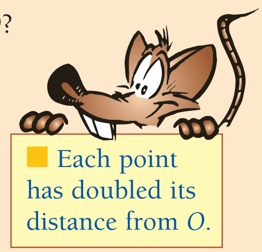
a Which angle corresponds to:
i ∠*A*? **ii** ∠*B*? **iii** ∠*C*? **iv** ∠*D*?

b Which side corresponds to:
i *AB*? **ii** *BC*? **iii** *CD*? **iv** *AD*?

c By measurement, calculate the ratios:
i $\frac{EF}{AB}$ **ii** $\frac{FG}{BC}$ **iii** $\frac{GH}{CD}$ **iv** $\frac{HE}{DA}$

d **i** How far is *E* from *O*?
ii How far is *A* from *O*?
iii Is $EO = 2 \times AO$?

e **i** Is $FO = 2 \times BO$?
ii Is $GO = 2 \times CO$?
iii Is $HO = 2 \times DO$?



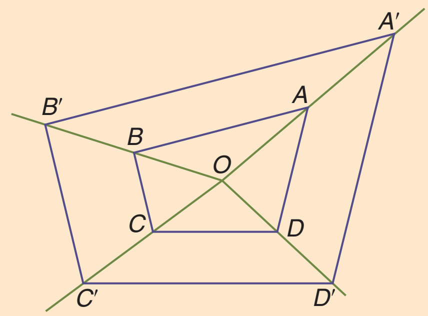
- To draw a figure similar to *ABCD* by enlargement, follow these steps.

Step 1 Select a point *O* to be the centre of enlargement.

Step 2 Draw rays from *O* through each of the vertices *A*, *B*, *C* and *D*.

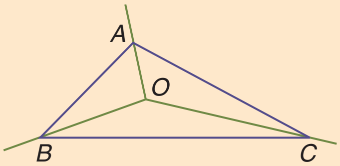
Step 3 Decide on an enlargement factor, say 2.

Step 4 Move each vertex to a new position on its ray which is twice its present distance from *O*. For example, if *A* is 18 mm from *O*, then it becomes 36 mm from *O*.

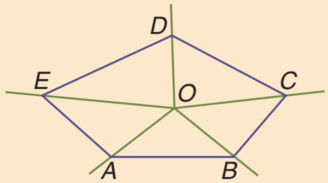


- Trace each of the figures below and then make a similar figure by the enlargement method, using an enlargement factor of 2.

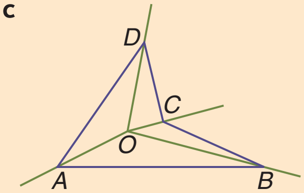
a



b



c

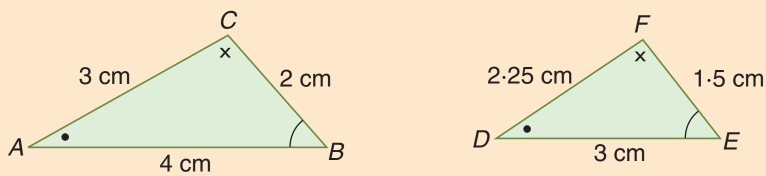


11:03 | Using the Scale Factor to Find Unknown Sides

prep quiz 11:03

Complete the following:

1 In any triangle, the longest side is opposite the ____________ angle.
2 In any triangle, the smallest side is opposite the ____________ angle.
3 If two triangles are similar, the two longest sides are matching. True or false?
4 If two triangles are similar, the two shortest sides are matching. True or false?
5 Matching sides are opposite matching angles. True or false?



The triangles above are similar.

6 Which is the matching side to *AB*?
7 What is the reduction factor?

In the triangles above, which side in △*EDF* is in a matching position to:

8 *BC*? 9 *AC*?
10 What is the ratio of matching sides?

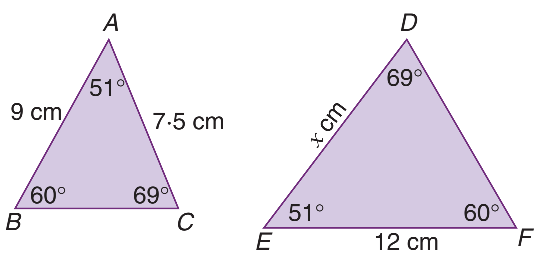
In 10:01, we saw that if two figures are similar, then the ratios of matching sides are equal. This ratio is also the scale factor (either an enlargement or reduction factor).

Once the scale factor is known, it can be used to calculate the lengths of unknown sides. This is shown in the following examples.

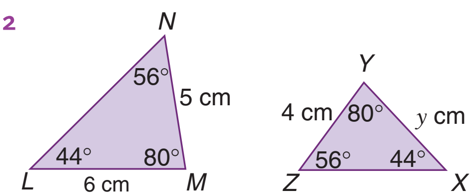
worked examples

Find the value of the pronumeral in each of the following.

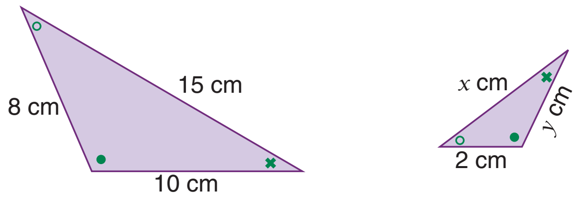
1



2



3



Solutions

1 $\Delta ABC \;|||\; \Delta EFD$
as matching angles are equal.
The scale factor is equal to the ratio of matching sides.
As EF matches AB,

$$\text{Scale factor} = \frac{EF}{AB} = \frac{12}{9}$$

Now DE (x cm) matches with AC (7.5 cm)

$$\therefore x = 7{\cdot}5 \times \frac{12}{9} = 10 \text{ cm}$$

■ The symbol ||| means 'is similar to'.

■ Follow these steps:
Step 1 Use a pair of known matching sides to calculate the enlargement or reduction factor.
Step 2 Find the unknown side and its matching side.
Step 3 Multiply the known side by the scale factor.

2 $\Delta LMN \;|||\; \Delta XYZ$
(matching angles are equal)
MN and YZ are matching sides.

$$\therefore \text{Scale factor} = \frac{YZ}{MN} = \frac{4}{5}$$

YX (y cm) and ML (6 cm) are matching sides.

$$\therefore y = 6 \times \frac{4}{5} = 4{\cdot}8 \text{ cm}$$

3 The 2 cm and 8 cm sides are matching sides (both opposite the angle marked with an X).

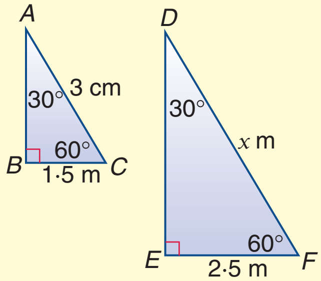
$$\therefore \text{Scale factor} = \frac{2}{8}$$

$$\therefore x = 15 \times \frac{2}{8} = 3{\cdot}75 \text{ cm}$$

$$\therefore y = 10 \times \frac{2}{8} = 2{\cdot}5 \text{ cm}$$

Exercise 11:03

1 For each of the following, copy and complete the working.

a 

$\Delta ABC \parallel\!| \Delta DEF$

AB matches with …

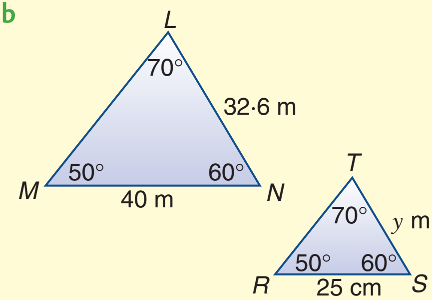
BC matches with …

AC matches with …

Enlargement factor = $\dfrac{2{\cdot}5}{\dots}$

$\therefore\ x = \dots \times \dfrac{2{\cdot}5}{\dots}$

$\therefore\ x = \dots$

b 

$\Delta LMN \parallel\!| \Delta TRS$

LM matches with …

LN matches with …

MN matches with …

Reduction factor = $\dfrac{25}{\dots}$

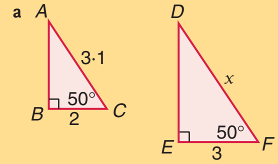
$\therefore\ y = \dots \times \dfrac{25}{\dots}$

$\therefore\ y = \dots$

Foundation Worksheet 11:03

Finding unknown sides in similar triangles

1 For each of the following, copy and complete the working.

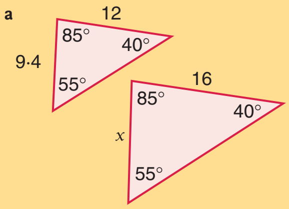
a 

BC matches with …

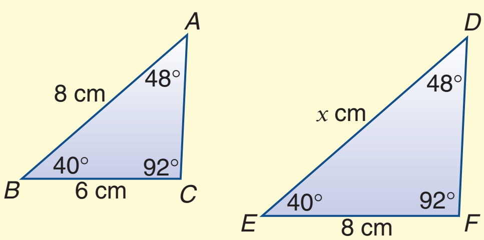
Enlargement factor = …

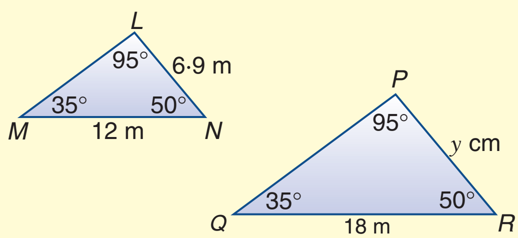
x =

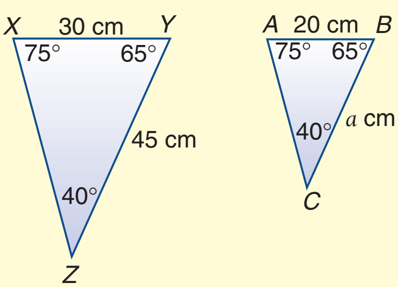
2 Find the value of the pronumerals.

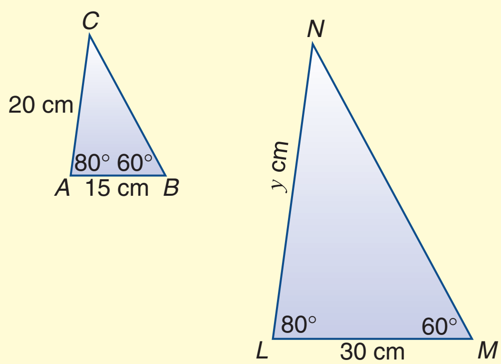
a 

2 Find the values of the pronumerals, correct to one decimal place where necessary.

a 

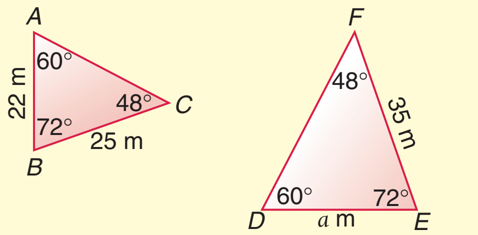
b 

c 

d 

3 Find the values of the pronumerals, correct to one decimal place where necessary.

a



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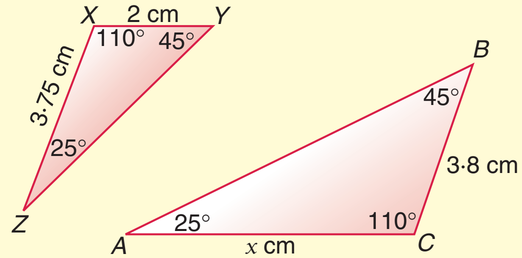

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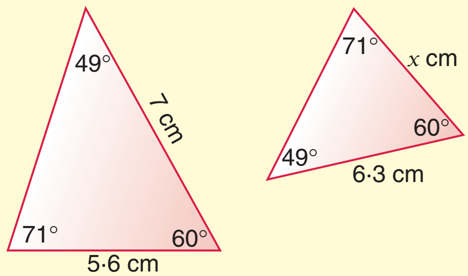
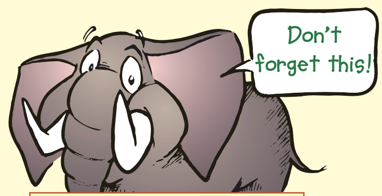
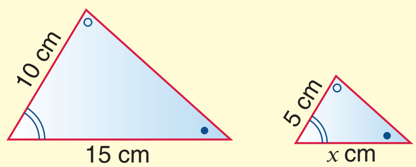




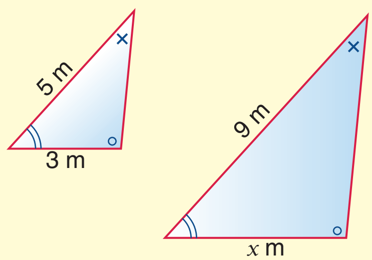
Matching sides are opposite equal angles.

4 Find the value of the pronumeral in each of the following. (Equal angles are marked with the same symbols.)

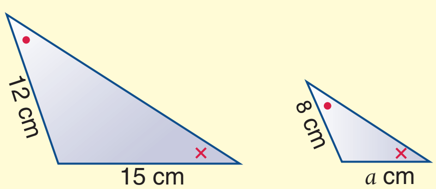
a



b



c



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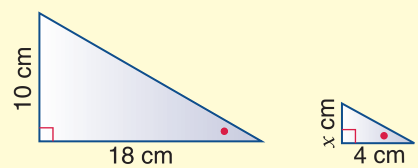
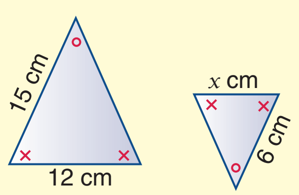


5 Find the value of the pronumeral in each of the following.

a



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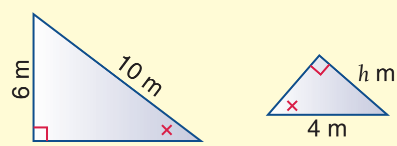

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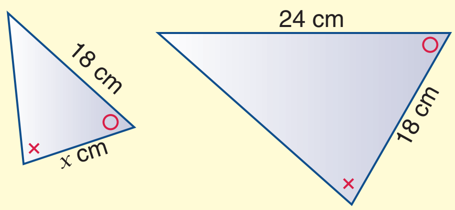
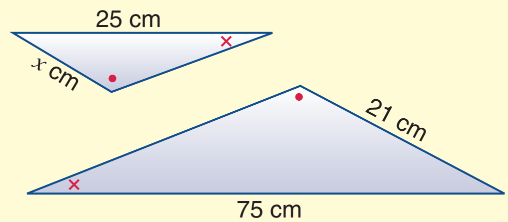
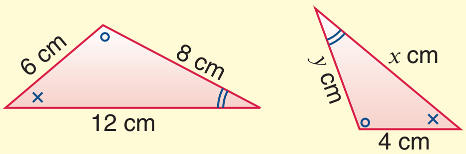


d

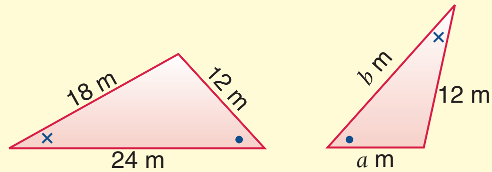


6 Find the values of the pronumerals in each of the following.

a



b



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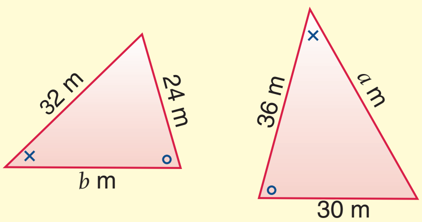

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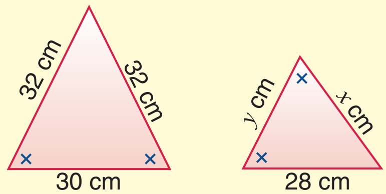
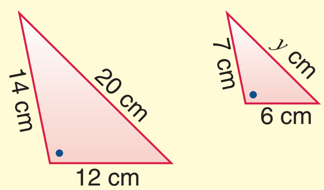


e



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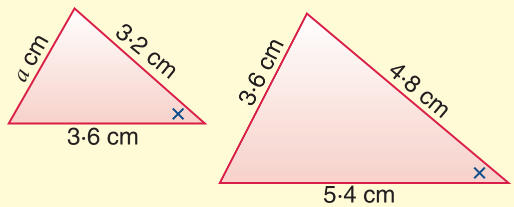
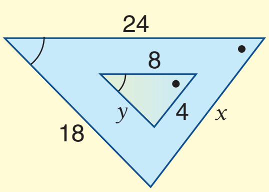
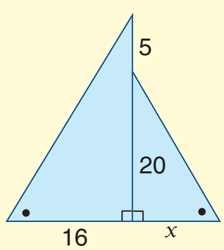


7 Find the values of the pronumerals in each of the following.

a



b



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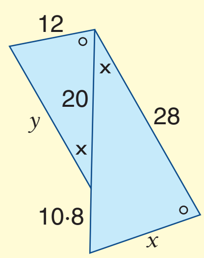


Fun Spot 11:03 | What happened to the mushroom that was double parked?

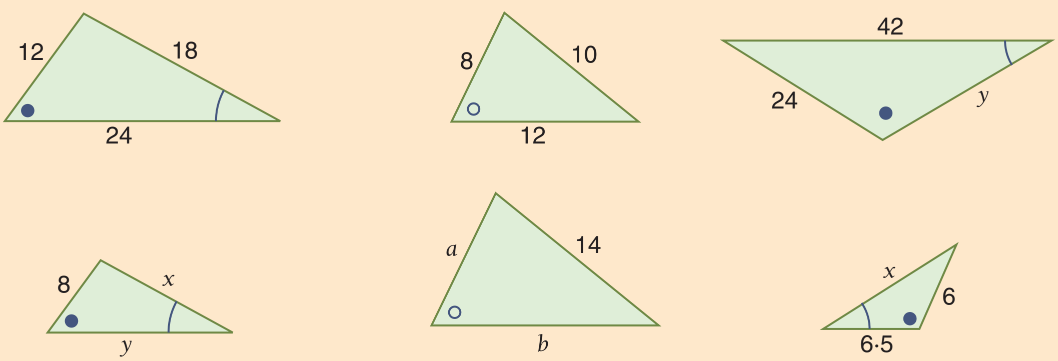
Work out the answer to each part and put the letter for that part in the box that is above the correct answer.

Simplify each ratio:

A 20 : 4 **A** 8 : 16 **B** 8 : 6 **C** 1 : 0·5 **D** $\frac{8}{10}$ **E** $\frac{9}{6}$

Solve:

E $\frac{x}{2} = \frac{3}{2}$ **E** $\frac{x}{3} = \frac{1}{3}$ **I** $\frac{x}{3} = 9$ **L** $\frac{x}{2} = \frac{3}{5}$



M Reduction factor =
S x =
W y =

O Enlargement factor =
T a =
O b =

O Enlargement factor =
T x =
T y =

$x = 27$	26	4 : 3	$\frac{3}{2}$	2 : 1	5 : 1	$\frac{2}{3}$	$x = 1$	1 : 2

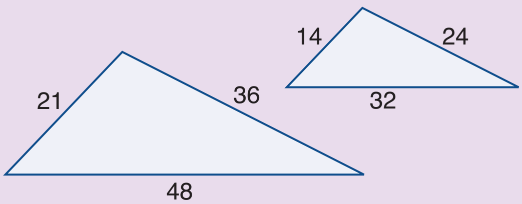
11·2	16·8	16	$x = 3$	$\frac{4}{5}$	$x = 12$	10·5	1·4	4	$x = 1·2$

11:04 | Similar Triangle Proofs

The conditions needed for two triangles to be similar are summarised below. Conditions **1** to **3** were investigated in 11:02. The fourth condition is a special case of condition **1** when it is applied to right-angled triangles.

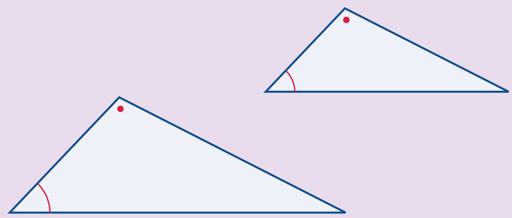
Two triangles are similar if:

1 the lengths of matching sides are in the same ratio.



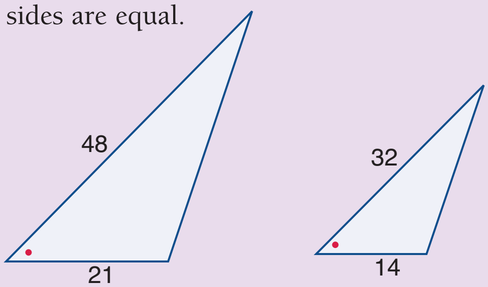
$$\frac{21}{14} = \frac{36}{24} = \frac{48}{32}$$

2 two angles of one are equal to two angles of the other.



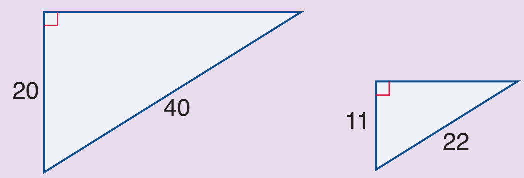
(The third angles are also equal.)

3 the lengths of two pairs of sides are in the same ratio and the angles included by these sides are equal.



$$\frac{48}{32} = \frac{21}{14}$$

4 both triangles are right-angled and the ratio of the hypotenuse to one side in one triangle equals the ratio of the hypotenuse to one side in the other triangle.

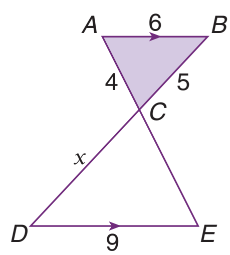


$$\frac{40}{20} = \frac{22}{11}$$

In the following exercise, the similarity conditions will be used to:
- write formal proofs of similarity of triangles and hence find unknown lengths and angles
- prove and apply further theorems.

worked examples

1 $AB \parallel DE$. Prove that $\Delta ABC \parallel\!| \Delta EDC$. Hence, find the value of x. Units are in metres.



Solution

In Δs ABC and EDC

$\angle ABC = \angle EDC$ (alternate angles $AB \parallel DE$)

$\angle ACB = \angle ECD$ (vertically opposite angles)

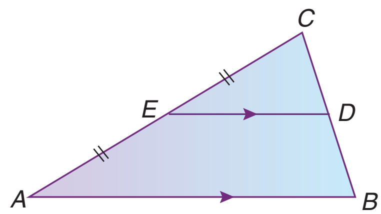
$\therefore \Delta ABC \parallel\!| \Delta EDC$ (equiangular)

$\therefore \frac{x}{5} = \frac{9}{6}$ (ratio of matching sides are equal)

$x = \frac{5 \times 9}{6}$

$\therefore x = 7\frac{1}{2}$

2 In the diagram shown, E is the midpoint of AC and ED is parallel to AB.
Prove that $CD = DB$.



Solution

In Δs CED and CAB

$\angle CED = \angle CAB$ (corresp. $\angle$s, $ED \parallel AB$)

$\angle CDE = \angle CBA$ (corresp. $\angle$s, $ED \parallel AB$)

$\therefore \Delta CED \; ||| \; \Delta CAB$ (2 pairs of equal angles)

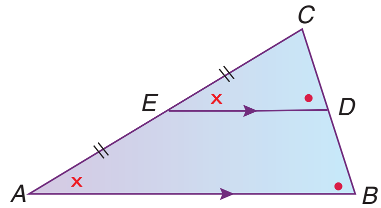
$\therefore \dfrac{CB}{CD} = \dfrac{CA}{CE}$ (ratio of matching sides are equal)

$\therefore \dfrac{CB}{CD} = \dfrac{2}{1}$ ($CA = 2 \times CE$)

$\therefore CB = 2 \times CD$

But $CB = CD + DB$

$\therefore CD = DB$

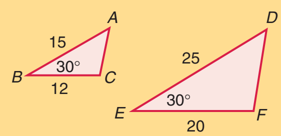


This example proves the following theorem:

A line drawn through the midpoint of a side of a triangle parallel to another side bisects the third side.

Exercise 11:04

Foundation Worksheet 11:04
Similar triangle proofs
1 Complete the proof to show that $\Delta ABC \; ||| \; \Delta DEF$.



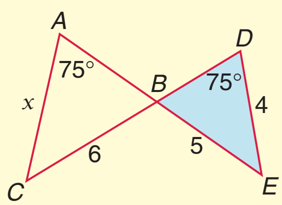
1 $\angle ABC = \ldots$ (data)

2 $\dfrac{AB}{DE} = \dfrac{CB}{FE} = \dfrac{\ldots}{\ldots}$

$\therefore \Delta ABC \; ||| \; \Delta DEF$ (. . .)

1 For each part, prove that the triangles are similar and then find the value of the pronumeral.

a



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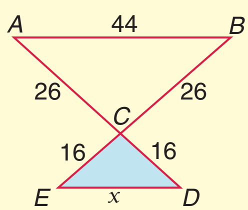
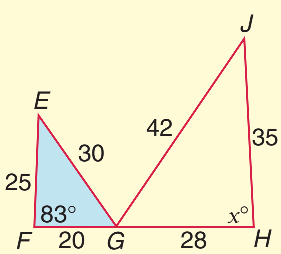
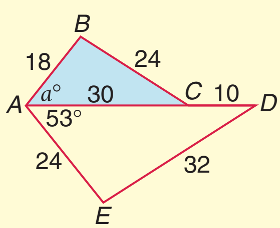


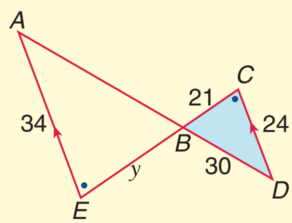
c



d

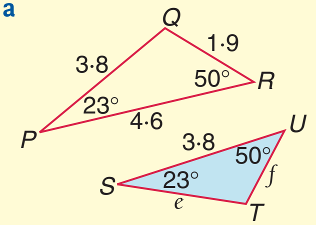


e



2 State what condition could be used to prove that the triangles are similar and then find the value of the pronumerals.

a



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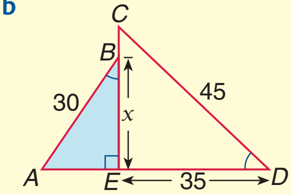
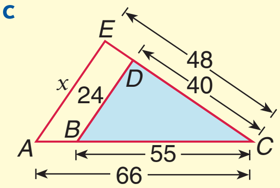
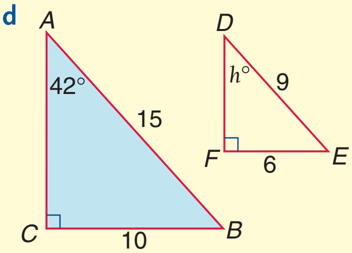


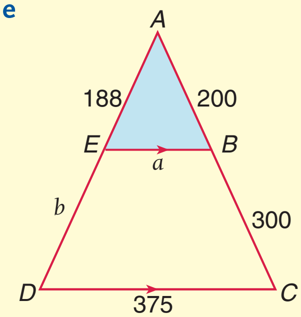
c



d



e



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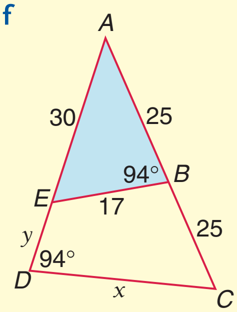
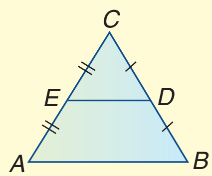


3 In the diagram, E and D are the midpoints of AC and BC respectively. Prove that $ED \parallel AB$ and $ED = \frac{1}{2}AB$.

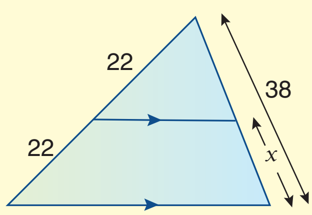


Question **3** *proves the following theorem:*

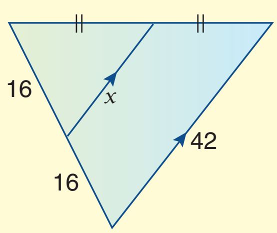
The interval joining the midpoints of two sides of a triangle is parallel to the third side and half its length.

4 Use the theorem above and the theorem on page 260 to find the value of the pronumerals in each of the following.

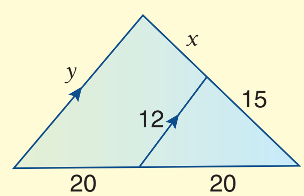
a



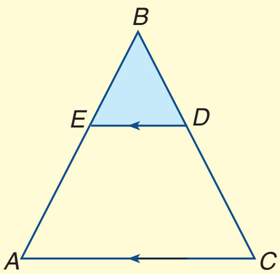
b



c



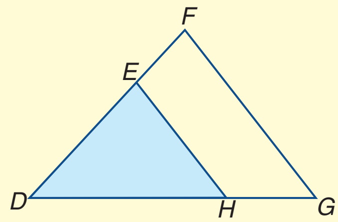
5 **a**



D divides *BC* in the ratio 4 : 5.
$AC \parallel ED$, $BA = 35$ m.

i Prove that $\Delta EBD \parallel\!| \Delta ABC$, giving reasons.
ii Find the ratio of *BD* to *BC*.
iii Find the length of *BE*.
iv Find the length of *EA*.
v Find the ratio of *BE* to *EA*.
vi Show that *ED* divides *BA* and *BC* in the same ratio, ie show that $\dfrac{BE}{EA} = \dfrac{BD}{DC}$.

b



$DE : EF = 3 : 2$, $DH : HG = 3 : 2$.

i Find $DE : DF$.
ii Find $DH : DG$.
iii Prove that $\Delta EDH \parallel\!| \Delta FDG$.
iv Is $\angle DEH$ equal to $\angle DFG$? Why?
v Is $\angle DHE$ equal to $\angle DGF$? Why?
vi Is $EH \parallel FG$? Why?

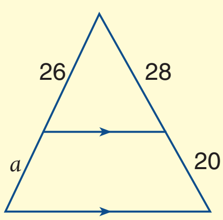
You have shown that if a line divides two sides of a triangle in the same ratio it must be parallel to the third side.

From **5a** above, we see that:

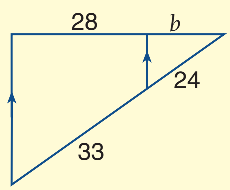
A line parallel to one side of a triangle divides the other two sides in the same ratio.

6 Find the value of the pronumeral in each of the following. Lengths are in metres.

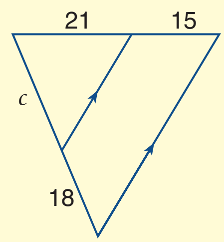
a



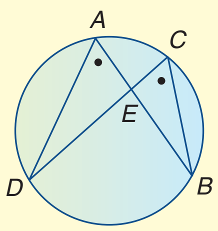
b



c

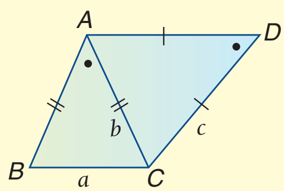


7 **a**



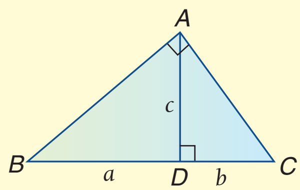
Prove that
$\Delta ADE \parallel\!| \Delta CBE$
and hence that
$AE \times EB = CE \times ED$

b



Δs *ABC* and *ADC* are isosceles with $\angle BAC = \angle ADC$.
Prove that $ac = b^2$

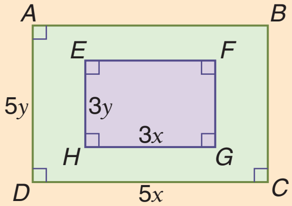
c



$\angle BAC = 90°$ and $AD \perp BC$.
Prove that $ab = c^2$

11:05 | Sides and Areas of Similar Figures

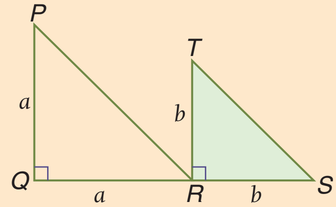
prep quiz 11:05



The rectangles are similar. Find the ratio:

1 $AD : EH$

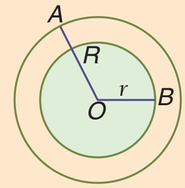
2 area $ABCD$: area $EFGH$.



The triangles are similar. Find the ratio:

3 $PQ : TR$

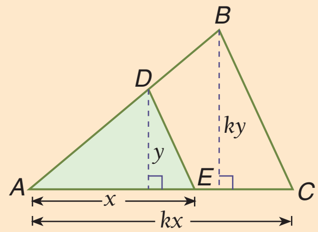
4 area ΔPQR : area ΔTRS.



The circles are similar. Find the ratio:

5 $OA : OB$

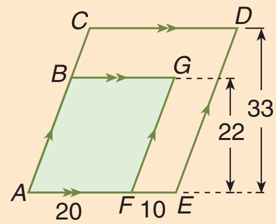
6 area of large circle to area of small circle.



$\Delta ABC \;|||\; \Delta ADE$.
Find the ratio:

7 $AC : AE$

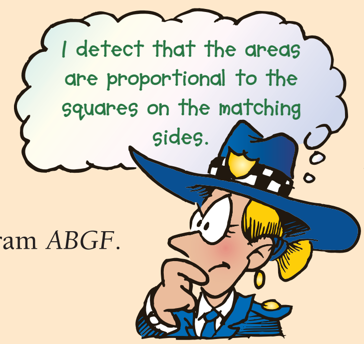
8 area ΔABC : area ΔADE.



Parallelogram $ACDE \;|||$ parallelogram $ABGF$.
Find the ratio:

9 $AE : AF$

10 area $ACDE$: area $ABGF$.



In the Prep Quiz, we investigated the relationship between the sides of similar figures and the areas of those same figures.

Results:

Similar figures	*Ratio of sides*	*Ratio of areas*
Rectangles	$5 : 3$	$5^2 : 3^2$
Triangles	$a : b$	$a^2 : b^2$
Circles	$R : r$	$R^2 : r^2$
Triangles	$k : 1$	$k^2 : 1^2$
Parallelograms	$3 : 2$	$3^2 : 2^2$

In similar figures:

if the ratio of matching sides is $a : b$
then the ratio of their areas is $a^2 : b^2$

ie The areas of similar figures are proportional to the squares on matching sides.

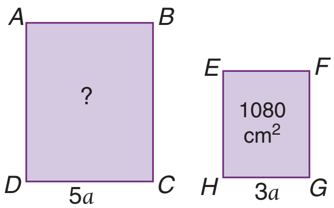
worked examples

1 The pages of two morning newspapers are similar in shape and the widths are in the ratio 3 : 5. Find the area of the larger page if the smaller one has an area of 1080 cm^2.

2 Two similar decorative stickers of boomerangs were produced for the Australia Day celebrations. The area of the larger boomerang sticker is 108 cm^2 and the area of the smaller is 48 cm^2. If the length of the larger is 18 cm, find the length of the smaller.

Solutions

1



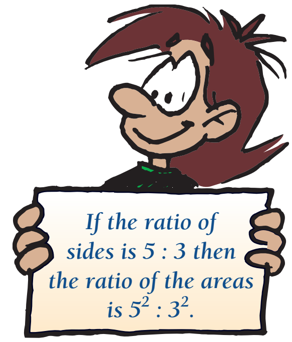
$$\frac{\text{area of } ABCD}{\text{area of } EFGH} = \frac{5^2}{3^2}$$

$$\therefore \frac{\text{area of } ABCD}{1080 \text{ cm}^2} = \frac{25}{9}$$

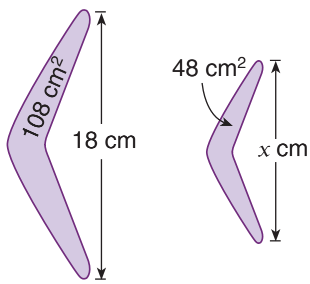
$$\text{area of } ABCD = \frac{25 \times 1080 \text{ cm}^2}{9}$$

$$= 3000 \text{ cm}^2$$

∴ the area of the larger page is 3000 cm^2



2



Let the length of the smaller sticker be x cm.

$$\therefore \frac{x^2}{18^2} = \frac{48}{108}$$

$$x^2 = \frac{48 \times 18^2}{108}$$

$$x^2 = 144$$

$$\therefore x = \pm 12$$

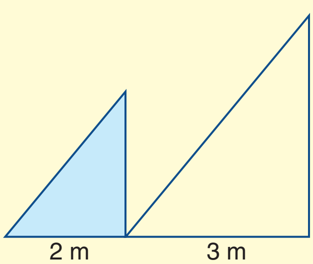
∴ the length of the smaller sticker is 12 cm.

Since a length must be positive, we ignore the −12.

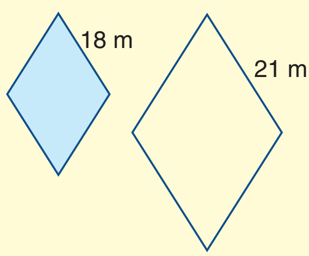
Exercise 11:05

1 In each, the figures are similar. Find the ratio of the smaller area to the larger.

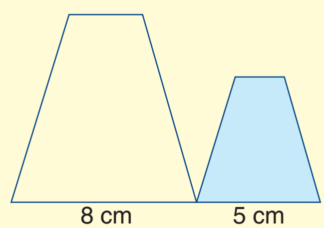
a



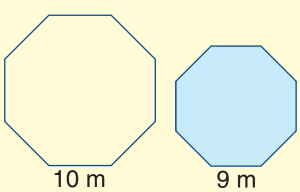
b



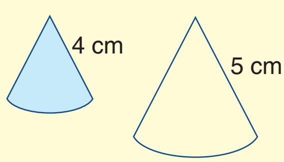
c



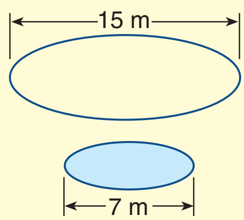
d



e



f



2 **a** In **1a**, the area of the larger triangle is 5·85 m². Find the area of the smaller triangle.

b In **1b**, the area of the larger rhombus is 367·5 cm². Find the area of the smaller rhombus.

c In **1c**, the area of the larger trapezium is 40 cm². Find the area of the smaller trapezium.

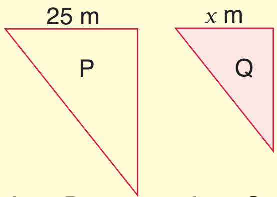
d In **1d**, the area of the smaller octagon is $162(1 + \sqrt{2})$ m². Find the area of the larger octagon.

e In **1e**, the area of the smaller sector is $\frac{16\pi}{6}$ cm². Find the area of the larger sector.

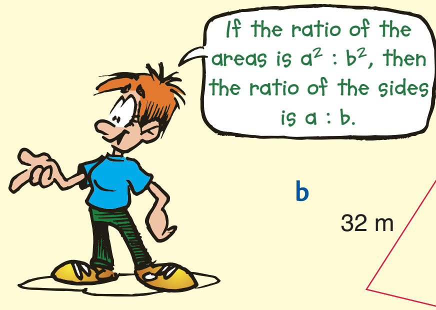
f In **1f**, the area of the smaller ellipse is 17·0 m² (to 1 dec. pl.). Find the area of the larger ellipse correct to one decimal place.

3 In each, the figures are similar. Find the ratio of a side on the smaller figure to the matching side on the larger, then find the value of the pronumeral.

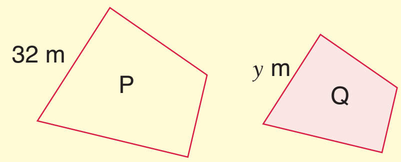
a



Area P = 375 m² Area Q = 135 m²

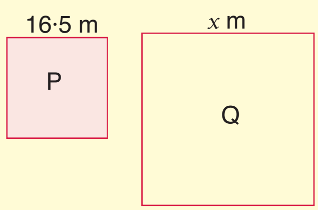


b



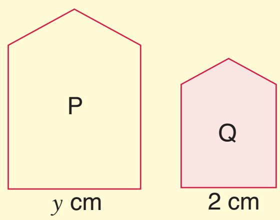
Area P = 592 m² Area Q = 333 m²

c



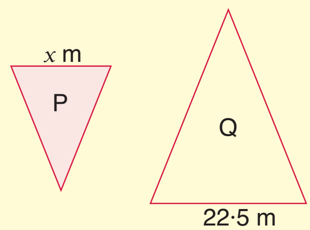
Area P = 272·25 m² Area Q = 900 m²

d



Area P = 6·875 cm² Area Q = 4·4 cm²

e



Area P = 135 m² Area Q = 303·75 cm²

4 **a** If the dimensions of an equilateral triangle are trebled, how is the area affected?

b If the areas of similar quadrilaterals are in the ratio 16 : 25, what is the ratio of matching sides?

c The ratio of matching dimensions of a home and its house plan is 150 : 1. Find the ratio of matching areas.

d The ratio of matching sides of two similar figures is 4 : 7. What is the ratio of the perimeters of the similar figures?

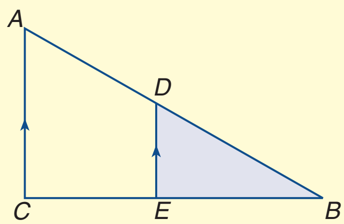
5 The matching sides of two similar kites are in the ratio 11 : 16. Find the area of the smaller if the larger has an area of 1·44 m².

6 **a** A photograph has a width of 9·5 cm and an area of 104·5 cm^2. An enlargement is to be made that has a width of 19 cm. What will be the area of the enlargement?

b A postcard is 8 cm wide and has an area of 96 cm^2. If the postcard is enlarged so that its area is 1536 cm^2, what is the width of the enlargement?

7 My neighbour, whose lawn is a similar shape to mine but has dimensions $\frac{2}{3}$ as large, was surprised to hear that I used 90 kg of ammonium sulfate to treat my lawn. How much would he need to treat his lawn in the same way?

8



$DE \parallel AC$ and $CE : EB = 2 : 3$. Find:

a $AD : DB$

b $AC : DE$

c area DBE : area ABC

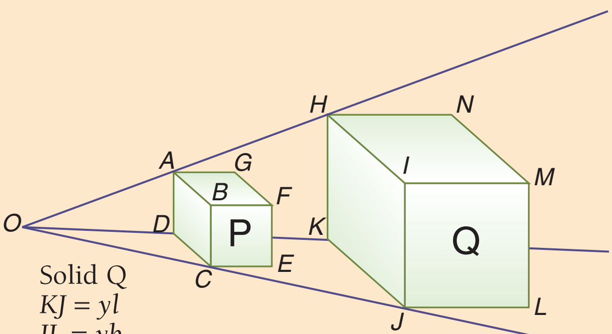
d area DBE : area $ADEC$

e area ABC : area $ADEC$

11:06 | Similar Solids

Similar solids have the same shape,
matching angles are equal,
the lengths of matching edges are in the same ratio.

prep quiz 11:06



Solid P
$DC = xl$
$CE = xb$
$CB = xh$

Solid Q
$KJ = yl$
$JL = yb$
$JI = yh$

These rectangular prisms are similar.

1 What is the ratio $DC : KJ$?

2 What is the ratio $CE : JL$?

3 What is the ratio area $ABCD$: area $HIJK$?

4 What is the ratio area $BCEF$: area $IJLM$?

5 What is the ratio volume of P : volume of Q?

On the right are two similar cylinders.

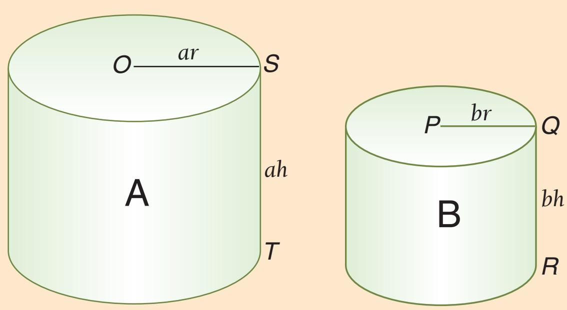
6 What is the ratio $ST : QR$?

7 What is the ratio $OS : PQ$?

8 What is the ratio cross-section of A to cross-section of B?

9 What is the ratio curved surface area of A to curved surface area of B?

10 What is the ratio volume of A : volume of B?



O and *P* are centres.

The Prep Quiz suggests certain relationships in the dimensions of similar solids.

For similar solids:
- **corresponding areas are proportional to the squares on matching sides**
- **corresponding volumes are proportional to the cubes on matching sides.**

• A doll twice the height of another would have eight times its volume

worked examples

1 Two cubes have the lengths of matching edges in the ratio 4 : 3. Find:

a the ratio of the areas of matching faces

b the ratio of their volumes

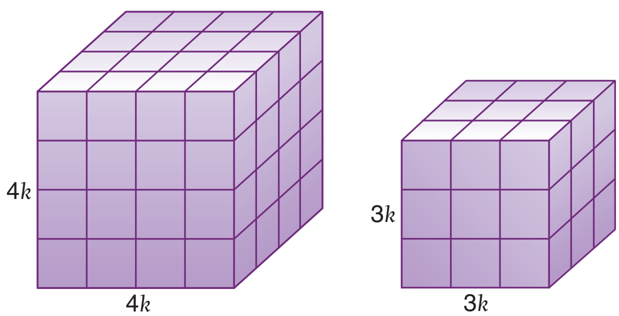
2 Two similar statues were to be made at Naomi's studio. She first completed the smaller one and painted it with a special gold paint. The smaller statue was 1·6 metres tall, had a volume of 0·384 m^2 and needed 400 mL of gold paint to complete the required two coats.

a What volume will the larger statue have if it is to be 2 metres tall?

b How much gold paint will be required to give the larger statue two coats?

Solutions

1



A B

All cubes are similar.

a Areas are proportional to the squares of sides.

$$\therefore \frac{\text{area A}}{\text{area B}} = \frac{(4k)^2}{(3k)^2} = \frac{16}{9}$$

(*Note*: The area of each face of A is $16k^2$. The area of each face of B is $9k^2$.)

b Volumes are proportional to the cubes of sides.

$$\therefore \frac{\text{volume A}}{\text{volume B}} = \frac{4^3}{3^3} = \frac{64}{27}$$

(*Note*: The volume of A is 64 m^3.
The volume of B is 27 m^3.)

2

The statues are similar solids.

a Volumes are proportional to the cubes of lengths.

$$\therefore \frac{\text{volume L}}{\text{volume S}} = \frac{2^3}{1{\cdot}6^3}$$

$$\therefore \frac{\text{volume L}}{0{\cdot}384\ \text{m}^3} = \frac{2^3}{1{\cdot}6^3}$$

$$\therefore \text{volume L} = \frac{2^3 \times 3{\cdot}384}{1{\cdot}6^3}\ \text{m}^3 = 0{\cdot}75\ \text{m}^3$$

b Paint required is proportional to area to be covered.
Areas are proportional to the squares of length.

$$\therefore \frac{\text{paint for L}}{\text{paint for S}} = \frac{2^2}{1{\cdot}6^2}$$

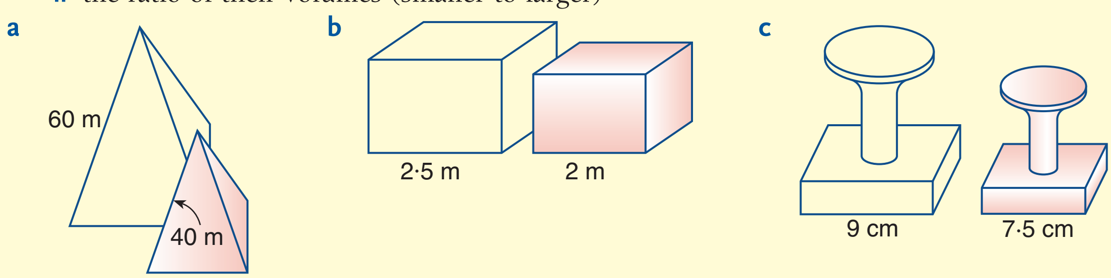
$$\therefore \frac{\text{paint for L}}{400\ \text{mL}} = \frac{2^2}{1{\cdot}6^2}$$

$$\therefore \text{paint for L} = \frac{2^2 \times 400}{1{\cdot}6^2}\ \text{mL} = 625\ \text{mL}$$

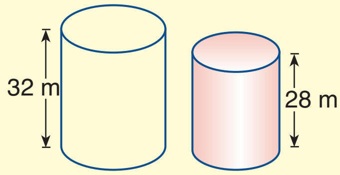
Exercise 11:06

1 In each, the solids are similar. Find:

i the ratio of matching areas (smaller to larger)
ii the ratio of their volumes (smaller to larger)



d



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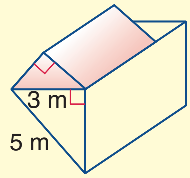
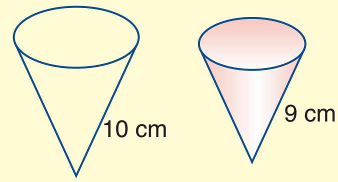


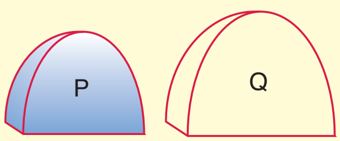
f



2 **a** In **1a**, the area of the base of the smaller solid is 780 m^2. Find the area of the base of the larger solid.
b In **1a**, the volume of the larger solid is 30 564 m^3. Find the volume of the smaller solid.
c In **1b**, the volume of the smaller solid is 10·08 m^3. What is the volume of the larger solid?
d In **1c**, the volume of the larger solid is 243 cm^3. What is the volume of the smaller solid?
e In **1d**, the cross-section of the larger cylinder has an area of 400 m^2. Find the cross-sectional area of the smaller cylinder.
f In **1e**, the surface area of the smaller triangular prism is 27 m^2. What is the surface area of the larger triangular prism?
g In **1e**, the volume of the smaller prism is 8·1 m^3. Find the volume of the larger prism.
h In **1f**, the surface area of the larger ice-cream cone is 226 cm^2. Find the surface area of the smaller ice-cream cone correct to the nearest square centimetre.
i In **1f**, the volume of the larger solid is 287 cm^3. What is the volume of the smaller solid correct to the nearest cubic centimetre?

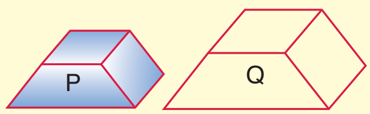
3 In each part, the solids are similar. Find the ratio of a side on the smaller solid to the matching side on the larger solid. Hence, find the ratio of their surface areas.

a



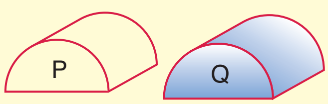
Volume P = 27 cm^3 Volume Q = 32·768 cm^3

b



Volume P = 0·875 cm^3 Volume Q = 7 m^3

c



Volume P = 512π m^3 Volume Q = 216π m^3

4 **a** In **3a**, the surface area of solid P is 63 cm^2. Find the surface area of solid Q.
b In **3b**, the length from end to end (ie the height) on prism P is 0·7 m. What is the matching length on prism Q?
c In **3b**, the area of the under-surface of solid Q is 5·04 m^2. Find the area of the matching face of solid P.
d In **3c**, the diameter of the cross-section of prism P is 16 m. Find the diameter of the cross-section of prism Q.
e In **3c**, the curved surface area of solid Q is 72π m^2. What is the curved surface area of solid P?

5 Two similar vases were to be made in Naomi's studio. She first completed the smaller one and gave it two coats of lacquer. The smaller vase was 40 cm tall, had a volume of 2625 cm^3 and needed 100 mL of lacquer to complete the two coats.

a What volume will the larger vase have if it is to be 60 cm tall?

b How much lacquer will be required to give the larger vase two coats?

6 The volume of metal in a scale model of a metal high-tension tower is 54 cm^3. The scale used was 1 : 50. It required 50 mL of paint to give the model one coat.

a What volume of metal would be required to build the high-tension tower?

b How much paint would be required to give the real tower one coat?

7 Let us assume that a boy and his father are similar solids. Each is wearing similar swimming costumes. The boy is one-quarter as tall as his father.

a If the boy loses 1 unit of heat from his exposed skin, how many units of heat would you expect his father to lose?

b If the father displaces 72 000 cm^3 of water when he dives into the pool, what volume of water would the son displace?

8 Two similar glasses have heights in the ratio 3 : 5. If, when using the larger glass, it takes 54 glasses of water to fill a fish tank, how many glasses of water would be needed to fill it using the smaller glass?

9 This Airbus A380 is 50 times bigger than the one I have in my room. Its length is 73 metres.

Complete:

a The surface area of an actual A380 is . . . times greater than that of the model in my room.

b The volume is . . . times greater than that of the model in my room.

10 

Each of these toy soldiers is $\frac{1}{32}$ of a real soldier's height.

Complete:

a The surface area of a toy soldier is . . . of the surface area of a real soldier.

b The volume of a toy soldier is . . . of the volume of a real soldier.

c If a real soldier has a volume of 90 000 cm^3, what would be the volume of a toy soldier? (Answer correct to two decimal places.)

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investigation 11:06

Investigation 11:06 | King Kong — Could he have lived?

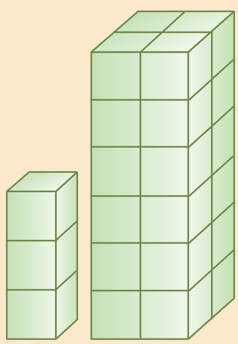
Please use the Assessment Grid on page 324 to help you understand what is required for this Investigation.

Could an ape be 10, 20, 30 or even 40 metres high? Would its enormous size cause it problems? The relationship between area and volume gives us the answer.

- A person who is forced to carry large weights for long periods is putting more weight on the bones and feet than they can comfortably handle.

Model of King Kong

- Let an ordinary ape be represented by a square prism with a height 3 times the width as shown. Its base of one square unit (feet) can comfortably carry the 3 units of mass above it. (Each unit of volume is one unit of mass.)



1 The model of the giant ape is twice as high as the model of a normal ape.
  - **a** What is its volume?
  - **b** What is its base area?
  - **c** How many units of mass must each unit of area in the base support now?

2 Imagine an ape 50 times as high as a normal ape.
  - **a** What is its volume?
  - **b** What is its base area?
  - **c** How many units of mass must each unit of area in the base support now?

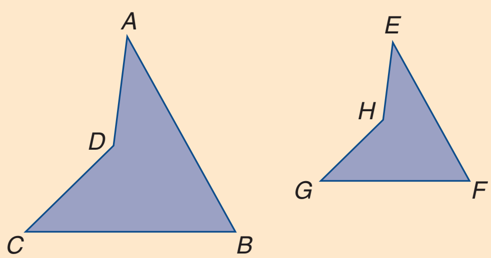
3 Use your results to suggest the enlargement factor of the area of a scaled-up object (ie if the object is enlarged *n* times, by how much does the area increase?). Is this the same as the enlargement factor for the volume of a scaled-up object?
4 If the enlargement factor is *F*, describe the change in the pressure on the bottom face.
5 Use another example to verify your answers to 3 and 4.
• Note: At some point, the bottom face (the skin and bone) will exceed its ability to withstand that pressure. Using this concept, Galileo was able to give a good estimate for the maximum height of trees. His estimate of 90 metres was close to the world's highest of 110 metres.
6 Does the above suggest that there may be a maximum height for mountains and buildings? Write down your opinion and give your reasons.
7 Consider the effect of enlarging a model aeroplane. The model plane may fly but when it is enlarged, the increase in mass (proportional to the cube of dimensions) is far more than the increase in wing-surface area (proportional to the square of the dimensions). How would you change the design of the enlargement so that it might fly?

(The ideas here are taken from the ABC series 'For All Practical Purposes: Growth and Form' and from the NSW Mathematics Syllabus.)

Mathematical Terms 11

mathematical terms 11

matching angles (or sides)
- Sides (or angles) that are in the same (or corresponding) positions.



Matching sides
- *AB* and *EF*
- *BC* and *FG*
- *CD* and *GH*
- *DA* and *HE*

Matching angles
- $\angle A$ and $\angle E$
- $\angle B$ and $\angle F$
- $\angle C$ and $\angle G$
- $\angle D$ and $\angle H$

scale drawing
- A drawing that is similar to the original.

scale factor
- The ratio of the lengths of matching sides on a pair of similar figures.
- It can be given as either an enlargement factor or a reduction factor.

similar figures
- Figures that have the same shape but a different size.
- They have matching angles equal and matching sides are in the same ratio.
- The ratio of matching sides gives us the scale factor.

similar triangles (tests for)
- A set of four tests that can be used to prove that two triangles are similar.

superimpose
- The placement of one figure upon another in such a way that the parts of one coincide with the parts of the other.

Technology Applications Drag and Drops
Mathematical terms 11

Assessment Grid for Investigation 11:06 | King Kong — Could he have lived?

The following is a sample assessment grid for this investigation. You should carefully read the criteria *before* beginning the investigation so that you know what is required.

Criterion		Assessment Criteria (B, C, D) for this investigation		Achieved ✓
Criterion B Investigating Patterns	a	None of the following descriptors have been achieved.	0	
	b	Some help was needed to complete some of the requirements.	1	
			2	
	c	The student independently completes most of the problems and attempts to explain their answers using appropriate patterns.	3	
			4	
	d	The student has described the patterns for area and volume, and given logical answers and conclusions to most problems.	5	
			6	
	e	The student has correctly described and verified the patterns for area and volume, and given appropriate answers and consistent conclusions to all problems.	7	
			8	
Criterion C Communication in Mathematics	a	None of the following descriptors have been achieved.	0	
	b	There is a basic use of mathematical language and notation, with some errors or inconsistencies evident. Lines of reasoning are insufficient.	1	
			2	
	c	There is sufficient use of mathematical language and notation. Explanations are clear with algebra skills applied well.	3	
			4	
	d	All explanations are thorough, clear and well written. Mathematical processes are well set out and easy to follow.	5	
			6	
Criterion D Reflection in Mathematics	a	None of the following descriptors have been achieved.	0	
	b	An attempt has been made to explain whether the results make sense and are consistent.	1	
			2	
	c	There is a correct but brief explanation of whether results make sense and how they were found. Parts 6 and 7 are answered well.	3	
			4	
	d	Parts 4, 6 and 7 are used to show the importance of the findings with real life examples and detailed explanations. The accuracy and reasonableness of results are considered.	5	
			6	

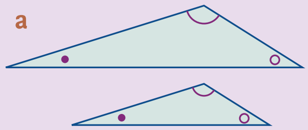
Diagnostic Test 11 | Similarity

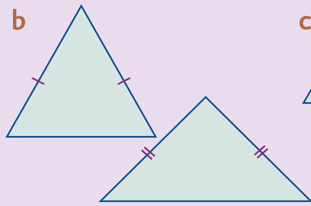
diagnostic test
11

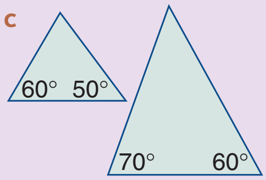
- These questions reflect the important skills introduced in this chapter.
- Errors made will indicate areas of weakness.
- Each weakness should be treated by going back to the section listed.

	Section
1 Are the following pairs of triangles similar?	11:02
2 Are the following pairs of triangles similar?	11:02
3 Are the following pairs of triangles similar?	11:02
4 Find the value of the pronumeral in each of the following.	11:03

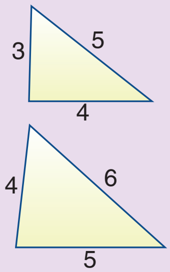
1 Are the following pairs of triangles similar?

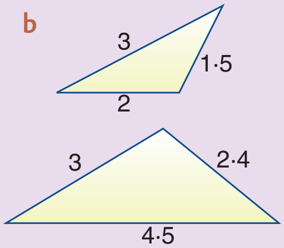
a


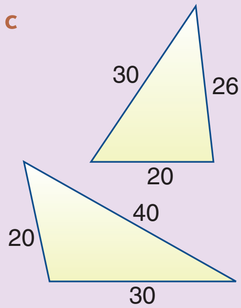
b


c


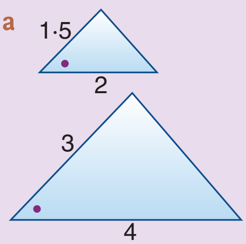
2 Are the following pairs of triangles similar?

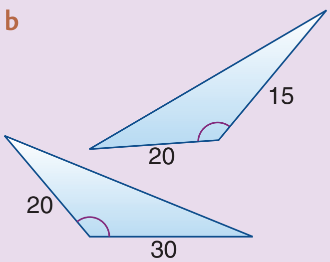
a


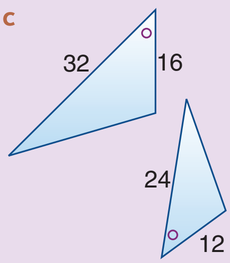
b


c


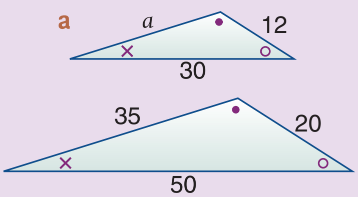
3 Are the following pairs of triangles similar?

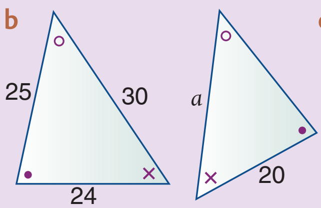
a


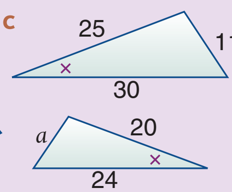
b


c


4 Find the value of the pronumeral in each of the following.

a


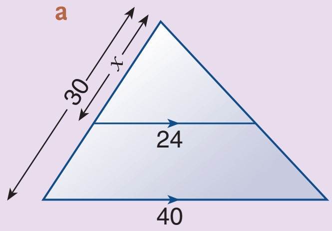
b


c


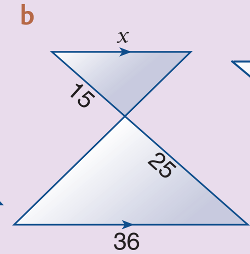
Section

5 Find the value of the pronumeral in each of the following. 11:04

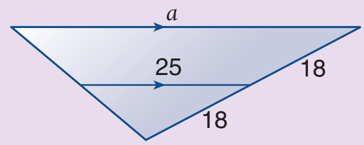
a



b

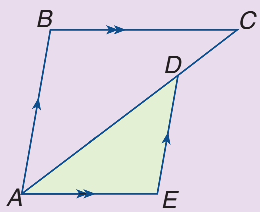


c



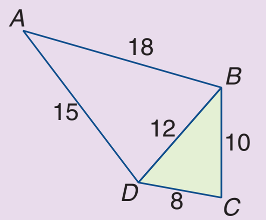
6 11:04

a



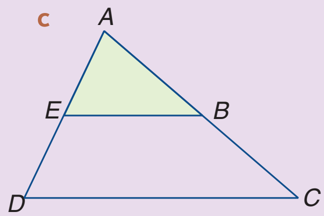
Prove that $\Delta ABC \;|||\; \Delta DEA$.

b



Lengths are in metres. Prove that $\Delta ABD \;|||\; \Delta BDC$.

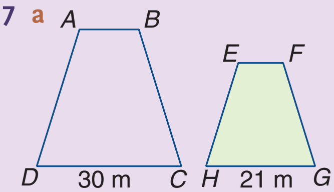
c



$AE = ED$, $AB = BC$. Prove that $\Delta AEB \;|||\; \Delta ADC$.

7 11:05

a

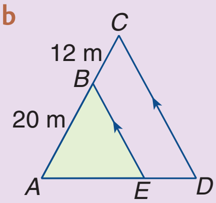


$ABCD \;|||\; EFGH$. Find:

i area $ABCD$: area $EFGH$

ii area $ABCD$ if area $EFGH$ is 269·5 m^2.

b

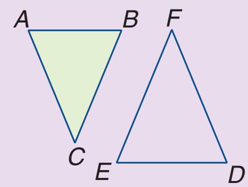


$BE \parallel CD$. Find:

i area ABE : area ACD

ii area ABE if area ACD is 512 m^2.

c

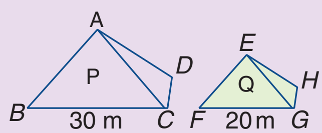


Area ABC = 243 mm^2.
Area EDF = 300 mm^2.

i Find $AB : ED$.

ii Find the length of EF if AC = 18 mm.

8 **a** Solids P and Q are similar. 11:06



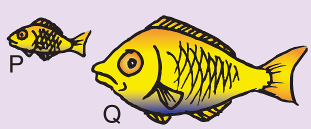
Find:

i area ABC : area EFG

ii volume P : volume Q

iii the volume of P if the volume of Q is 2000 m^3.

b



Surface area of P = 44·8 cm^2
Surface area of Q = 179·2 cm^2

These solid models of fish are similar.

i Find the ratio length of P : length of Q.

ii Find the ratio volume P : volume Q.

iii Find the volume of Q if the volume of P is 18·2 cm^3.

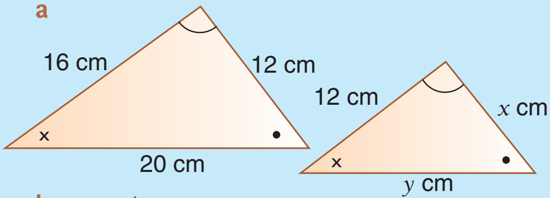
c The heights of two similar statues are 3·2 m and 3·6 m. If the volume of the smaller is 80·64 m^3, find the volume of the larger statue.

Chapter 11 | Revision Assignment

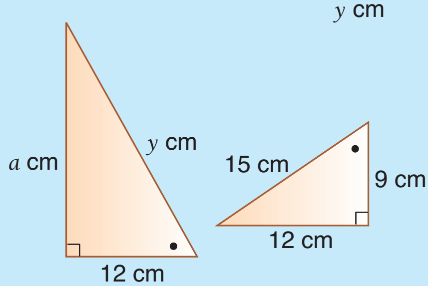
11A

1 Each pair of triangles is similar. Find the value of the pronumerals.

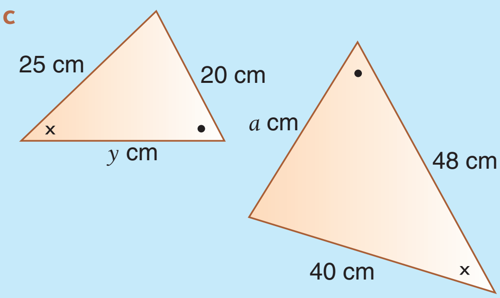
a



b

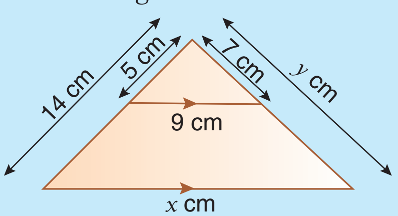


c

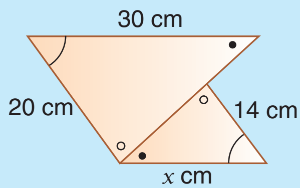


2 Find the value of the pronumerals in each of the following.

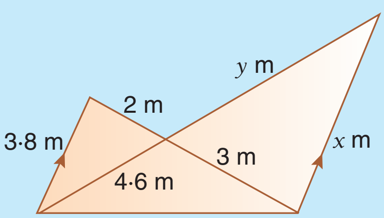
a



b



c



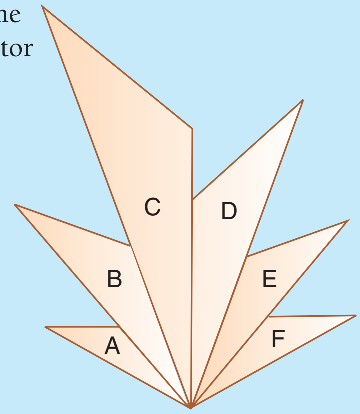
3 The diagram below shows a piece of jewellery based on similar triangles. Triangle A has sides of length 10 mm, 15 mm and 22 mm.

Triangles B and C have been produced from triangle A by successive enlargements using an enlargement factor of 1·6.

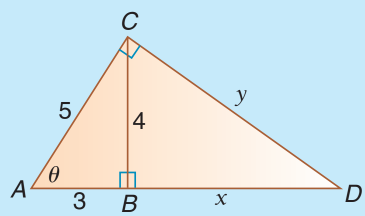
Triangle C has been reduced and reflected to produce triangle D. The reduction factor is 0·75.

Triangles E and F have been produced from triangle D by successive reductions using the same reduction factor of 0·75.

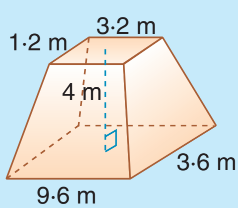
Calculate the side lengths of triangles C and F.



4 Prove that $\Delta ABC \,|||\, \Delta CBD$ and hence find the values of x and y.



5 The solid shown was made by removing the top part of a rectangular pyramid. Calculate its volume.



assignment 11B

Chapter 11 | Working Mathematically

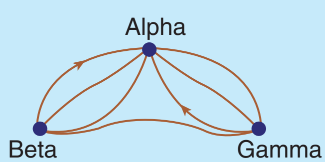
1 On a dart board, it is possible with one dart to score:
- a number from 1 to 20
- the double or triple of a number from 1 to 20
- 25 or 50 for an outer or inner bullseye.

Hence, the largest score possible with one dart is 60 and the smallest is 1. What scores between 1 and 60 are impossible?

2 In a recent class test, the average was 74·5%. If the 11 girls in the class averaged 80%, what was the boys' average mark if there were 28 students in the class?

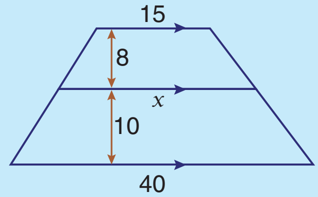
3 A glass of fruit juice is full. The total weight of the glass and juice is 400 g. When the glass is one-third full of fruit juice, its weight is 280 g. How much does the glass weigh?

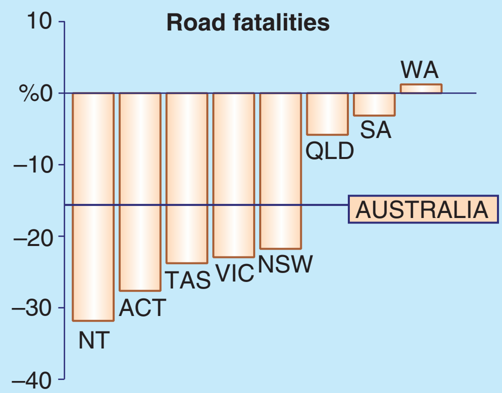
4 The diagram below shows the roads that run between Alpha, Beta and Gamma. How many ways are there to travel from:
- **a** Alpha to Beta?
- **b** Beta to Gamma?
- **c** Gamma to Beta and back again?
- **d** Gamma to Beta and back again without using any road twice?



Arrows indicate one way only.

5 Find the value of x.



6


This graph shows the percentage change in road fatalities in Australian regions for a 12-month period compared with the previous five-year average.
- **a** What does the dark line labelled 'Australia' refer to?
- **b** What percentage change has occurred in:
  i WA? **ii** Vic? **iii** NT?
- **c** How would you describe this year's statistics compared with those of the previous five years?

Technology Applications Activities — 1 Maths race, 2 Similar figures
Technology Applications Drag and Drops — Using the scale factor
Technology Applications Animations — Scale it
Chapter Review Questions

12

Further Trigonometry

Chapter Contents

Learning Outcomes

Students will be able to:

- Identify trigonometric relationships between acute and obtuse angles.
- Use the sine rule.
- Use the sine rule in the ambiguous case.
- Use the cosinerule.
- Find the area of a non-right-angled triangle.
- Apply all of the above to the solution of problems including bearings.

Areas of Interaction

Approaches to Learning (Knowledge Acquisition, Problem Solving, Communication, Logical Thinking, Information and Communication Technology Skills, Collaboration, Reflection), Human Ingenuity, Environments

12:01 | Trigonometric Ratios of Obtuse Angles

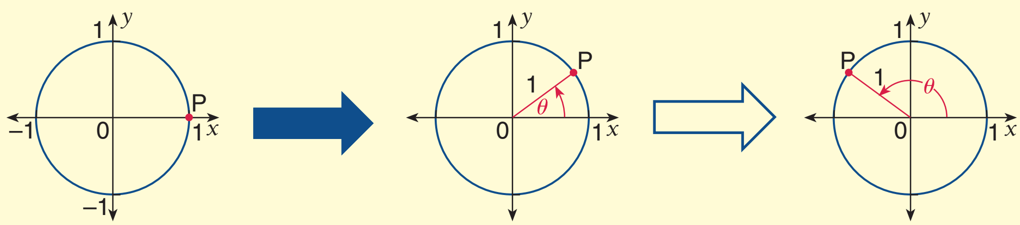
In Year 9, we needed to know the trigonometric ratios for acute angles only because all of the problems involved right-angled triangles.

In this chapter, we will see how trigonometry can be applied to any triangle, including those which have an obtuse angle.

A new definition for trigonometric functions

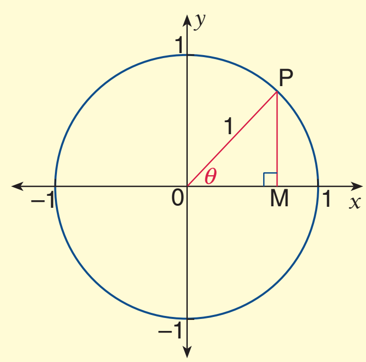
Sine, cosine and tangent ratios can be defined in terms of a circle of radius 1 unit.

- We refer to a **unit circle** that has its centre at (0, 0) on the number plane. If we take a radius OP initially along the *x*-axis and rotate it anticlockwise about O, of course point P will still lie on the circle.



The coordinates of P on the number plane will depend on the size of θ, the angle of rotation.

If a perpendicular PM is drawn as in this diagram, the method of defining the trig. ratios that we met in Year 9 would suggest that:



$$\cos\theta = \frac{OM}{1} \quad \left(\text{ie } \frac{\text{adj.}}{\text{hyp.}}\right) \qquad \sin\theta = \frac{PM}{1} \quad \left(\text{ie } \frac{\text{opp.}}{\text{hyp.}}\right)$$

$= OM \qquad\qquad = PM$

$= x$-coordinate of P $\qquad = y$-coordinate of P

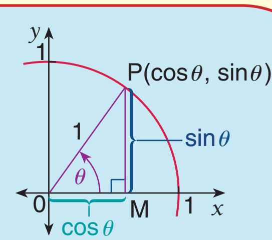
$\therefore$ P is the point $(\cos\theta, \sin\theta)$.

Hence, the previous method of defining the trig. ratios using side ratios suggests a new way of defining the trig. ratios as the coordinates of the point P.

This method is used to give the trig. ratios for acute angles and, in fact, angles of any size.

For a point P on a unit circle which has been rotated through an angle θ about O from the positive *x*-axis:

$\cos\theta$ = the *x*-coordinate of P,
$\sin\theta$ = the *y*-coordinate of P.

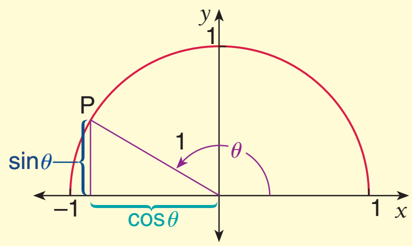


- The triangle above can also be used to redefine the tangent function.

$$\tan\theta = \frac{PM}{OM} \quad \left(\text{ie } \frac{\text{opp.}}{\text{adj.}}\right)$$
$$= \frac{y\text{-coordinate of P}}{x\text{-coordinate of P}}$$
$$= \frac{\sin\theta}{\cos\theta}$$

$\tan\theta = \dfrac{\sin\theta}{\cos\theta}$

- In practice, the values of the trig. ratios are obtained using a calculator.
- The diagram on the right shows that when θ is obtuse:
  the x-coordinate of P is negative,
  the y-coordinate of P is positive,
  the $\dfrac{y\text{-coordinate of P}}{x\text{-coordinate of P}}$ is negative.



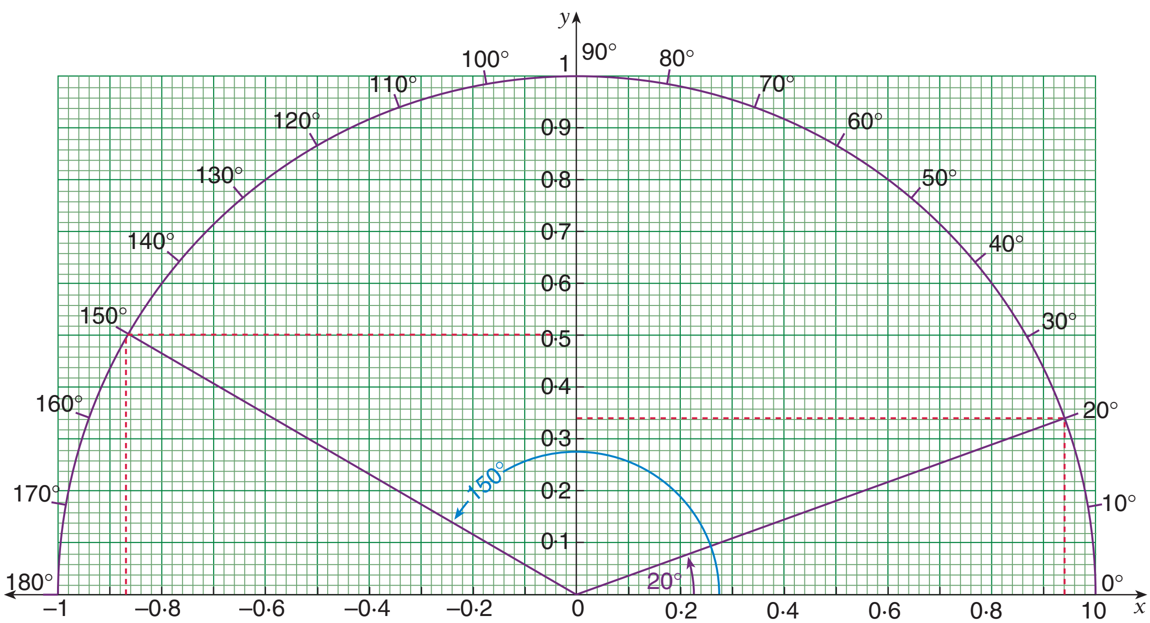
If θ is obtuse: **$\sin\theta$ is positive**
$\cos\theta$ is negative
$\tan\theta$ is negative

worked examples

1 Use the graph below to find the sine and cosine ratios of 20° and 150° correct to two decimal places. Using these values, find tan 20° and tan 150° correct to two decimal places.

2 Use a calculator to find the following correct to two decimal places:
a sin 120° **b** cos 150° **c** tan 95°

3 Use the graph to estimate the obtuse angle θ for which $\cos\theta = -0{\cdot}5$.



continued →→→

Solutions

1 Reading the appropriate coordinates off the graph:

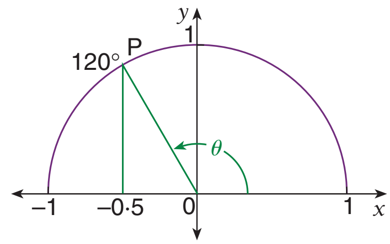
$\sin 20° = 0{\cdot}34$ $\quad\quad$ $\sin 150° = 0{\cdot}50$

$\cos 20° = 0{\cdot}94$ $\quad\quad$ $\cos 150° = -0{\cdot}87$

$$\tan 20° = \frac{0{\cdot}34}{0{\cdot}94} = 0{\cdot}36 \qquad \tan 150° = \frac{0{\cdot}50}{-0{\cdot}87} = -0{\cdot}57$$

2 From the calculator:
- a $\sin 120° = 0{\cdot}87$
- b $\cos 150° = -0{\cdot}87$
- c $\tan 95° = -11{\cdot}43$

3 We need to find the point P which has an x-coordinate of $-0{\cdot}5$. Drawing in the line $x = -0{\cdot}5$ gives us the point.
Hence, $\theta = 120°$.
(*Note:* There would be two points on the curve that have a y-coordinate of 0·5.)

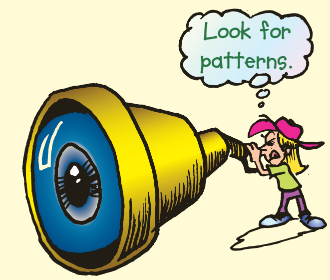


Exercise 12:01

1 By referring to the graph on the previous page, find the following correct to two decimal places.
- a $\sin 110°$ b $\cos 110°$ c $\sin 160°$ d $\cos 160°$

2 Use your answers from question 1 to find the value of the following, correct to one decimal place.
- a $\tan 110°$ b $\tan 160°$

3
- a Estimate the values of sin 75° and sin 105°. What do you notice?
- b Estimate the values of sin 80° and sin 100°. What do you notice?
- c Estimate the values of cos 70° and cos 110°. What do you notice?
- d Estimate the values of cos 75° and cos 105°. What do you notice?



4 Find one value of θ, to the nearest degree, for which:
- a $\cos \theta = 0{\cdot}5$ b $\cos \theta = 0{\cdot}34$ c $\cos \theta = 0{\cdot}94$
- d $\cos \theta = 0{\cdot}26$ e $\cos \theta = 0$ f $\cos \theta = -0{\cdot}26$
- g $\cos \theta = -0{\cdot}14$ h $\cos \theta = -0{\cdot}34$ i $\cos \theta = -0{\cdot}7$

5 Find the two values of θ, to the nearest degree, for which:
- a $\sin \theta = 0{\cdot}34$ b $\sin \theta = 0{\cdot}64$

6 For θ between 0° and 180° inclusive, find values of θ if:
- a $\sin \theta = 0{\cdot}5$ b $\sin \theta = 0{\cdot}26$ c $\sin \theta = 0{\cdot}94$
- d $\cos \theta = -0{\cdot}57$ e $\cos \theta = 0{\cdot}71$ f $\cos \theta = -0{\cdot}82$
- g $\sin \theta = 0{\cdot}64$ h $\sin \theta = 0{\cdot}91$ i $\sin \theta = 0{\cdot}31$
- j $\sin \theta = 0$ k $\sin \theta = 0{\cdot}54$ l $\sin \theta = 1$

7 Use your calculator to evaluate correct to three decimal places:
a sin 152° **b** cos 128° **c** tan 100° **d** cos 140° **e** tan 105° **f** sin 95°

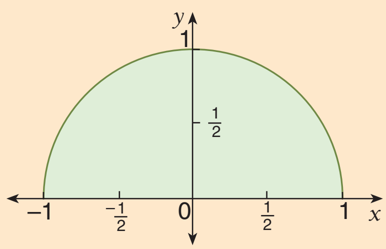
8 Why can there be no value for tan 90°?

9 For θ between 0° and 180° inclusive, what is the largest value of sin θ, and for what value of θ does it occur?

12:02 | Trigonometric Relationships between Acute and Obtuse Angles

prep quiz 12:02

Copy the diagram, and, by drawing an appropriate line on it, state whether the following equations would have 1 or 2 solutions



1 $\sin\theta = \frac{1}{2}$ **2** $\cos\theta = \frac{1}{2}$

3 $\cos\theta = -\frac{1}{2}$

For what angle or angles does:

4 $\sin\theta = 1$?

5 $\cos\theta = 1$?

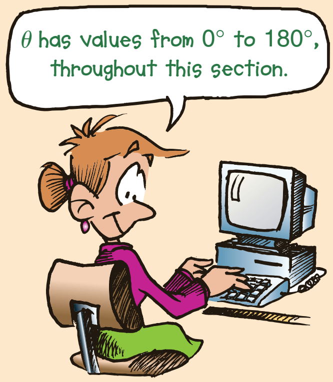
6 $\sin\theta = 0$?

7 $\cos\theta = 0$?

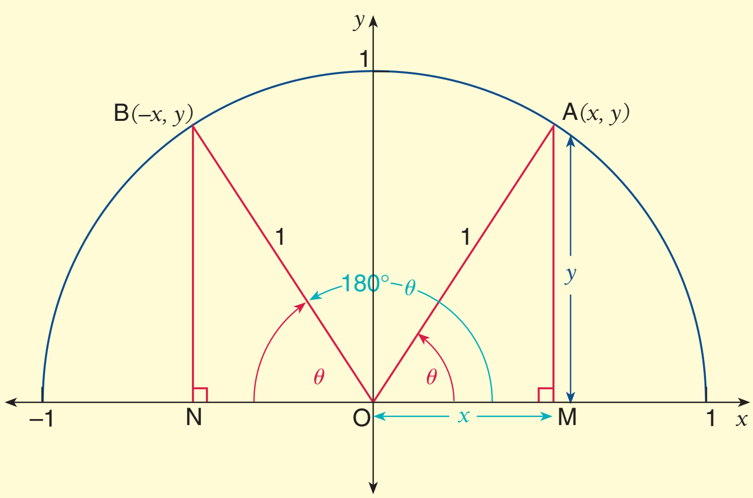
8 $\cos\theta = -1$?

Use your calculator to evaluate the following, correct to four decimal places.

9 sin 40° **10** sin 140°



- In the last exercise and the Prep Quiz above, you should have seen that it is possible for an acute angle and an obtuse angle to have the same sine.
- *There is a relationship between the trig. ratios of acute and obtuse angles and this is now investigated.* OA and OB have been drawn at an angle θ to the positive arm of the x-axis and the negative arm of the x-axis respectively. Hence, $\angle BOx = 180° - \theta$



∴ The coordinates of A are $(\cos\theta, \sin\theta)$.
The coordinates of B are $(\cos(180° - \theta), \sin(180° - \theta))$.

Now, Δs AMO and BNO are congruent (AAS).
∴ AM = BN and OM = ON.
If A is the point (x, y) then B is the point $(-x, y)$.

Now, since A has coordinates $(\cos\theta, \sin\theta)$ or (x, y), then:

$\cos\theta = x$, $\sin\theta = y$ and $\tan\theta = \frac{y}{x}$.

Since B has coordinates $(\cos(180° - \theta), \sin(180° - \theta))$ or $(-x, y)$, then:

$\cos(180° - \theta) = -x$, $\sin(180° - \theta) = y$ and $\tan(180° - \theta) = \frac{y}{-x}$.

Equating the two sets of ratios gives these results.

$\sin(180° - \theta) = \sin\theta$, $\cos(180° - \theta) = -\cos\theta$, $\tan(180° - \theta) = -\tan\theta$

This means, for example, that:

$$\sin 120° = \sin(180° - 60°) = \sin 60°$$
$$\cos 160° = \cos(180° - 20°) = -\cos 20°$$
$$\tan 141° = \tan(180° - 39°) = -\tan 39°$$

worked examples

1 Which acute angle has the same sine as 150°?

2 If θ is an angle between 0° and 180°, find θ to the nearest degree if:
a $\cos\theta = 0{\cdot}6$ **b** $\tan\theta = -0{\cdot}8$

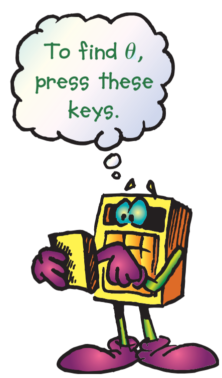
3 Give an acute and obtuse value of θ (to the nearest minute) for which $\sin\theta = 0{\cdot}354$.

Solutions

1 $\sin(180° - \theta) = \sin\theta$
$180° - \theta = 150°$
$\theta = 30°$
∴ $\sin 30° = \sin 150°$

2 **a** $\cos\theta = 0{\cdot}6$
(∴ θ is acute.)
$\theta = 53{\cdot}13°$
$= 53°$ (to the nearest degree)

Press:
2nd F cos 0·6 =



b $\tan\theta = -0{\cdot}8$

($\therefore$ θ is obtuse)

Let $\theta = 180° - \alpha$

$\tan(180° - \alpha) = -0{\cdot}8$

$-\tan\alpha = -0{\cdot}8$ $\quad[\tan(180° - \alpha) = -\tan\alpha]$

$\tan\alpha = 0{\cdot}8$

$\alpha = 38°40'$ (from the calculator)

But $\quad\theta = 180° - \alpha$

$\therefore\ \theta = 180° - 38°40'$

$= 141°$ (to the nearest degree)

3 $\sin\theta = 0{\cdot}354$

Press [2nd F] [sin] 0·354 [=] [D°M'S]

The calculator gives 20°44′ to the nearest minute.

Now $\sin(180° - \theta) = \sin\theta$

If $\theta = 20°44'$ then $180° - \theta = 159°16'$.

$\therefore\ \theta = 20°44'$ or $159°16'$

■ If θ is obtuse, solve $\tan\theta = 0{\cdot}8$ and subtract the answer from 180°.

Exercise 12:02

Foundation Worksheet 12:02

Trig. ratios of obtuse angles

1 Use a calculator to find:
 a sin 40°, sin 140° **b** sin 60°, sin 110°

2 In which part of question **1** were the ratios equal? For the ratios that were equal, what did you notice about the angles?

3 Which acute angle has the same sine as:
 a 120°? **b** 135°?

4 If θ is between 0° and 180°, find θ to the nearest degree if:
 a $\sin\theta = 0{\cdot}8$ **b** $\cos\theta = -0{\cdot}5$

1 Use a calculator to evaluate the following correct to three decimal places.

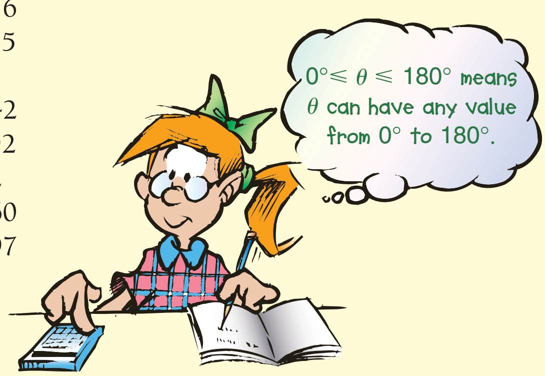
a sin 167°30′ **b** cos 140°20′ **c** tan 150°19′

2 Use the results on page 282 to express each of the following as a ratio of an acute angle.

a sin 150° **b** cos 100° **c** tan 140°
d sin 125° **e** sin 140°32′ **f** cos 167°31′
g tan 101°03′ **h** cos 92°50′

3 If θ is an angle between 0° and 180°, find θ to the nearest minute if:

a $\cos\theta = 0{\cdot}716$ **b** $\cos\theta = -0{\cdot}716$
c $\tan\theta = 8{\cdot}215$ **d** $\tan\theta = -8{\cdot}215$
e $\cos\theta = -0{\cdot}5$ **f** $\tan\theta = -1$
g $\cos\theta = 0{\cdot}906$ **h** $\cos\theta = -0{\cdot}342$
i $\tan\theta = 1{\cdot}881$ **j** $\tan\theta = -1{\cdot}192$
k $\cos\theta = -0{\cdot}966$ **l** $\cos\theta = 0{\cdot}602$
m $\tan\theta = -0{\cdot}754$ **n** $\cos\theta = -0{\cdot}760$
o $\tan\theta = 3{\cdot}323$ **p** $\cos\theta = -0{\cdot}997$



4 If $0° \leqslant \theta \leqslant 180°$, give two possible values for θ (to the nearest minute) if $\sin\theta$ is equal to:

a 0·5 **b** 0·73 **c** 0·36
d 0·453 **e** 0·990 **f** 0·3665
g 0·7083 **h** 0·0567

5 Complete the table below and use it to sketch $y = \sin x°$ for $0° \leqslant x° \leqslant 180°$.

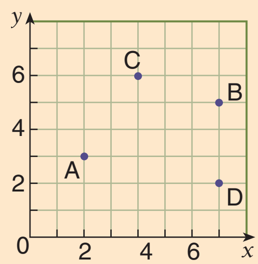
$x°$	0°	20°	40°	60°	80°	90°	100°	120°	140°	160°	180°
$\sin x°$											

6 Make a sketch of $y = \cos x°$ for $0° \leqslant x° \leqslant 180°$.

fun spot 12:02

Fun Spot 12:02 | Why are camels terrible dancers?

Answer each question and put the letter for that question in the box above the correct answer.



Find:

E the midpoint of AB
T the distance from A to B
L the gradient of CB
E the equation of BD
T the equation of the x-axis
A the equation of AC
F the area of ΔABD.
E Through which of the points A, B, C or D does the line $2x - 5y = 4$ pass?

Find the equation linking x and y for each table.

H

x	0	1	2	3
y	3	4	5	6

L

x	0	1	2	3
y	2	5	8	11

T

x	2	3	4	5
y	4	9	16	25

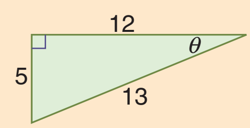
Use the triangle to find the value of:

E $\sin\theta$ **A** $\cos\theta$ **F** $\tan\theta$

H $\dfrac{1}{\sin\theta}$ **L** $\dfrac{1}{\cos\theta}$ **O** $\dfrac{1}{\tan\theta}$

E $(\sin\theta)^2 + (\cos\theta)^2$

T $(\tan\theta)^2 - \left(\dfrac{1}{\cos\theta}\right)^2$



Solve:

V $x^2 = 49$ **W** $\dfrac{1}{x} = \dfrac{7}{2}$ **Y** $x^2 - 6x + 9 = 0$

$\sqrt{29}$	$\frac{13}{5}$	$(4\frac{1}{2}, 4)$	$x = 3$	$\frac{12}{13}$	$\frac{13}{12}$	$-\frac{1}{3}$	$y = x + 3$	$y = \frac{3}{2}x$	$y = \pm 7$	$x = 7$	$y = x^2$	$x = \frac{2}{7}$	$\frac{12}{5}$	$y = 3x + 2$	$\frac{5}{13}$	$\frac{5}{12}$	$y = 0$	7·5 units^2	1	D	−1

12:03 | The Sine Rule

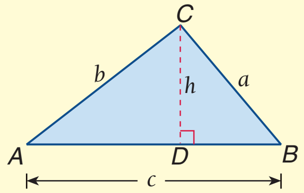
Now that the trig. ratios can be calculated for obtuse angles, trigonometry can be applied to any type of triangle.

In the triangles below:

- side a is opposite $\angle A$
- side b is opposite $\angle B$
- side c is opposite $\angle C$.

Note: $\sin(180° - B) = \sin B$

A



ABC is an acute-angled triangle.
CD is perpendicular to *AB*.

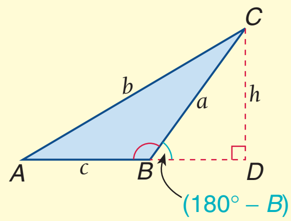
In ΔACD, $\sin A = \dfrac{h}{b}$

$\therefore\ h = b \sin A \ldots.\ \text{(i)}$

In ΔBCD, $\sin B = \dfrac{h}{a}$

$\therefore\ h = a \sin B \ldots.\ \text{(ii)}$

B



ABC is an obtuse-angled triangle.
CD is perpendicular to *AB* (produced to *D*).

In ΔACD, $\sin A = \dfrac{h}{b}$

$\therefore\ h = b \sin A \ldots.\ \text{(i)}$

In ΔBCD, $\sin(180° - B) = \dfrac{h}{a}$

$\therefore\ h = a \sin(180° - B)$

ie $h = a \sin B \ldots.\ \text{(ii)}$

Equating (i) and (ii) from either triangle gives:

$$a \sin B = b \sin A$$

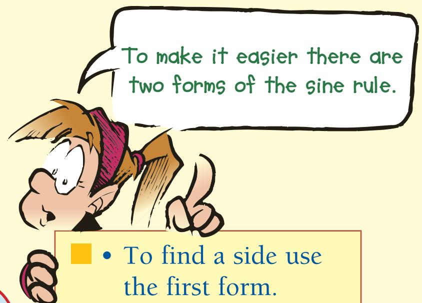
$$\text{ie}\quad \frac{a}{\sin A} = \frac{b}{\sin B}$$

Similarly, it can be shown that

$$\frac{b}{\sin B} = \frac{c}{\sin C}$$

Thus, we obtain:

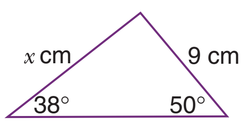
$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} \quad \textit{or} \quad \frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$$



- To find a side use the first form.
- To find an angle use the second form.

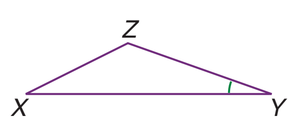
worked examples

1



Find the value of x in this triangle.
Answer correct to one decimal place.

2



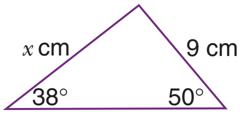
$\angle X = 32°50'$, $x = 15{\cdot}6$ cm, $y = 9{\cdot}7$ cm.
Find $\angle Y$ correct to the nearest minute.

3 Town B is 20 km due east of A. If the bearing of town C is N 35° E from A and N 65° W from B find the distance from A to C and from B to C.

continued →→→

Solutions

1 Here, two angles and two sides are involved. As we are finding a side, it is best to use the first form of the sine rule.



■ *Remember:*
side a is opposite $\angle A$
side b is opposite $\angle B$

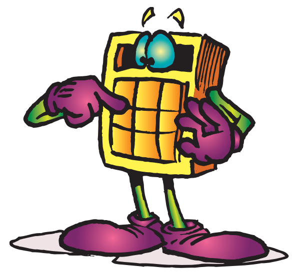
$$\frac{a}{\sin A} = \frac{b}{\sin B}$$

$$\frac{x}{\sin 50°} = \frac{9}{\sin 38°}$$

$$\therefore x = \frac{9 \sin 50°}{\sin 38°}$$

$$\doteqdot 11{\cdot}2 \text{ cm}$$

■ Press:
9 [×] [sin] 50 [÷] [sin] 38 [=]



2 Always draw a diagram if one is not given with the question. When finding an angle, the second form of the sine rule is easier to use.

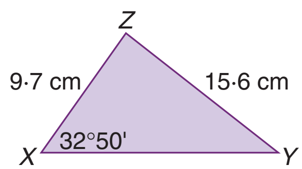
$$\frac{\sin Y}{y} = \frac{\sin X}{x}$$

$$\therefore \frac{\sin Y}{9{\cdot}7} = \frac{\sin 32°50'}{15{\cdot}6}$$

$$\sin Y = \frac{9{\cdot}7 \sin 32°50'}{15{\cdot}6}$$

$$= 0{\cdot}337\ 135\ 4$$

$$\therefore \angle Y = 19°42'$$

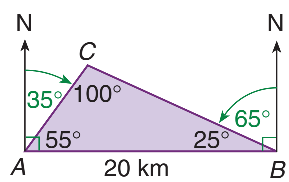


3

$\frac{b}{\sin B} = \frac{c}{\sin C}$	$\frac{a}{\sin A} = \frac{c}{\sin C}$
$\frac{AC}{\sin 25°} = \frac{20}{\sin 100°}$	$\frac{BC}{\sin 55°} = \frac{20}{\sin 100°}$
$AC = \frac{20 \sin 25°}{\sin 100°}$	$BC = \frac{20 \sin 55°}{\sin 100°}$
$\doteqdot 8{\cdot}6$ km	$\doteqdot 16{\cdot}6$ km

$\angle CAB = 90° - 35° = 55°$
$\angle CBA = 90° - 65° = 25°$
$\therefore \angle ACB = 180° - (55° + 25°)$
$= 100°$

So, the distance AC is 8·6 km and the distance BC is 16·6 km (given correct to one decimal place).



To use the *sine rule*, at least one angle and the side opposite it must be known.

Exercise 12:03

Foundation Worksheet 12:03

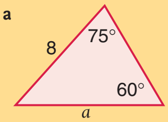
The sine rule

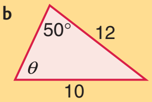
1 Solve the following equations:

a $\frac{x}{6} = 8$ b $\frac{a}{0{\cdot}756} = 4{\cdot}654$

2 a If $\frac{a}{\sin A} = \frac{b}{\sin B}$, find a when:
$A = 30°$, $B = 70°$ and $b = 15$

3 Complete the sine rule substitution for each triangle shown.

a


b


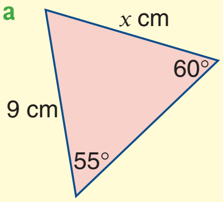
4 For each part in question 3, find the value of the pronumerals.

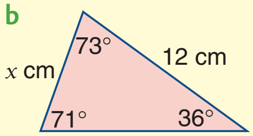
1 Solve the following equations.

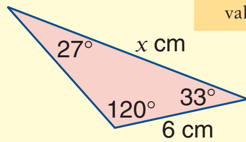
a $\frac{a}{0{\cdot}866} = \frac{12{\cdot}5}{0{\cdot}48}$ b $\frac{a}{0{\cdot}4356} = \frac{7{\cdot}8}{0{\cdot}648}$

c $\frac{a}{\sin 20°} = \frac{12}{\sin 60°}$ d $\frac{a}{\sin 70°} = \frac{20{\cdot}6}{\sin 50°}$

2 For each triangle, part of the sine rule has been written. Copy and complete the rule, then find x correct to one decimal place.

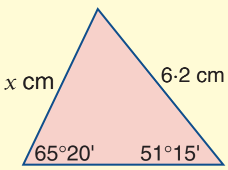
a


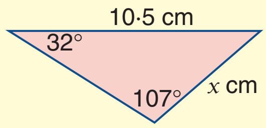
b


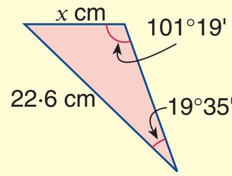
c


$\frac{x}{\sin 55°} = \frac{9}{\ldots}$ $\frac{x}{\ldots} = \frac{12}{\sin 71°}$ $\frac{x}{\ldots} = \frac{6}{\ldots}$

3 Find x in the following diagrams, correct to one decimal place.

a


b


c


4 In each of the following, find the value of θ.

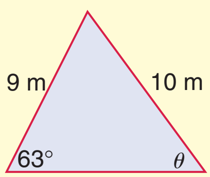
i correct to the nearest degree

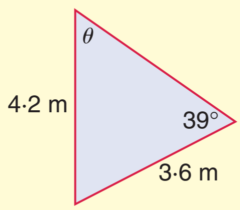
ii correct to the nearest minute

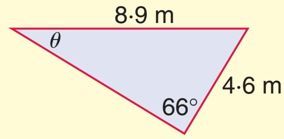
a $\sin\theta = 0{\cdot}7178$ b $\sin\theta = 0{\cdot}8164$ c $\sin\theta = 0{\cdot}2$

d $\sin\theta = \frac{14{\cdot}6 \sin 48°}{12{\cdot}6}$ e $\frac{\sin\theta}{11{\cdot}4} = \frac{\sin 63°}{16{\cdot}2}$

5 For each triangle, part of the sine rule has been written. Copy and complete the rule and then find the angle θ, correct to the nearest degree.

a


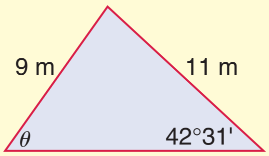
b


c


$\frac{\sin\theta}{9} = \frac{\sin 63°}{\ldots}$ $\frac{\sin\theta}{\ldots} = \frac{\sin 39°}{\ldots}$ $\frac{\sin\theta}{\ldots} = \frac{\sin 66°}{\ldots}$

6 Find the value of θ to the nearest minute.

a



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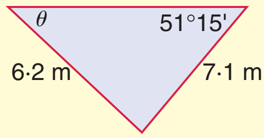

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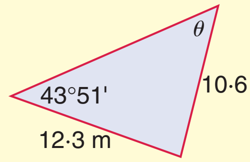
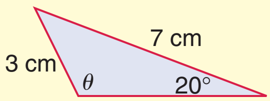


7 Find θ to the nearest degree, noting that in each triangle θ is obtuse.

a



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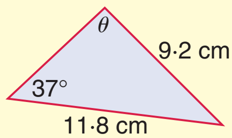

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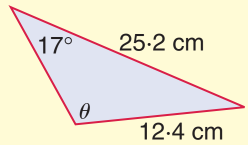


8 Answer the following by first drawing a diagram.

a In ΔABC, $\angle A = 30°$, $\angle B = 52°$, $a = 9$ cm; find b (to one decimal place).

b In ΔPQR, $\angle P = 51°$, $\angle R = 77°$, $r = 10{\cdot}2$ cm; find p (to one decimal place).

c In ΔXYZ, $\angle Y = 19°$, $y = 19{\cdot}1$ cm, $x = 15{\cdot}7$ cm; find $\angle X$ (to the nearest minute).

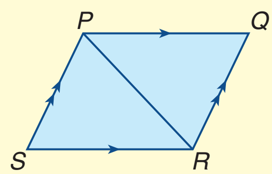
d In ΔLMN, $\angle M = 37°$, $m = 8{\cdot}7$ m, $n = 4{\cdot}6$ m; find $\angle N$ (to the nearest minute).

9 $PQRS$ is a parallelogram, where $PQ = 8$ cm, $QR = 5$ cm, and $\angle PRQ = 68°$.

Find: **a** $\angle RPQ$ (to the nearest degree)

b $\angle PQR$ (to the nearest degree)

c diagonal PR (to one decimal place)



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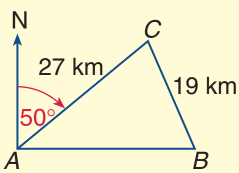


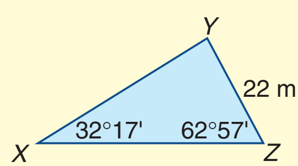
Three towns are situated so that the distance from A to C is 27 km, the distance from B to C is 19 km and the bearing of C from A is N 50° E. If B is due east of A, find:

a $\angle ABC$ (to the nearest degree)

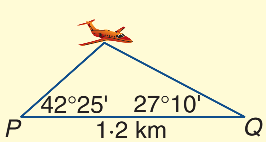
b $\angle ACB$ (to the nearest degree)

c distance of B from A (to two significant figures).

11 Three posts X, Y and Z are situated so that $\angle X$ is 32°17′ and $\angle Z$ is 62°57′. If the distance between Y and Z is 22 m, find the distance between X and Y to the nearest metre.



12



A man at P observes a plane to the south at an angle of elevation of 42°25′, while a second man at Q observes the same plane to the north at an angle of elevation of 27°10′. If the distance between the observers is 1·2 km, find the distances of each observer from the plane, to the nearest metre.

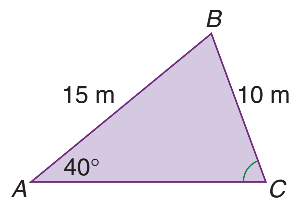
12:04 | The Sine Rule: The Ambiguous Case

- We have seen that when an equation like $\sin \theta = 0{\cdot}5$ is solved, there is an acute and an obtuse answer.
- Hence, when the sine rule is used to find an angle, it is possible for there to be two solutions. When this occurs, we have what is called the ambiguous case.
- While the acute angle will always be a solution, the obtuse angle will only be a solution if it gives an angle sum less than 180° when added to the other angle in the triangle.

Both the one-solution and two-solution cases are shown in the examples below.

worked examples

1 Use the sine rule to find the size of angle C.



Solution

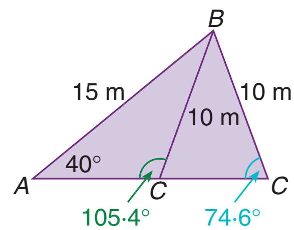
$$\frac{\sin C}{c} = \frac{\sin A}{a}$$

$$\frac{\sin C}{15} = \frac{\sin 40°}{10}$$

$$\sin C = \frac{15 \sin 40°}{10}$$

$$= 0{\cdot}964$$

$\therefore C = 74{\cdot}6°$ (from the calculator)



However:

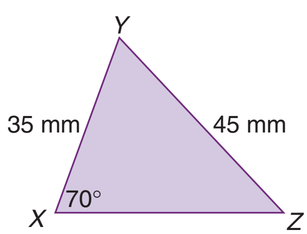
$\sin (180° - 74{\cdot}6°)$ is also equal to 0·964.

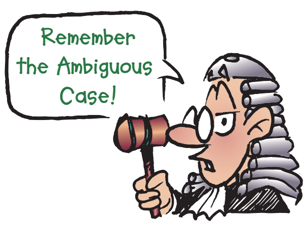
$\therefore C = 74{\cdot}6°$ or $105{\cdot}4°$

As $105{\cdot}4° + 40° < 180°$, then 105·4° is also a solution.

The diagram shows the two possible solutions.

2 Find the size of angle Z.





Now $\dfrac{\sin Z}{35} = \dfrac{\sin 70°}{45}$

$$\sin Z = \frac{35 \sin 70°}{45}$$

$$\doteqdot 0{\cdot}731$$

$$Z \doteqdot 47°$$

$\sin (180° - 47°)$ is also equal to 0·73

$\therefore Z = 47°$ or $133°$ (to the nearest degree)

Because $70° + 133°$ is greater than 180°, it is impossible for Z to be 133°.

We therefore reject this answer.

$\therefore Z = 47°$

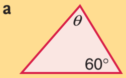
Exercise 12:04

Foundation Worksheet 12:04

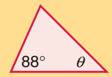
Sine rule — the ambiguous case

1 a Find θ if $\sin\theta = 0{\cdot}8$ and θ is:
   i acute   ii obtuse

2 From the information shown, is there one or two possible solutions for θ?

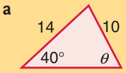
a 

$\theta = 70°$ or $110°$

b 

$\theta = 30°$ or $150°$

3 Find the acute and obtuse solution for θ in the triangle shown.

a 

1 Find the acute and obtuse angles that are solutions to the equations.

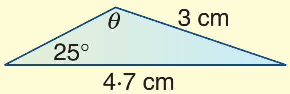
a $\sin\theta = 0{\cdot}8563$

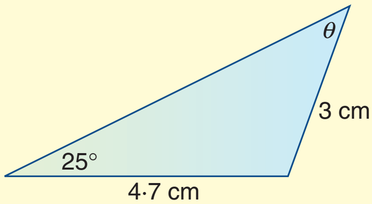
b $\sin\theta = 0{\cdot}216$

c $\sin\theta = \dfrac{15\sin 60°}{20}$

d $\sin\theta = \dfrac{20\sin 15°}{15}$

2



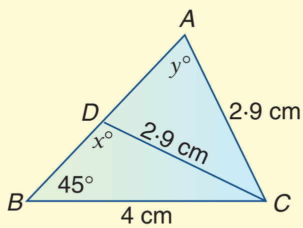


In both the triangles above, $\sin\theta = \dfrac{4{\cdot}7\sin 25°}{3}$. Find θ in both triangles. Give answers to the nearest degree.

3 Triangles ABC and DBC both have sides 4 cm and 2·9 cm in length and a non-included angle of 45°.

a Show that $\sin x°$ and $\sin y°$ both equal $\dfrac{4\sin 45°}{2{\cdot}9}$.

b Find $x°$ and $y°$ to the nearest degree.



4 Find (to the nearest minute) the angle required in each of the following.

a In ΔXYZ, $\angle X = 30°$, $x = 5$ cm, $y = 9$ cm; find $\angle Y$.

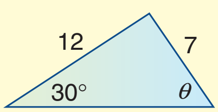
b In ΔPQR, $\angle Q = 19°20'$, $q = 2{\cdot}9$ cm, $r = 3{\cdot}7$ cm; find $\angle R$.

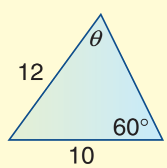
c In ΔABC, $\angle B = 32°17'$, $b = 10{\cdot}7$ cm, $a = 12{\cdot}1$ cm; find $\angle A$.

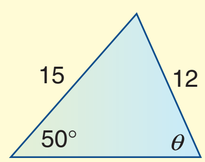
d In ΔKLM, $\angle M = 27°51'$, $m = 8{\cdot}7$ cm, $k = 9{\cdot}8$ cm; find $\angle K$.

Naming sides and angles in triangles:
x is opposite $\angle X$, y is opposite $\angle Y$
m is opposite $\angle M$, k is opposite $\angle K$

5 Find the possible values of θ in each of the following.

a 

b 

c 

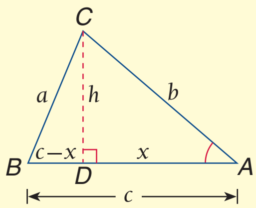
12:05 | The Cosine Rule

Like the sine rule, the cosine rule can be used to find unknown sides or angles in triangles. It is applied to situations where three sides and one angle are involved.

In the triangles below:

- CD is perpendicular to AB.
- Let AD be equal to x.
- In triangle **A**, $BD = c - x$.

A



Using Pythagoras' theorem,
From ΔBCD,

$$a^2 = (c - x)^2 + h^2$$
$$= c^2 - 2cx + x^2 + h^2$$

Now, from ΔACD, $b^2 = x^2 + h^2$

so $a^2 = c^2 - 2cx + b^2$

$$a^2 = b^2 + c^2 - 2cx \ . \ . \ . \ . \ . \ . \ . \ . \ (\text{i})$$

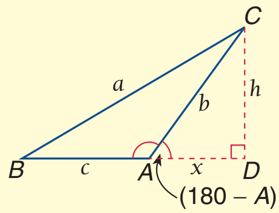
Now, in ΔACD, $\dfrac{x}{b} = \cos A$

$$x = b \cos A$$

Substituting in (i) gives:

$$a^2 = b^2 + c^2 - 2bc \cos A$$

B



From ΔBCD,

$$a^2 = (c + x)^2 + h^2$$
$$= c^2 + 2cx + x^2 + h^2$$

Now, from ΔACD, $b^2 = x^2 + h^2$

so $a^2 = c^2 + 2cx + b^2$

$$a^2 = b^2 + c^2 + 2cx \ . \ . \ . \ . \ . \ . \ . \ . \ (\text{ii})$$

Now, in ΔACD, $\dfrac{x}{b} = \cos(180° - a)$

$$x = b \cos(180° - A)$$
$$= -b \cos A$$

Substituting in (ii) gives:

$$a^2 = b^2 + c^2 - 2bc \cos A$$

Thus, for any triangle ABC:

$$\boldsymbol{a^2 = b^2 + c^2 - 2bc \cos A}$$

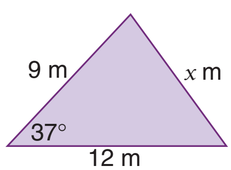
To find an angle, this rule can be rearranged to give:

$$\boldsymbol{\cos A = \frac{b^2 + c^2 - a^2}{2bc}}$$

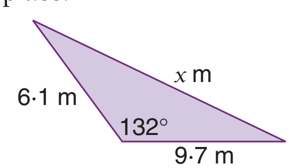
worked examples

1 Find x in each triangle, correct to one decimal place.

a



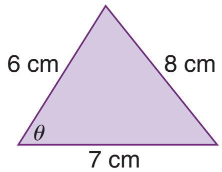
b



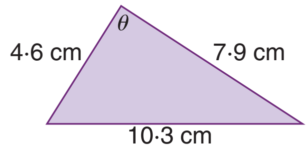
continued →→→

2 Evaluate θ, correct to the nearest minute.

a



b



Solutions

1 a $a^2 = b^2 + c^2 - 2bc \cos A$

Applying this formula gives:

$$\begin{aligned} x^2 &= 12^2 + 9^2 - 2 \times 12 \times 9 \cos 37° \\ &= 144 + 81 - 216 \cos 37° \\ &= 52{\cdot}494\,729 \end{aligned}$$

$$\begin{aligned} \therefore x &= \sqrt{52{\cdot}494\,729} \\ &= 7{\cdot}2 \text{ (to 1 dec. pl.)} \end{aligned}$$

b $a^2 = b^2 + c^2 - 2bc \cos A$

That is:

$$\begin{aligned} x^2 &= 6{\cdot}1^2 + 9{\cdot}7^2 - 2 \times 6{\cdot}1 \times 9{\cdot}7 \cos 132° \\ &= 37{\cdot}21 + 94{\cdot}09 - 118{\cdot}34 \cos 132° \\ &= 210{\cdot}484\,91 \end{aligned}$$

$$\begin{aligned} x &= \sqrt{210{\cdot}484\,91} \\ &= 14{\cdot}5 \text{ (to 1 dec. pl.)} \end{aligned}$$

2 a $\cos A = \dfrac{b^2 + c^2 - a^2}{2bc}$

Applying this formula gives:

$$\begin{aligned} \cos \theta &= \frac{6^2 + 7^2 - 8^2}{2 \times 6 \times 7} \\ &= \frac{36 + 49 - 64}{84} \\ &= 0{\cdot}25 \end{aligned}$$

$\therefore \theta = 75°31'$ (to the nearest minute)

b $\cos A = \dfrac{b^2 + c^2 - a^2}{2bc}$

That is:

$$\begin{aligned} \cos \theta &= \frac{4{\cdot}6^2 + 7{\cdot}9^2 - 10{\cdot}3^2}{2 \times 4{\cdot}6 \times 7{\cdot}9} \\ &= \frac{21{\cdot}16 + 62{\cdot}41 - 106{\cdot}09}{72{\cdot}68} \\ &= -0{\cdot}309\,851\,4 \end{aligned}$$

$\therefore \theta = 108°3'$ (to the nearest minute)

The *cosine rule* can be used, in any triangle, to find:
- **the third side, given the other sides and their included angle**
- **any angle, given the three sides.**

Exercise 12:05

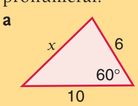
Foundation Worksheet 12:05

The cosine rule

1 Find a if $a^2 = b^2 + c^2 - 2bc \cos A$ and:
 a $b = 10$, $c = 12$, $A = 60°$

2 Find A if:
 a $\cos A = 0{\cdot}5$ **b** $\cos A = -0{\cdot}5$

3 Complete the cosine rule substitution for each triangle and then find the value of the pronumeral.
 a



1 Find a (correct to 1 dec. pl.) if $a^2 = b^2 + c^2 - 2bc \cos A$ and:

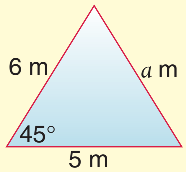
a $b = 10$, $c = 12$, $A = 60°$
b $b = 6{\cdot}8$, $c = 9{\cdot}4$, $A = 120°$
c $b = 15$, $c = 12$, $A = 75°$

2 Find A (to the nearest degree) if:

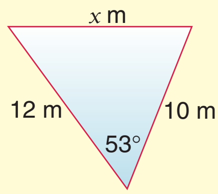
a $\cos A = 0{\cdot}6$
b $\cos A = -0{\cdot}6$
c $\cos A = \dfrac{6^2 + 10^2 - 8^2}{2 \times 6 \times 10}$
d $\cos A = \dfrac{6^2 + 8^2 - 12^2}{2 \times 6 \times 8}$

3 Find the value of the pronumeral in the following (correct to 1 dec. pl.).

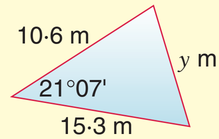
a



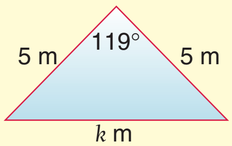
b



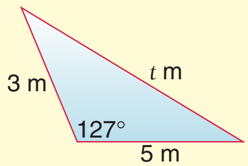
c



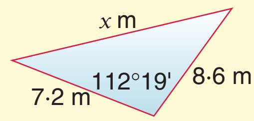
d



e

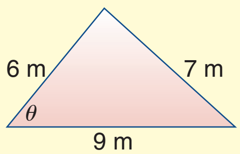


f

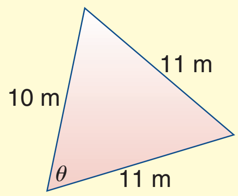


4 Find the size of angle θ, correct to the nearest minute.

a



b



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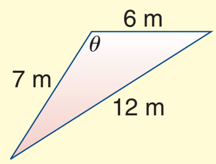
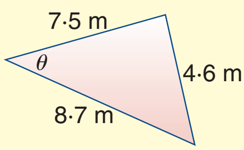


d



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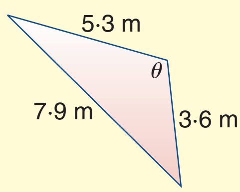

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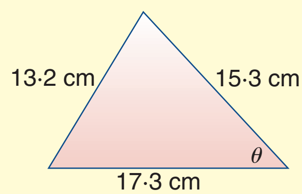


5 Answer the following by first drawing a diagram.

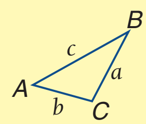
a In ΔABC, $\angle A = 75°$, $b = 9$ cm, $c = 5$ cm; find a (correct to 1 dec. pl.).

b In ΔDEF, $\angle F = 61°$, $d = 2{\cdot}3$ m, $e = 3{\cdot}1$ m; find f (correct to 1 dec. pl.).

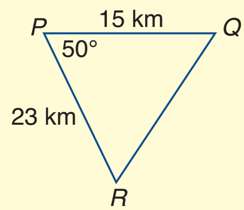
c In ΔLMN, $\angle M = 163°$, $l = 9{\cdot}4$ cm, $n = 8{\cdot}2$ cm; find m (correct to 1 dec. pl.).

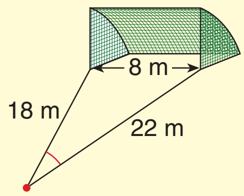
d In ΔPQR, $p = 9$ m, $q = 7$ m, $r = 8$ m; find $\angle P$ (correct to the nearest degree).

e In ΔTUV, $t = 2{\cdot}3$ cm, $u = 1{\cdot}9$ cm, $v = 1{\cdot}7$ cm; find $\angle V$ (correct to the nearest minute).

> Angle *A* is opposite side *a*.
>
> 
>

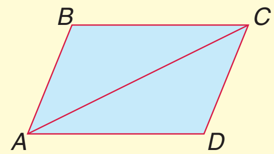
6 Three towns, *P*, *Q* and *R*, are connected by straight roads. The distance from *P* to *Q* is 15 km and the distance from *P* to *R* is 23 km. If the roads meet at *P* at an angle of 50°, how far is town *Q* from town *R*, to the nearest kilometre?



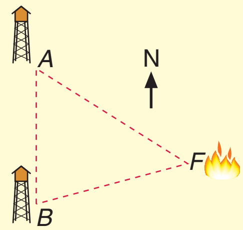
7 

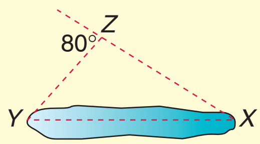
If a soccer goal is 8 m wide and a player shoots for goal when he is 18 m from one post and 22 m from the other, within what angle must the shot be made to score the goal? (Give your answer correct to the nearest degree.)

8 In a parallelogram *ABCD*, *AB* is 9 cm, *BC* is 15 cm, and $\angle ABC$ is 130°. Find the length of the diagonal *AC*, correct to the nearest millimetre.



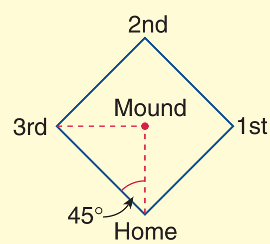
9 An observer at fire tower *A* observes a fire 18 km away to have a bearing S67°E. How far is the fire from tower *B* if this tower is 12 km due south of tower *A*?



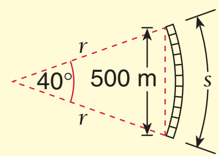
10 

To find the length of a pond, a surveyor walks 350 metres from point *X* to point *Z*, then turns 80° and walks 290 metres to point *Y*. Find the length of the pond to the nearest metre.

11 In a (square) baseball diamond with sides of length 27·4 metres, the centre of the pitcher's mound is 18·4 metres from home plate.
How far is it from the centre of the pitcher's mound to third base? Give your answer correct to one decimal place.

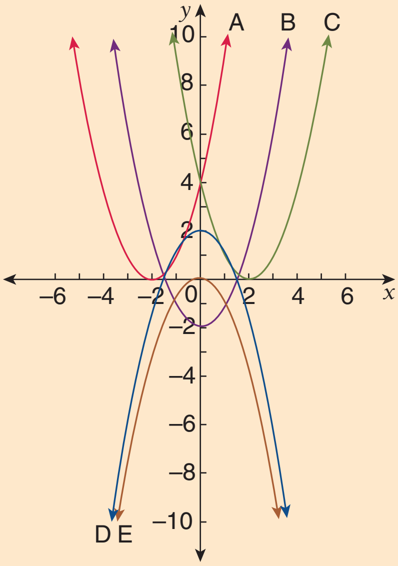


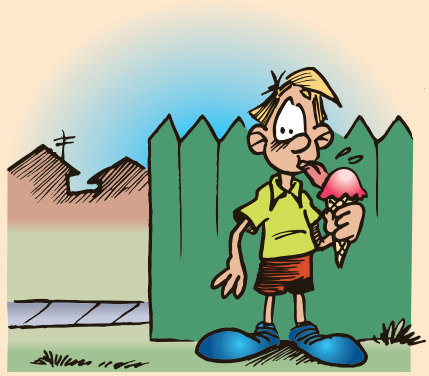
12 The section of railway track shown is a circular arc that has a chord length of 500 metres and subtends an angle of 40° at the centre of the circle. Find the radius *r* of the circular arc and the length *s* of that arc.



Fun Spot 12:05 | Why did Tom's mother feed him Peter's ice-cream?

Answer each question and put the letter for that question in the box above the correct answer.





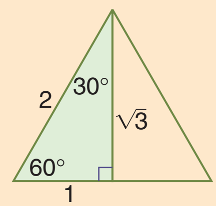
Match each of the parabolas A to E with its equation.

E $y = (x - 2)^2$ **T** $y = -x^2$ **F** $y = (x + 2)^2$ **H** $y = x^2 - 2$ **O** $y = 2 - x^2$

Use the diagrams on the right to give the exact value of:

E $\cos 60°$ **T** $\sin 60°$ **S** $\tan 60°$ **F** $(\sin 30°)^2$

H $(\cos 30°)^2$ **I** $(\tan 30°)^2$ **E** $(\sin 30°)^2 + (\cos 30°)^2$



Give the resulting coordinates if the point A is:

E reflected in the x-axis

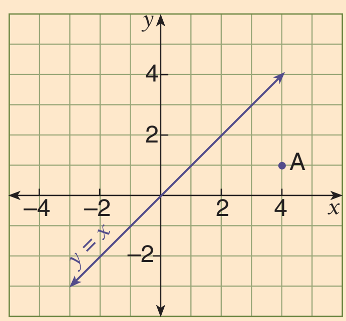
T translated 4 units left

K rotated through 90° about (0, 0)

M reflected in the y-axis

O reflected in the line $y = x$

P translated two units left, then 3 units up.



For each point below, what is its reflection in the line $y = x$?

E (2, 0) **R** (5, –2) **S** (–1, –2) **T** (0, –1)

E	(1, 4)	(–1, 4)	C	1	(2, 4)	$\frac{3}{4}$	$\frac{1}{3}$	(–4, 1)	D	$\frac{1}{4}$	A

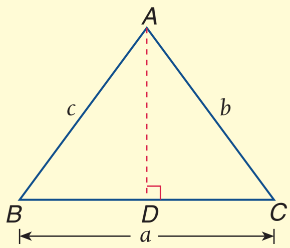
(0, 4)	B	(4, –1)	(–2, –1)	$\frac{\sqrt{3}}{2}$	(–2, 5)	(0, 2)	$\frac{1}{2}$	(–1, 0)	$\sqrt{3}$

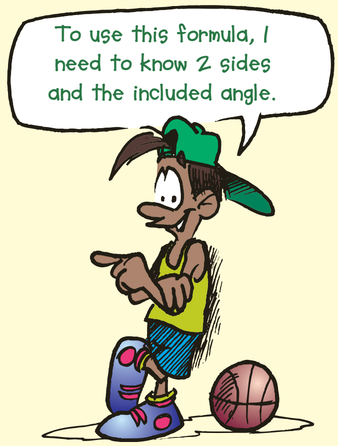
fun spot 12:05

12:06 | Area of a Triangle

To calculate the area of a triangle, the formula $A = \frac{1}{2}bh$ is used, where b is the length of the base and h is the perpendicular height. So, to calculate the area of any triangle, these two measurements needed to be known.

Now, consider the triangle ABC, below.





AD is perpendicular to BC.
So, the area of ΔABC is given by

$\text{Area} = \frac{1}{2}BC \times AD$ . . . . (i)

Now, from ΔACD, $\frac{AD}{AC} = \sin C$

ie $AD = AC \sin C$

Substituting this into (i) above,

$\text{Area} = \frac{1}{2}BC \times AC \sin C$

$= \frac{1}{2}ab \sin C$

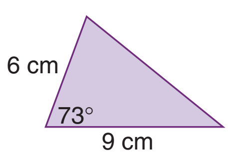
So, another formula for the area of a triangle is:

Area = $\frac{1}{2}ab \sin C$
where a and b are two sides and C is the angle included by them.

worked examples

Find the area of each triangle correct to the nearest cm^2.

1



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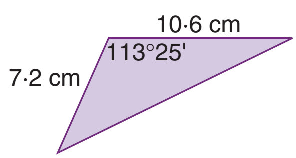


Solutions

1 $A = \frac{1}{2}ab \sin C$

That is:

$A = \frac{1}{2} \times 6 \times 9 \times \sin 73°$

$= 25{\cdot}820\,228 \text{ cm}^2$

$= 26 \text{ cm}^2$ (to nearest cm^2)

2 $A = \frac{1}{2}ab \sin C$

That is:

$A = \frac{1}{2} \times 10{\cdot}6 \times 7{\cdot}2 \times \sin 113°25'$

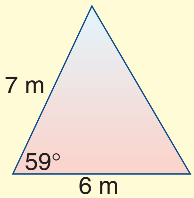
$= 35{\cdot}017\,106 \text{ cm}^2$

$= 35 \text{ cm}^2$ (to nearest cm^2)

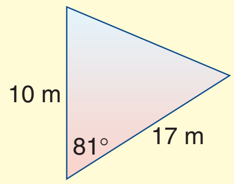
Exercise 12:06

1 Find the area of each triangle, correct to one decimal place.

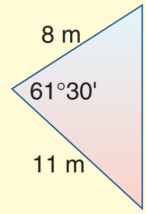
a



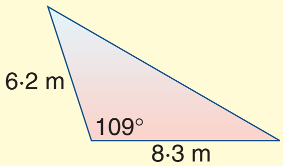
b



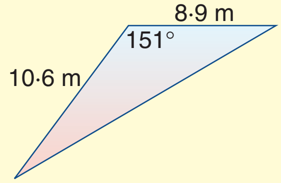
c



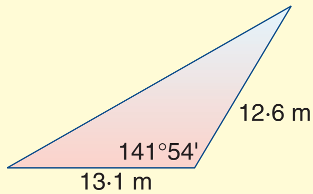
d



e

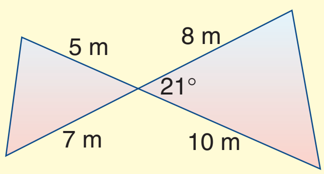


f



2 Find the area enclosed by each figure (to nearest cm^2).

a



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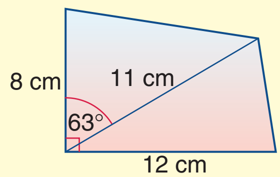


3 For ΔXYZ, find its area, to the nearest square unit, if:

a $x = 7$, $y = 10$, $Z = 47°$

b $x = 14·6$, $y = 17·2$, $Z = 72°31'$

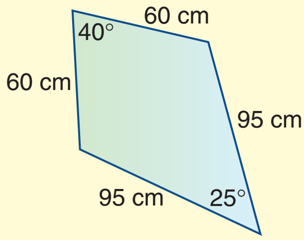
c $y = 2·3$, $z = 3·9$, $X = 62°$

d $y = 52$, $z = 63$, $X = 127°55'$

e $x = 20$, $z = 31$, $Y = 53°24'$

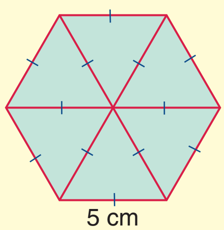
f $x = 72·3$, $z = 91·6$, $Y = 142°07'$

4 **a**



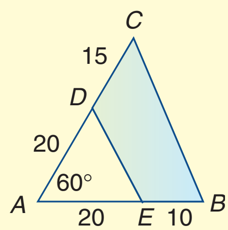
Calculate the area of the kite correct to the nearest square centimetre.

b



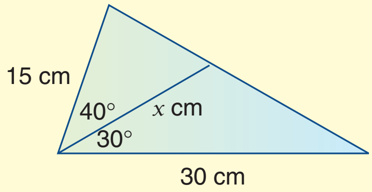
Calculate the area of the hexagon correct to one decimal place.

c



Calculate the area of the quadrilateral *BCDE*. All measurements are in centimetres.

5 Find the value of x correct to one decimal place.



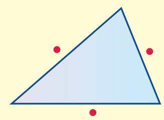
12:07 | Miscellaneous Problems

When doing a set of mixed problems, it is important to remember when the sine and cosine rules can be used.

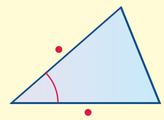
There is an interesting connection between the information that is needed for the sine and cosine rules and the standard congruence tests for triangles.

To use the cosine rule you need:

- Three sides (SSS).
  This allows you to find all the angles.

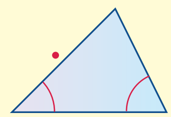


- Two sides and an included angle (SAS).

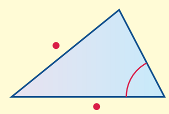


To use the sine rule you need:

- Two angles and a side (AAS).
  This allows you to find the other angle and the other sides.



- Two sides and a non-included angle (SSA).
  This allows you to find the unknown angle opposite the known side. As there could be two solutions for the angle, this is the ambiguous case. (Hence, SSA is *not* a congruence test.)



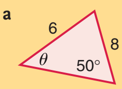
Exercise 12:07

Foundation Worksheet 12:07

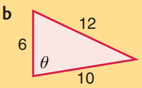
Sine rule or cosine rule?

1 Use the AAS, SSA, SAS, SSS tests to decide whether each of the following problems requires the sine rule or the cosine rule.

a



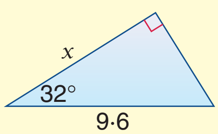
b



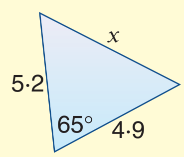
2 Find the value of the pronumerals in question **1**.

1 Use the information above to identify whether the sine rule or the cosine rule is needed and then find the value of the pronumeral correct to one decimal place.

a



b



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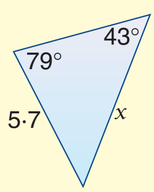
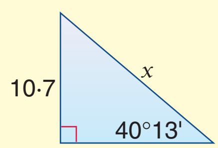
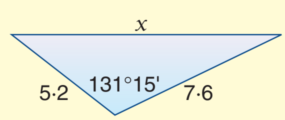


d

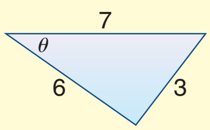


e

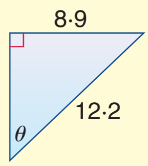


2 Evaluate θ, correct to the nearest minute.

a



b



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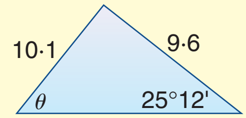
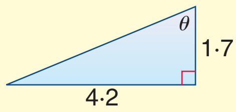


d



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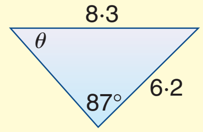

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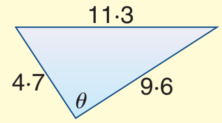
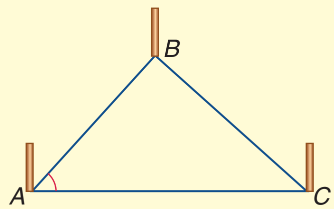


3 For a triangle XYZ,

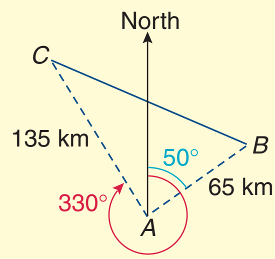
- **a** $x = 9{\cdot}2$, $y = 5{\cdot}6$, $\angle Z = 90°$; find $\angle X$ correct to the nearest minute.
- **b** $\angle X = 36°$, $\angle Y = 47°$, $y = 9{\cdot}6$; find x correct to one decimal place.
- **c** $\angle Z = 14°25'$, $x = 4{\cdot}2$, $y = 3{\cdot}7$; find z correct to one decimal place.
- **d** $\angle Y = 95°17'$, $\angle Z = 47°05'$, $y = 11{\cdot}3$; find x correct to one decimal place.
- **e** $\angle X = 90°$, $\angle Y = 23°17'$, $y = 5{\cdot}2$; find x correct to one decimal place.
- **f** $x = 10{\cdot}3$, $y = 9{\cdot}6$, $z = 8{\cdot}7$; find $\angle Y$ correct to the nearest minute.

4 Three posts, A, B and C are positioned so that AB is 33 m, BC is 45 m and AC is 51 m. Find the angle subtended at post A by BC (to the nearest minute).

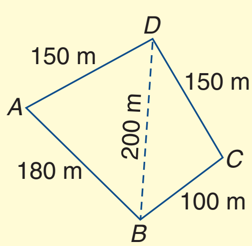


5 P and Q are two points on the shore 950 m apart. R is a buoy out at sea so that the angles RPQ and RQP are 73°19′ and 68°32′, respectively. Find the distance of P from R.

6 From town A, the bearings of towns B and C are 050° and 330° respectively. If A is 65 km from B and 135 km from C, how far is town B from town C?



7

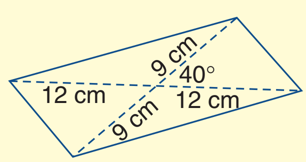


A farmer has a piece of land as shown in the diagram. What is the area of the land, to the nearest square metre?

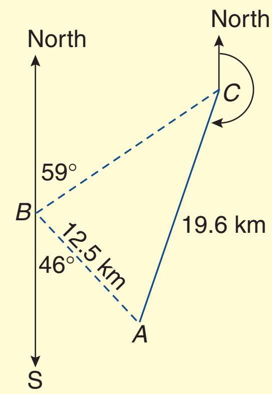
8 The diagonals of a parallelogram bisect each other at an angle of 40°. If the diagonals are 24 cm and 18 cm long, find:

- **a** the area of the parallelogram
- **b** the lengths of the sides of the parallelogram.

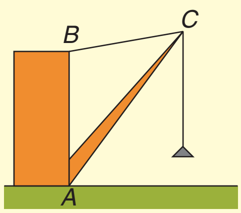
Give answers correct to one decimal place.



9 Town A is 12·5 km S46°E of town B; town C is 19·6 km from town A and C bears N59°E from B. Find the bearing of A from C.



10



The diagram represents a crane where AB is 7·5 m, AC is 12·6 m and BC is 9·6 m. Find the angle ABC and the height of C above the ground.

12:08 | Problems Involving More than One Triangle

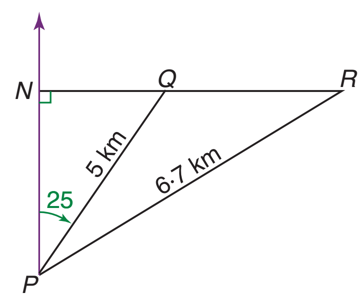
All the trigonometry met so far can be used in more complicated problems. In these problems, the unknown side (or angle) cannot be found directly. Extra information, such as the lengths of other sides or the sizes of angles, will need to be calculated from other triangles. This can then be used to calculate the unknown side (or angle).

In these problems, there are often different methods of finding the solution.

worked examples

Example 1

P, Q and R are three villages. Q is 5 km and N25°E from P. R is east of Q and is 6·7 km from P. What is the bearing of R from P, to the nearest degree?



Solution 1

To find the bearing of R from P, we need to find the size of angle NPR. In ΔNPR we know the length of PR, but we need to know one of the other sides, either NR or NP. Side NP can be calculated using ΔNPQ.

In ΔNPQ: $\quad \dfrac{NP}{5} = \cos 25°$

ie $\quad NP = 5 \cos 25°$

In ΔNPR: $\quad \cos \angle NPR = \dfrac{NP}{6 \cdot 7}$

$$= \frac{5 \cos 25°}{6 \cdot 7}$$

$$= 0 \cdot 676\ 349$$

$\therefore\ \angle NPR = 47°$ (to the nearest degree)

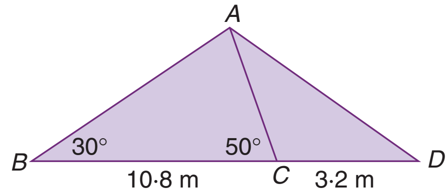
$\therefore$ The bearing of R from P is N47°E.

Example 2

A builder has to construct a section of roof as shown in the diagram.

a Find the length of AB correct to three decimal places.

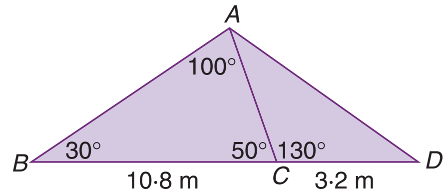
b Use your answer from part **a** to find AD correct to three decimal places.



Solution 2

In ΔABC, $\angle BAC = 100°$

Using the sine rule in ΔABC,



$$\frac{AB}{\sin 50°} = \frac{10{\cdot}8}{\sin 100°}$$

$$AB = \frac{10{\cdot}8 \sin 50°}{\sin 100°}$$

$$= 8{\cdot}401 \text{ m (correct to 3 dec. pl.)}$$

Now, using the cosine rule in ΔABD

$$AD^2 = AB^2 + 14^2 - 2 \times 14 \times AB \times \cos 30°$$

$$= 8{\cdot}401^2 + 14^2 - 28 \times 8{\cdot}401 \times \cos 30°$$

$$= 62{\cdot}8633\ldots$$

$$\therefore AD = \sqrt{62{\cdot}8633\ldots}$$

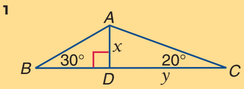
$$= 7{\cdot}929 \text{ m (correct to 3 dec. pl.)}$$

Exercise 12:08

Foundation Worksheet 12:08

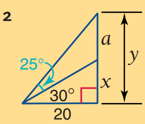
Problems with more than one triangle

1



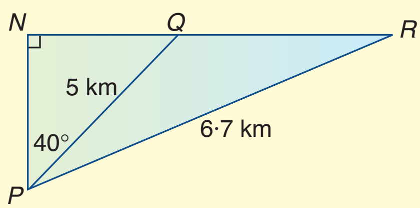
a Use ΔABD to find x.
b Use ΔADC to find y.

2

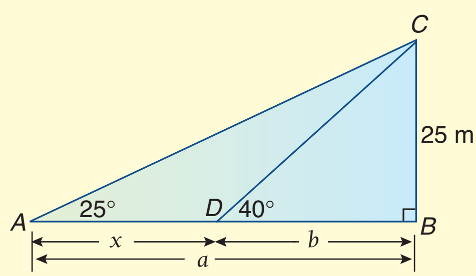


Use the fact that $a = y - x$ to find the value of a.

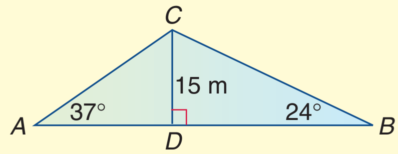
1 **a** Use ΔPNQ to find NP correct to one decimal place.
b Use ΔPNR to find $\angle NPR$ to the nearest degree.



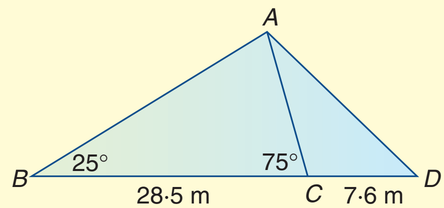
2 **a** Use ΔABC to find the value of a correct to two decimal places.
b Use ΔDBC to find the value of b correct to two decimal places.
c Find x (correct to one decimal place).



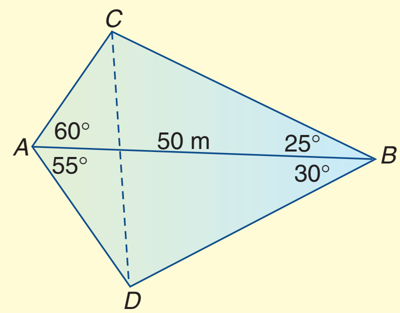
3 In ΔABC, $CD \perp AB$ and $CD = 15$.
$\angle ABC = 24°$ and $\angle CAB = 37°$.
Find:
a AD **b** DB **c** AB



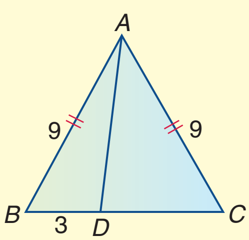
4 **a** Use the sine rule in ΔABC to find AB correct to one decimal place.
b Using your answer from part **a**, find the length of AD correct to one decimal place.



5 **a** Use the sine rule in ΔABC to find AC correct to one decimal place.
b Use the sine rule in ΔADB to find AD correct to one decimal place.
c Using your answers from parts **a** and **b**, find CD correct to one decimal place.

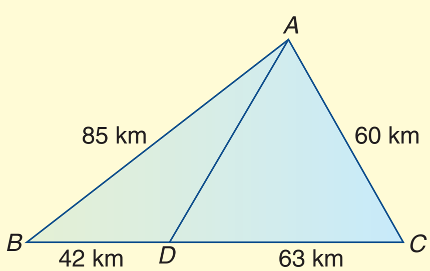


6



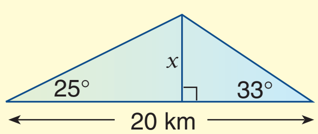
ΔABC is an equilateral triangle with sides 9 cm long.
D is a point on BC, 3 cm from B.
Find $\angle ADC$.

7

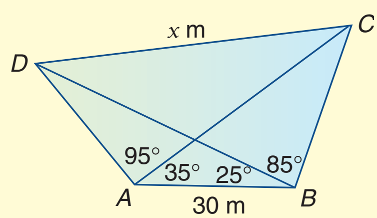


Three towns A, B and C are joined by straight roads. A straight road also runs from A to D. If the distances between the towns are as shown in the diagram, find the length of AD to the nearest kilometre.

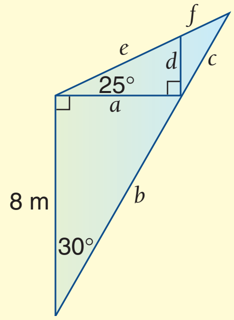
8 Find the value of x correct to one decimal place.



9 Find the value of x correct to one decimal place.

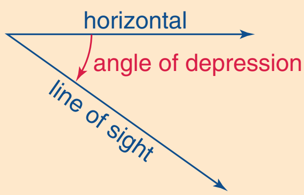


10 Find the value of the pronumerals correct to one decimal place.

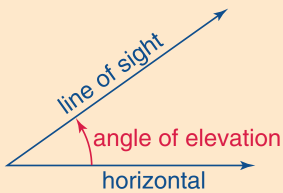


Mathematical Terms 12

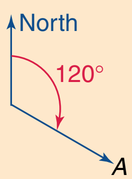
angle of depression
- When looking down, the angle between the line of sight and the horizontal.



angle of elevation
- When looking up, the angle between the line of sight and the horizontal.



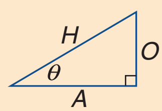
bearing
- An angle used to measure the direction of a line from north.
- Bearings can be recorded in two ways.
  eg 120° or S60°E



cosine rule
- A rule that is used to find either the third side of a triangle when the other two sides and the included angle (SAS) are known, or an angle when the three sides are known (SSS).

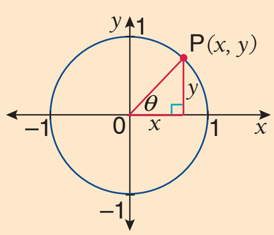
sine rule
- A rule used to find an angle or side of a triangle when either two angles and a side (AAS), or two sides and a non-included angle (SSA), are known.

trigonometric (trig.) ratios
- A set of ratios (sine, cosine and tangent) that have constant values for any particular angle.
- For acute angles, these can be defined in terms of the side lengths of a right-angled triangle.



$$\sin\theta = \frac{O}{H},\ \cos\theta = \frac{A}{H},\ \tan\theta = \frac{O}{A}$$

- For acute or obtuse angles, they are defined in terms of the coordinates of a point P which has its position on a unit circle determined by a radius drawn at an angle of θ to the horizontal.



$$\sin\theta = y,\ \cos\theta = x,\ \tan\theta = \frac{y}{x}$$

Mathematical terms 12

Diagnostic Test 12 | Further Trigonometry

- Each section of the diagnostic test has similar items that test a certain question type.
- Errors made will indicate areas of weakness.
- Each weakness should be treated by going back to the section listed.

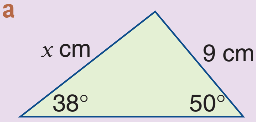
	Section
1 Which obtuse angle has the same sine ratio as: **a** 30°? **b** 50°? **c** 80°?	12:02
2 Write as the trig. ratio of an acute angle: **a** sin 140° **b** cos 140° **c** tan 140°	12:02
3 If $90° \leqslant \theta \leqslant 180°$, find θ correct to the nearest minute, if: **a** $\cos \theta = -0{\cdot}625$ **b** $\sin \theta = 0{\cdot}257$ **c** $\cos \theta = -0{\cdot}018$	12:02
4 Find the value of x, correct to one decimal place.	12:03
5 Find the value of θ, correct to the nearest degree.	12:03, 12:04
6 Find the value of x, correct to one decimal place.	12:05
7 Evaluate θ, correct to the nearest minute.	12:05
8 Find the area of each triangle (correct to the nearest cm²).	12:06

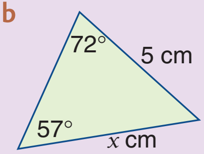
1 Which obtuse angle has the same sine ratio as:
a 30°? **b** 50°? **c** 80°?

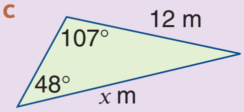
2 Write as the trig. ratio of an acute angle:
a sin 140° **b** cos 140° **c** tan 140°

3 If $90° \leqslant \theta \leqslant 180°$, find θ correct to the nearest minute, if:
a $\cos \theta = -0{\cdot}625$ **b** $\sin \theta = 0{\cdot}257$ **c** $\cos \theta = -0{\cdot}018$

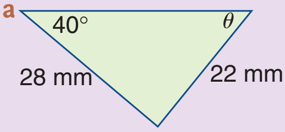
4 Find the value of x, correct to one decimal place.

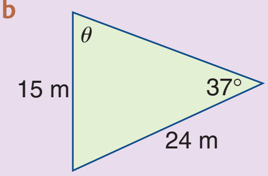
a


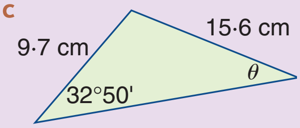
b


c


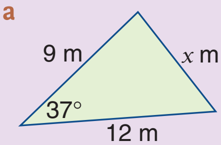
5 Find the value of θ, correct to the nearest degree.

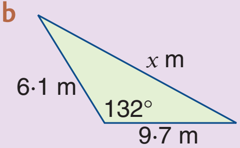
a


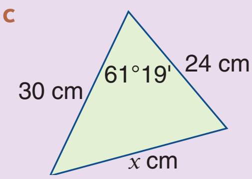
b


c


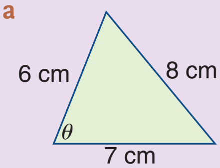
6 Find the value of x, correct to one decimal place.

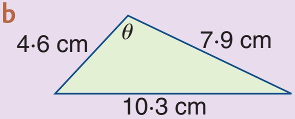
a


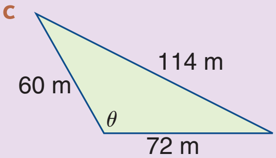
b


c


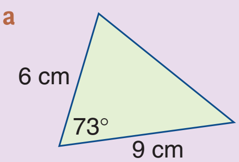
7 Evaluate θ, correct to the nearest minute.

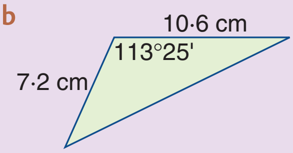
a


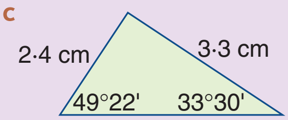
b


c


8 Find the area of each triangle (correct to the nearest cm²).

a


b


c


Chapter 12 | Revision Assignment

12A

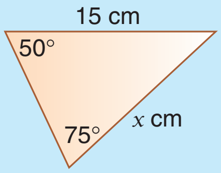
1 Find θ if θ is obtuse and:

a $\sin\theta = 0{\cdot}5$ b $\cos\theta = -0{\cdot}5$

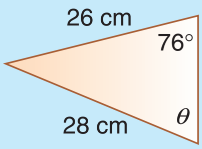
c $\tan\theta = -1$

2 Use the sine rule to find the value of the pronumeral in each of the following.

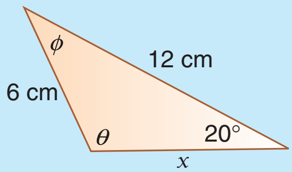
a



b



3

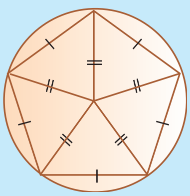


Find θ, ϕ and x if:

i θ is acute ii θ is obtuse

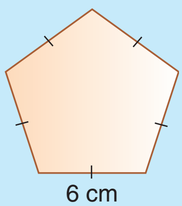
4 Find the area of the following pentagons.

a



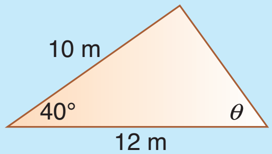
The radius of the circle is 6 cm.

b

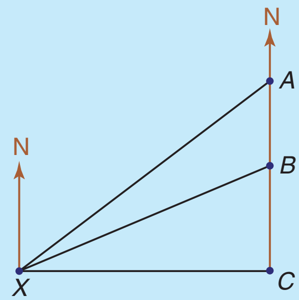


5 a A triangle has sides which are 8 cm, 12 cm and 10 cm in length. Find the size of the largest angle.

b Find the size of θ to the nearest degree.



6

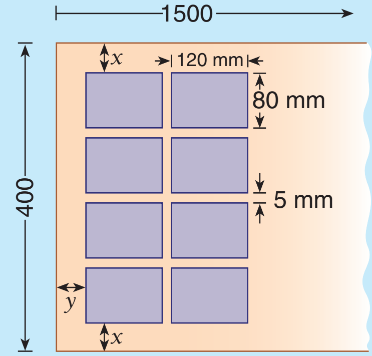


A, B and C are three towns where A and B are due north of C. From a position X on a map, A has a bearing of N27°E and B has a bearing of N67°E. Town C is due east of X and 7·5 km from it. Find the distance, correct to one decimal place, between A and B.

- A compass is needed to get our bearings.

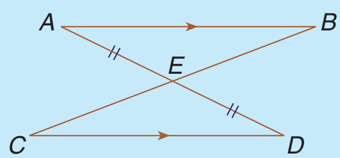
Chapter 12 | Working Mathematically

1

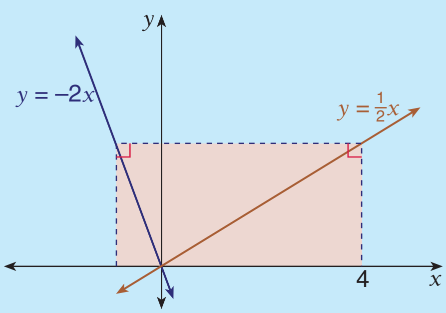


A carpenter has to make a table top that is rectangular in shape. It has to be 1500 mm long and 400 mm wide. He has to place 4 rows of 11 tiles in the centre of the table top. Each tile is rectangular and is 120 mm by 80 mm. He also has to leave a 5 mm gap between each row of tiles. How far must the outer row of tiles be from each side of the table? (Find x and y in the diagram.)

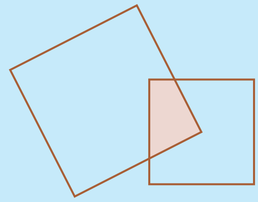
2 Study the diagram and give three different descriptions of how it could be drawn.



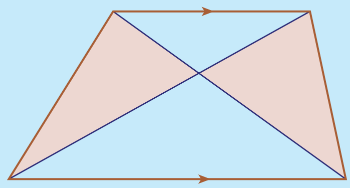
3 Use the information on the diagram to find the area of the shaded rectangle.



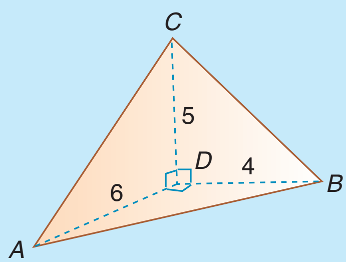
4 In the diagram, the side length of the smaller square is 40 cm and the side length of the larger square is 60 cm. The large square intersects the smaller square $\frac{1}{4}$ of the way along the two sides, as shown. What is the area of the overlapping region if the vertex of the large square is at the centre of the small square?



5 The diagram shows a trapezium divided into four triangles by its diagonals. Prove that the areas of the coloured triangles are equal.



6 Calculate the surface area of the pyramid.



Chapter Review Questions

Technology Applications Activities
1 Sine rule
2 Investigating sine curves

Technology Applications Drag and Drops
1 Sine rule
2 Cosine rule

Technology Applications Animations
Unit circle

13

Circle Geometry

Chapter Contents

Learning Outcomes

Students will be able to:

- Apply deductive reasoning to prove circle theorems and to solve problems.
- Describe a variety of parts of a circle.
- Investigate and prove the relationships between the properties of a circle, including chords, angles and tangents.

Areas of Interaction

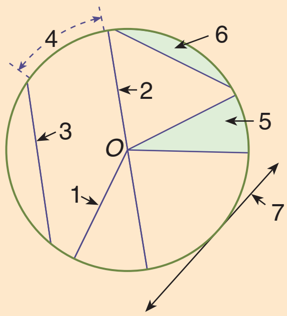
Approaches to Learning (Knowledge Acquisition, Problem Solving, Logical Thinking, Communication, Reflection), Human Ingenuity, Environments

#Optional topics as further preparation for the Mathematics Extension courses in Stage 6.

13:01 | Circles

prep quiz

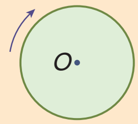
13:01



O is the centre.

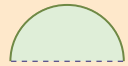
For 1 to 9, give the name of that part.

8

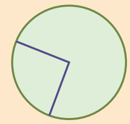


The distance around the circle.

9



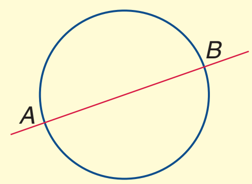
10 How many sectors are shown?



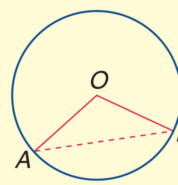
The Prep Quiz above has reviewed the terms associated with circles with which you should be familiar.

In later sections we will be investigating some relationships between angles in circles. Before we can do this we need to define some new terms.

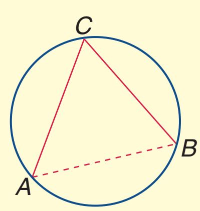
Secant: A line that intersects a curve in at least two places.



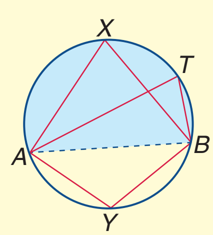
Angle at the centre: An angle formed by joining the ends of an arc or chord to the centre of a circle. (We say ∠*AOB* is an angle at the centre standing on the arc or chord *AB*.)



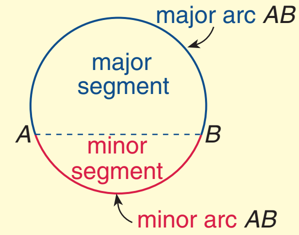
Angle at the circumference: An angle formed by joining the ends of an arc or chord to another point on the circumference. (We say ∠*ACB* is an angle at the circumference standing on the arc or chord *AB*.)

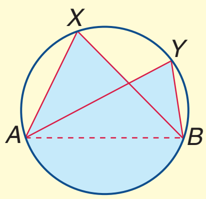


Angles in the same segment: A chord divides a circle into two segments. The larger segment is called the *major segment* and the smaller is called the *minor segment*. In the diagram, ∠*AXB* is in the major segment while ∠*AYB* is in the minor segment. Angles *AXB* and *ATB* are in the same segment.

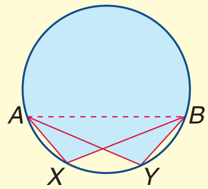


Angles standing on the same arc: Two points *A* and *B* divide a circle into two arcs. The larger arc is called the *major arc* and the smaller is called the *minor arc*. When the ends of an arc or chord are joined to two different points in the same segment, the angles are said to be standing on the same arc.

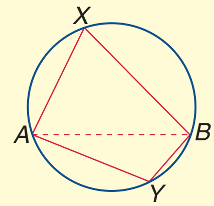




∠s *AXB* and *AYB* are standing on the major arc *AB*.

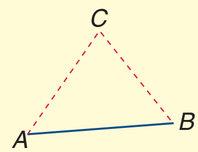


∠s *AXB* and *AYB* are standing on the minor arc *AB*.



∠s *AXB* and *AYB* are not standing on the same arc.

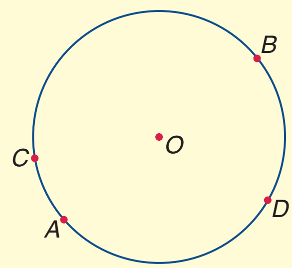
Subtend: If the ends of an interval *AB* are joined to a point *C*, the angle formed (∠*ACB*) is the angle subtended at *C* by the interval *AB*. In circle geometry, we speak about angles subtended at the circumference by the arc (or chord) *AB* or angles subtended at the centre by the arc (or chord) *AB*.



Exercise 13:01

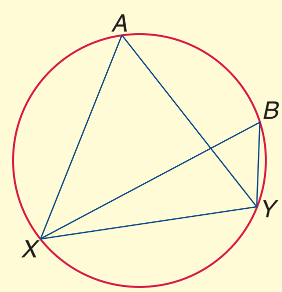
1 Make tracings of the diagram, and on separate drawings show:

- **a** the angle at the centre standing on the arc *AC*
- **b** the angle at the centre subtended by the arc *AD*
- **c** the angle subtended at *D* by the arc *BC*
- **d** the angle at *C* standing on the arc *AD*

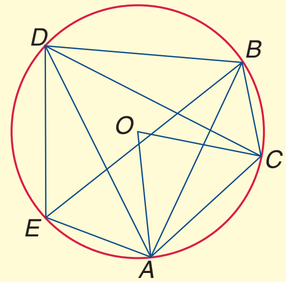


2 **a** Name an angle at the circumference that is standing on the arc:
i *AX* **ii** *BY*

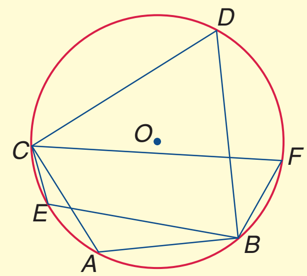
b Name two angles at the circumference that are standing on the arc:
i *XY* **ii** *AB*



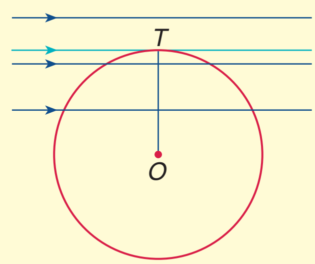
3 **a** Which two angles are standing on the arc *EA*?
b Which three angles are standing on the arc *AC*?
c How many angles does the arc *DB* subtend at the circumference?
d The chord *DA* divides the circle into minor and major segments. Name the angle in the minor segment.
e Only one angle is subtended at the centre. On which chord is it standing?

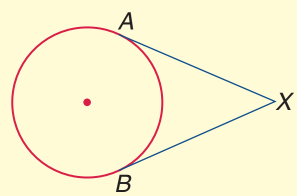


4 **a** Name two angles:
  i standing on the minor arc *BC*
  ii standing on the major arc *BC*
b The chord *BC* divides the circle into major and minor segments. Name two angles in:
  i the minor segment
  ii the major segment



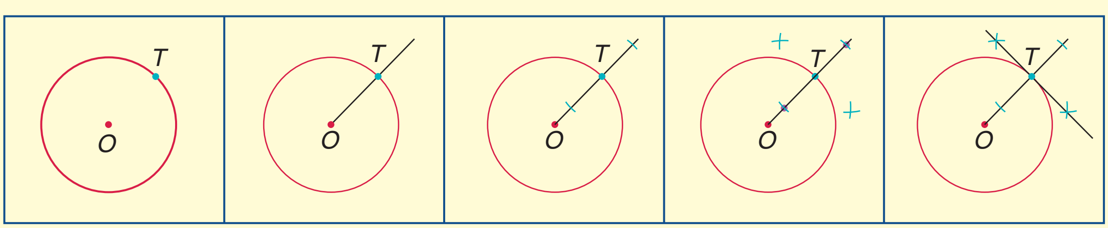
5 The diagram shows a number of parallel lines and a circle. The tangent is coloured green. *T* is the point of contact (where the tangent and the circle meet) and *O* is the centre of the circle.
a Use the diagram to complete the following:
The angle between a tangent and the radius drawn to the point of contact is . . .
b Draw a circle and mark a point, *X*, outside the circle. Use a ruler to draw two tangents to the circle through *X*. Join the points of contact to the centre and measure the angle between the tangent and the radii. Are they right angles?



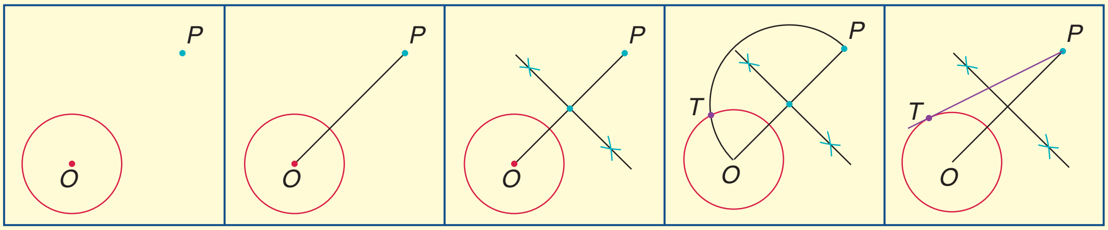


6 Use a ruler and compasses to perform the following constructions involving tangents.

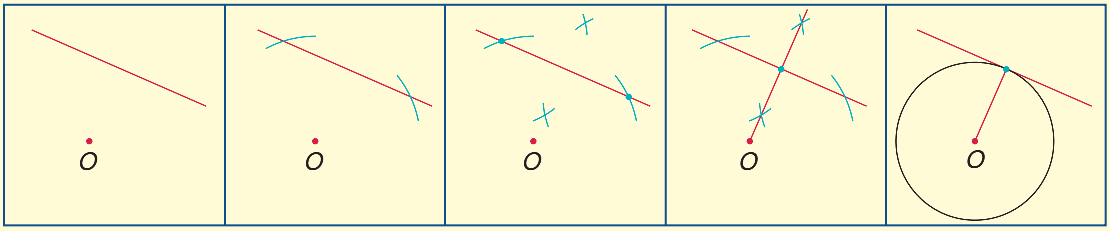
a Construct a tangent to a circle at a given point on the circle.



b Construct the tangent to a given circle from an external point.



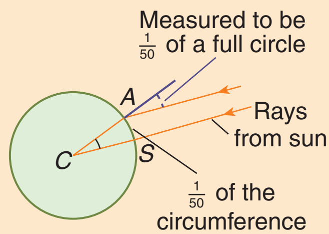
c Construct a circle given its centre and a tangent.



Reading Mathematics 13:01 | Circles in space

Historical note

The properties of circles have been used to discover measurements related to the earth and the moon.



- *Eratosthenes*, in 200 BC, noticed that the sun was directly overhead at a certain time in Syene in Egypt. At exactly the same time in Alexandria, which was due north of Syene, the sun was about $\frac{1}{50}$ of a circle south of the zenith (ie $\frac{1}{50}$ of a full circle away from being directly overhead). He reasoned therefore that the distance between the cities must be about $\frac{1}{50}$ of the earth's circumference.
- *Aristarchus* devised an ingenious method for determining the ratio between the distance to the moon and the radius of the earth using the average duration of a lunar eclipse and the length of the month. A simple property of circles was used in his derivation, namely that the arc of a circle subtended by an angle at its centre is proportional to the radius of the circle. That is, if you double the radius, you double the arc length.

  Aristarchus' method, involving simple triangle and circle geometry, resulted in an estimate for the moon's distance from earth of 80 earth radii. The actual distance is in fact about 60 earth radii.

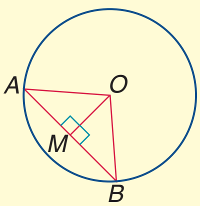
Modern measurements of astronomical distances

Measure	*Measurement (km)*
Radius of the sun	695 000
Radius of the earth	6 378
Radius of the moon	1 738
Earth to the sun (centre to centre, average distance)	149 594 000
Earth to the moon (centre to centre, average distance)	384 393

13:02 | Chord Properties of Circles (1)

A perpendicular drawn to a chord from the centre of a circle bisects the chord, and the perpendicular bisector of a chord passes through the centre.

We can use congruent triangles to prove this result.



Prove that $AM = MB$, if O is the centre and OM is perpendicular to the chord AB.

Construction: Draw in OA and OB.

Proof: In the Δs OAM and OBM

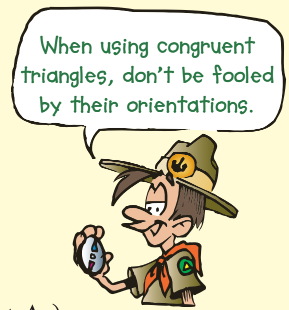
1 $\angle AMO = \angle BMO = 90°$ ($OM \perp AB$)

2 $AO = BO$ (radii of the circle)

3 OM is common.

$\therefore\ \Delta OAM \equiv \Delta OBM$ (RHS)

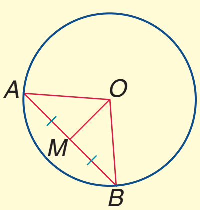
$\therefore\ AM = MB$ (corresponding sides of congruent Δs)



■ '$\perp$' means 'is perpendicular to'.

$\therefore$ The perpendicular bisects the chord. Q.E.D. ('which was to be demonstrated'). Now, since there is only one perpendicular bisector of a chord, it must be the one here that passes through the centre.

The line from the centre of a circle to the midpoint of the chord meets the chord at right angles.



We can use congruent triangles to prove this result.

Given: O is the centre of the circle and M is the midpoint of the chord AB.

Aim: To prove that OM is perpendicular to AB.

Construction: Draw OA and OB.

Proof: In the Δs OAM and OBM

1 $AM = MB$ (M is the midpoint of AB)

2 $AO = BO$ (radii of the circle)

3 OM is common

$\therefore\ \Delta OAM \equiv \Delta OBM$ (SSS)

$\therefore\ \angle OMA = \angle OMB$ (corresponding angles of congruent Δs)

But $\angle OMA + \angle OMB = 180°$ (adjacent angles on a straight line)

$\therefore\ \angle OMA = 90°$

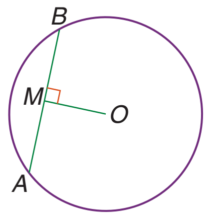
$\therefore\ OM \perp AB$ Q.E.D.

■ *Note:* Unless otherwise stated, O will be the centre of the circle.

■ Q.E.D. *quod erat demonstrandum* 'which was to be demonstrated'

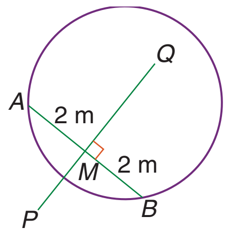
worked examples

1



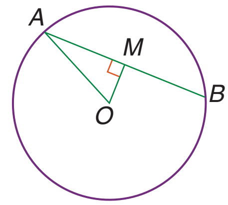
$AB = 15$ cm. Find the length of MB, giving reasons.

2



Give reasons why PQ must pass through the centre of the circle.

3



$AO = 26$ cm, $OM = 10$ cm. Find the length of AB, giving reasons.

Solutions

1 $MB = \frac{1}{2}$ of AB. (The perpendicular from O bisects chord AB.)

$\therefore MB = 7{\cdot}5$ cm

2 PQ passes through the centre because it is the perpendicular bisector of chord AB.

3 $OA^2 = AM^2 + OM^2$ (Pythagoras)

$26^2 = AM^2 + 10^2$

$\therefore AM^2 = 576$

$\therefore AM = \sqrt{576}$

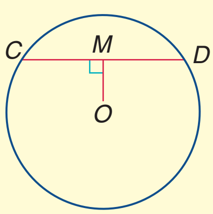
$= 24$ cm

Now $AB = 2 \times AM$ (as OM is the perpendicular bisector of AB)

$\therefore AB = 48$ cm

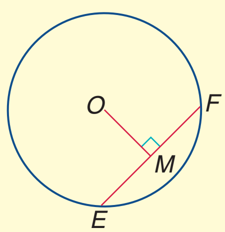
Exercise 13:02

1 **a**



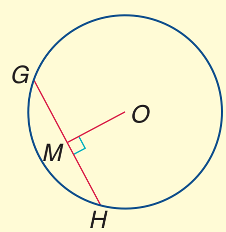
$CD = 35$ cm. Find the length of MD, giving reasons.

b



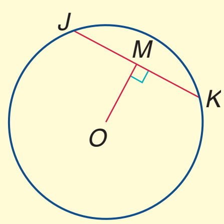
$EM = 27$ cm. Find the length of MF, giving reasons.

c



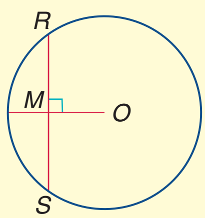
$GM = 18$ m. Find the length of GH, giving reasons.

d



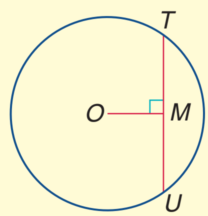
$JM = 9$ cm. Find the length of MK and JK.

e



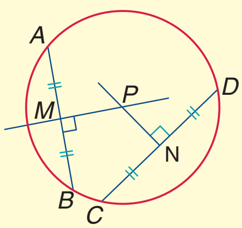
$MS = MR$. Find the size of $\angle RMO$.

f



$TU = 18{\cdot}6$ cm. Find the length of TM and MU.

2 **a**

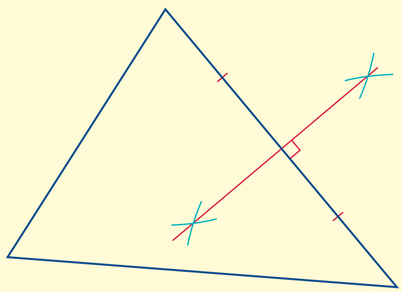


i Give reasons why *MP* passes through the centre of the circle.
ii Give reasons why *NP* passes through the centre of the circle.
iii Which point is the centre here? Why?

b Draw any circle. Use the method in **2a** to find the centre of your circle.

c The same method can be used to draw a circle that passes through any three non-collinear points. Choose any three non-collinear points, and by constructing two perpendicular bisectors, locate the centre and then draw the circle that passes through these points.

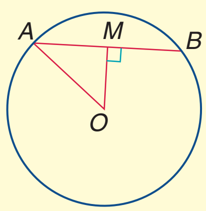
3



Draw any triangle and then construct the perpendicular bisectors of each side. These lines should be concurrent. The point of intersection is called the **circumcentre**. A circle (the circumcircle) can be drawn, with this point as its centre, that will pass through the three vertices of the triangle.

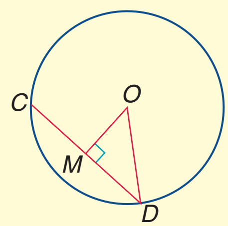
4 Use Pythagoras' theorem in the following. (Answer correct to 1 dec. pl.)

a



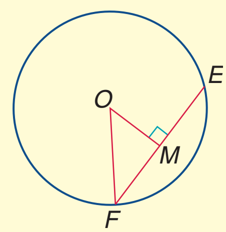
$OM \perp AB$. $AO = 5$ cm, $OM = 4$ cm. Find the length of AM and AB.

b



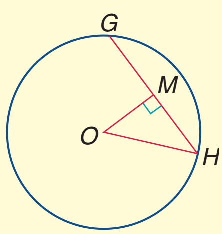
$OM \perp CD$. $MO = 6$ m, $OD = 8$ m. Find the length of MD and CD.

c



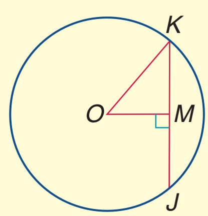
$OM \perp EF$. $OM = 12$ mm, $OF = 13$ mm. Find the length of FM and FE.

d



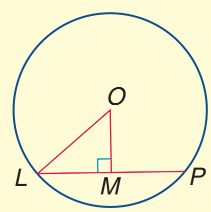
$OM \perp GH$. $MH = 8$ cm, $OM = 6$ cm. Find the length of OH and GH.

e



$OM \perp JK$. $KJ = 14$ cm, $OM = 3$ m. Find the length of KM and OK.

f



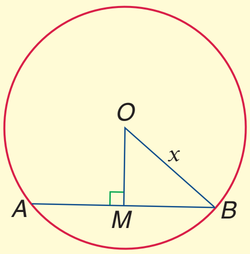
$OM \perp LP$. $OL = 10$ m, $LP = 18$ mm. Find the length of LM and OM.

g A chord of length 12 cm is drawn on a circle of radius 8 cm. How far is this chord from the centre of the circle?

h A chord of length 10 cm has a perpendicular distance of 4 cm from the centre of the circle. What is the radius of the circle?

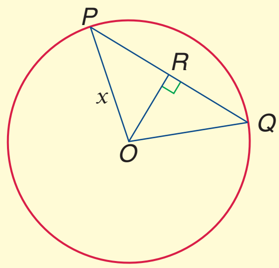
5 **a** Use trigonometry to find x in each diagram (correct to1 dec. pl.).

i



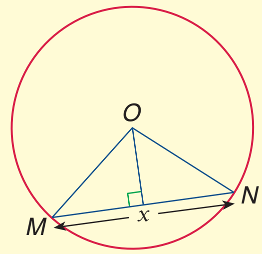
$\angle MOB = 50°$
$AB = 10$ cm

ii



$\angle POQ = 140°$
$PQ = 12{\cdot}6$ cm

iii

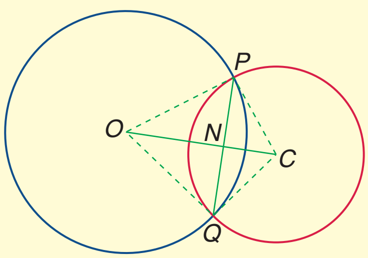


$\angle MON = 126°$
$OM = 7{\cdot}3$ cm

b Find the radius of a circle in which a chord of length 14 cm subtends an angle of 70° at the centre. (Give the answer correct to one decimal place.)

c A chord subtends an angle of 110° at the centre of a circle of radius 5·6 cm. Find the length of the chord correct to one decimal place.

6



These two circles have as their centres points O and C. PQ is the common chord joining the points of intersection of the two circles. N is the point where PQ intersects the line OC which joins the centres.

a Prove that the triangles POC and QOC are congruent.

b Hence, show that $\angle POC = \angle QOC$.

c Now, prove that the triangles PON and QON are congruent.

d Hence, show that N bisects PQ and that $PQ \perp OC$.

If you have completed question **6**, you have proved the following result.

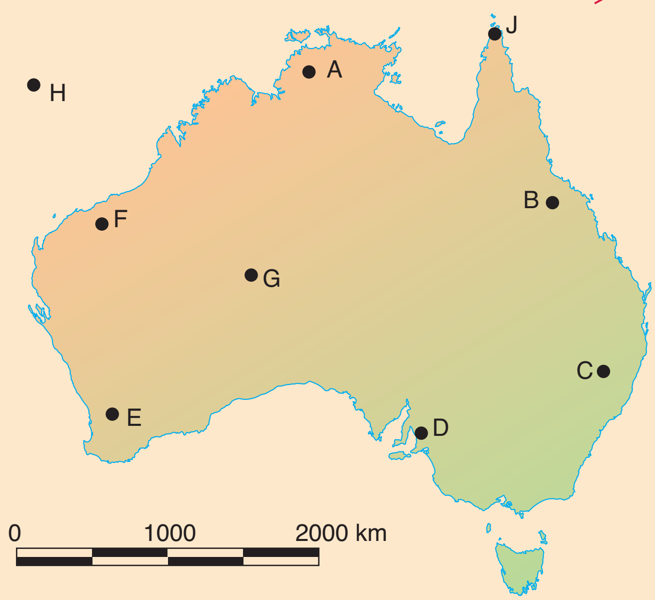
When two circles intersect, the line joining their centres bisects their common chord at right angles.

investigation 13:02

Investigation 13:02 | Locating the epicentre of earthquakes

The epicentre of an earthquake can be located using three intersecting circles.

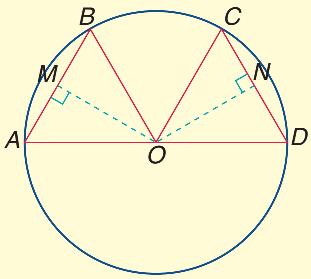
- At seismograph stations, the distance to the epicentre of earthquakes is calculated by examining the graphs of waves detected.
- By drawing circles with radii equal to the distances to the epicentre from three different stations, it is possible to determine the position of the epicentre.
- The point of intersection of the three circles is the epicentre.



- Use the scale of the map and the distances from stations given below, to find the point that is the epicentre of each earthquake.
  1. The epicentre is 1770 km from A, 1140 km from C and 3140 km from E.
  2. The epicentre is 1450 km from D, 1020 km from F and 1410 km from A.
  3. The epicentre is 1750 km from A, 4070 km from C and 2160 km from E.
  4. The epicentre is 2640 km from D, 2290 km from C and 2770 km from F.
- What is the Richter scale? Find out what you can about the scale and its use.

13:03 | Chord Properties of Circles (2)

Equal chords of a circle are the same distance from the centre and subtend equal angles at the centre.



Once again we use congruent triangles to prove these properties.

Data: *AB* and *CD* are equal chords of the same circle.

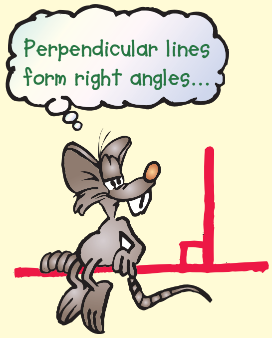
Aim: To prove that equal chords subtend equal angles at the centre of the circle and that these chords are the same distance from the centre.
(ie $\angle AOB = \angle COD$ and $OM = ON$)

> The altitude (height) of a triangle is perpendicular to the base. Each triangle has three altitudes.

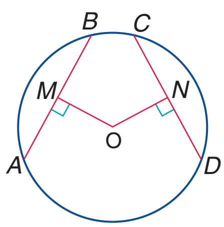
Proof: In the Δs *ABO* and *CDO*:

1 $AB = CD$ (given)
2 $OA = OC$ (radii of the circle)
3 $OB = OD$ (radii of the circle)

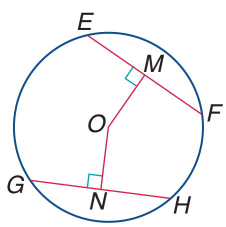
$\therefore \Delta ABO \equiv \Delta CDO$ (SSS)
$\therefore \angle AOB = \angle COD$ (corresponding angles of congruent Δs)
$\therefore$ Equal chords subtend equal angles at the centre. Q.E.D.
Now, since Δs *ABO* and *CDO* are congruent (the same shape and size) the height of each triangle (dotted line on the figure) must be the same.
ie $OM = ON$.
$\therefore$ The chords are the same distance from the centre. Q.E.D.



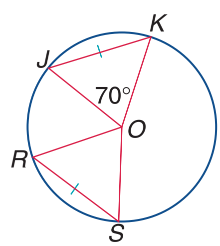
worked examples

1


$AB = CD$, $OM = 6$ cm.
Find the length of *ON*, giving reasons.

2


$OM = ON$, $EF = 13$ m.
Find the length of *GH*, giving reasons.

3


$JK = RS$, $\angle JOK = 70°$.
Find the size of $\angle ROS$, giving reasons.

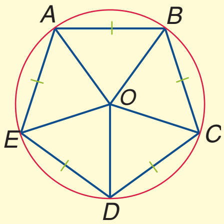
Solutions

1 $ON = 6$ cm, as equal chords of a circle are the same distance from the centre.

2 $GH = 13$ m, as chords that are equidistant from the centre are equal in length.

3 $\angle ROS = 70°$, as equal chords subtend equal angles at the centre of the circle.

To construct regular figures within a circle, construct the required number of equal sides by measuring equal angles at the centre of the circle.



The regular pentagon *ABCDE* has been drawn within the circle.

As the angles at the centre are subtended by equal chords, the five angles are equal.
∴ $5 \times$ angle size = 360° (angles at point)
∴ angle size = 72°

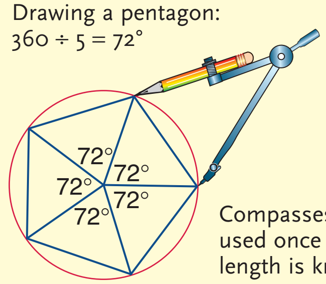
'are subtended by' means 'stand on'.

A regular figure has sides of equal length.

Steps for constructing regular figures within a circle

- Divide 360° by the number of sides in the regular figure to find the size of the angle subtended at the centre by each side.
- Use a protractor to draw the angles at the centre and extend the arms until they meet the circle.
- Join these points on the circle to form the regular figure.

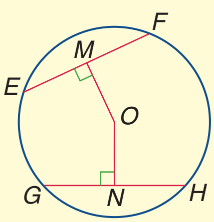
Drawing a pentagon:
$360 \div 5 = 72°$



Compasses can be used once one chord length is known.

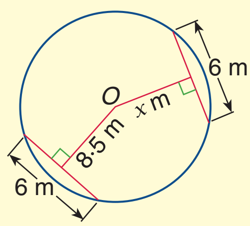
Exercise 13:03

1 **a**



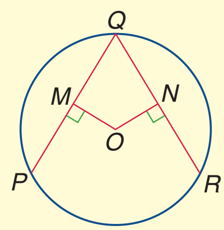
$EF = GH$, $ON = 8$ m.
Find the length of *OM*, giving reasons.

b



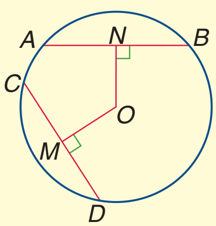
Find the value of *x*, giving reasons.

c



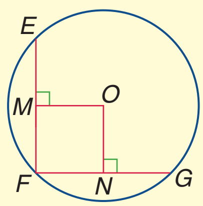
$PQ = QR = 15$ cm,
$OM = 3{\cdot}5$ cm.
Find the length of *ON*, giving reasons.

d



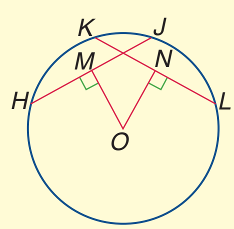
$OM = ON$, $AB = 11{\cdot}5$ m.
Find the length of *CD*, giving reasons.

e



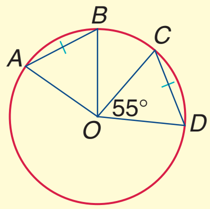
$OM = ON = 12$ m,
$EF = 20$ m.
Find the length of *FG*, giving reasons.

f



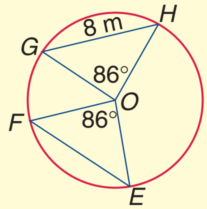
$OM = ON$, $HJ = 13{\cdot}8$ m.
Find the length of *KL*, giving reasons.

2 **a**



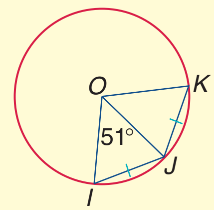
$AB = CD$, $\angle AOB = 55°$. Find the size of $\angle COD$, giving reasons.

b



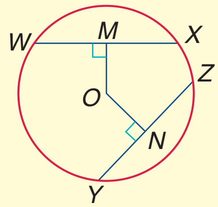
$\angle GOH = \angle FOE$, $GH = 8$ m. Find the length of FE, giving reasons.

c



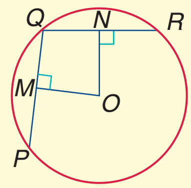
$IJ = JK$, $\angle IOJ = 51°$. Find the size of $\angle JOK$, giving reasons.

d



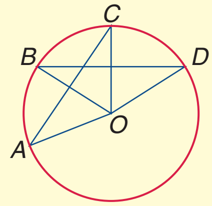
$WM = 6$ m, $YZ = 12$ m, $OM = 5$ m. Find the length of ON, giving reasons.

e



$OM = ON$, $QR = 14$ m. Find the length of QM, giving reasons.

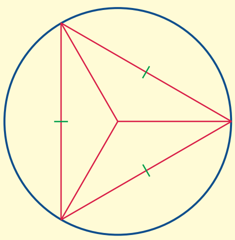
f



$AC = BD$, $\angle AOC = 125°$. Find the size of $\angle BOD$, giving reasons.

3 Find the size of the angle subtended at the centre by one side of each of these regular figures.

a



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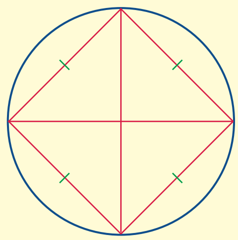

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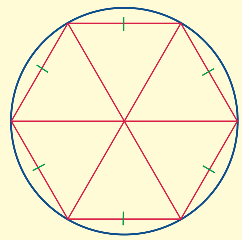
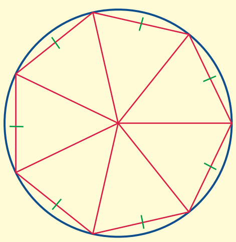
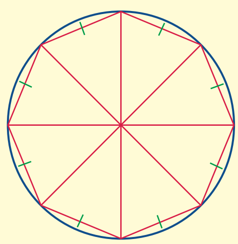


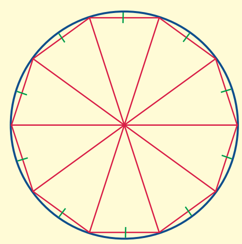
d



e



f

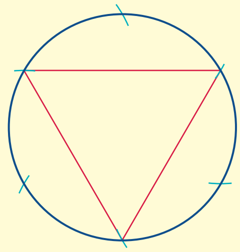


4 In circles of radius 3 cm, construct:

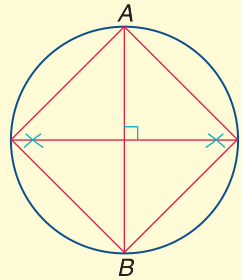
- **a** an equilateral triangle
- **b** a square
- **c** a regular hexagon
- **d** a regular octagon

5 Use ruler and compasses to copy the constructions of the following regular figures. *AB* is a diameter in **b** and **d**.

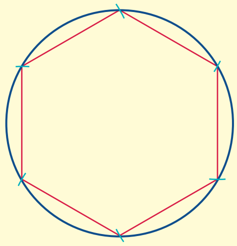
a

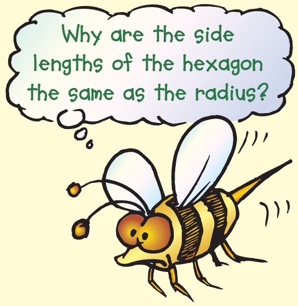


b

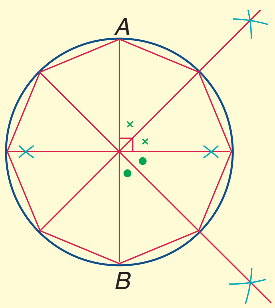


c





d

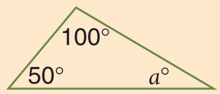


The method of constructing a regular pentagon with ruler and compasses is very complicated and the construction of a regular heptagon (7 sides) and a regular nonagon (9 sides) with rulers and compasses is impossible.

- An arc illuminated on the circumference of the earth.

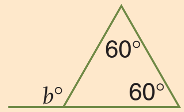
13:04 | Angle Properties of Circles (1)

1



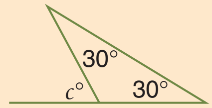
$a = \ldots\ldots$

2



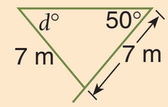
$b = \ldots\ldots$

3



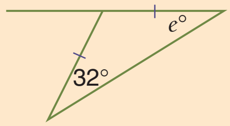
$c = \ldots\ldots$

4



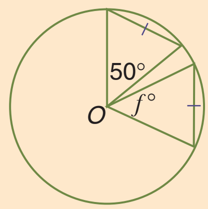
$d = \ldots\ldots$

5



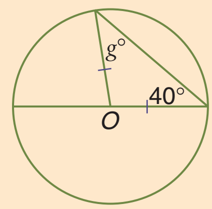
$e = \ldots\ldots$

6



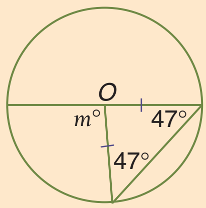
$f = \ldots\ldots$

7



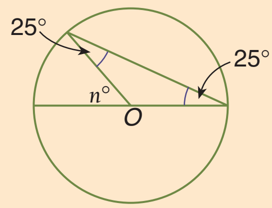
$g = \ldots\ldots$

8



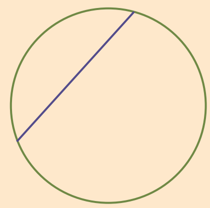
$m = \ldots\ldots$

9



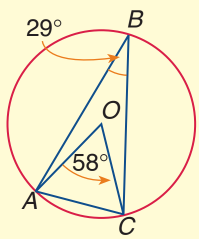
$n = \ldots\ldots$

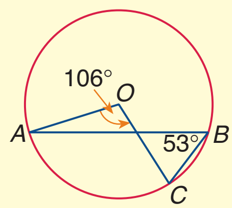
10

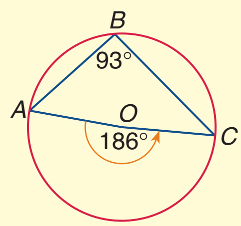


Into how many segments does a chord divide a circle?

The angle subtended by an arc (or chord) at the centre of a circle is double the angle subtended by the arc (or chord) at the circumference.







More than one proof will be required in order to cover all possibilities in establishing the above result. (Refer to Figures 1, 2 and 3 on the next page.)

Data: A, B, C and P are points on the circumference of a circle with centre O. $\angle AOC$ is subtended at centre O by arc APC, $\angle ABC$ is subtended at the circumference by arc APC. In Figure 2, A, O and B are collinear.

Aim: To prove that $\angle AOC = 2 \times \angle ABC$.

Construction: In Figures 1 and 3, join BO and produce to D.

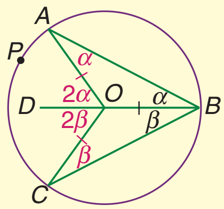


Figure 1

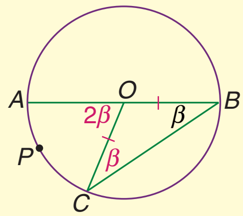


Figure 2

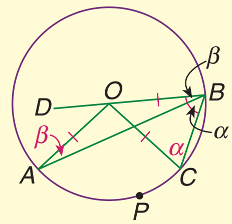


Figure 3

Proof:

In Figure 1

Let $\angle OBA$ be equal to α and $\angle OBC$ be equal to β.

Now, ΔAOB is isosceles ($OA = OB$, radii of circle)

$\therefore \quad \angle OAB = \alpha$ (base angles of isosceles Δ)

$\therefore \quad \angle AOD = \alpha + \alpha$ (exterior of ΔAOB)

$\qquad\qquad = 2\alpha$

Similarly,

$\quad \angle COD = 2\beta$

$\therefore \quad \angle AOC = 2\alpha + 2\beta$

$\qquad\qquad = 2(\alpha + \beta)$

and $\angle ABC = \alpha + \beta$

$\therefore \quad \angle AOC = 2 \times \angle ABC$

In Figure 2

Let $\angle ABC$ be equal to β.

Now, ΔBOC is isosceles ($OB = OC$, radii of circle)

$\therefore \quad \angle OCB = \beta$ (base angle of isosceles Δ)

$\therefore \quad \angle AOC = 2\beta$ (exterior angle of ΔBOC)

$\therefore \quad \angle AOC = 2 \times \angle ABC$

In Figure 3

Let $\angle OBC$ be equal to α and $\angle OBA$ be equal to β.

Now, ΔAOB is isosceles

($OB = OC$, radii of circle)

$\therefore \quad \angle OCB = \alpha$ (base angles of isosceles Δ)

$\therefore \quad \angle DOC = 2\alpha$ (exterior angle of ΔOBC)

Similarly,

$\quad \angle DOA = 2\beta$

$\therefore \quad \angle AOC = \angle DOC - \angle DOA$

$\qquad\qquad = 2\alpha - 2\beta$

$\qquad\qquad = 2(\alpha - \beta)$

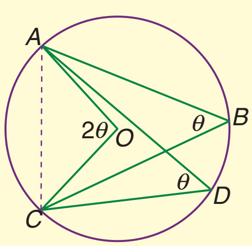
$\quad \angle ABC = \angle OBC - \angle OBA$

$\qquad\qquad = \alpha - \beta$

$\therefore \quad \angle AOC = 2 \times \angle ABC$

$\therefore$ The angle subtended at the centre is twice the angle subtended at the circumference. Q.E.D.

Angles subtended at the circumference by the same or equal arcs (or chords) are equal.



Since the angle subtended at the centre is twice the angle subtended at the circumference,

if $\angle ABC = \theta$

then $\angle AOC = 2\theta$ (angle at centre)

and $\angle ADC = \theta$ (angle at circumference)

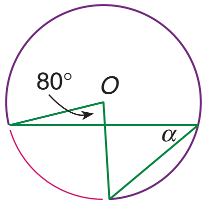
$\therefore \angle ABC = \angle ADC$

ie Angles are subtended at the circumference by the same or equal arcs (or chords) are equal.

worked examples

Find the value of the pronumerals, giving reasons.

1



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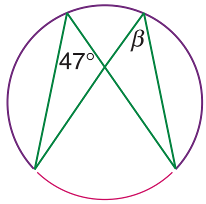

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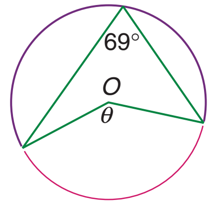


Solutions

1 $\alpha = 40°$ (Angle at the circumference is half the angle at the centre.)

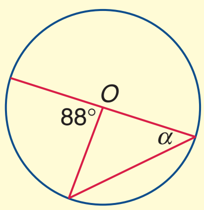
2 $\beta = 47°$ (Angles subtended at the circumference by the same arc.)

3 $\theta = 138°$ (Angle at the centre is twice the angle at the circumference.)

Exercise 13:04

1 Find the value of the pronumerals in each part.

a



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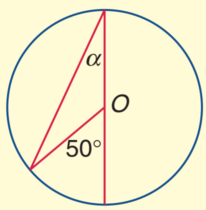

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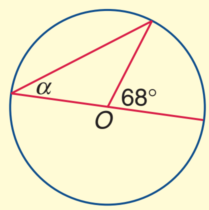

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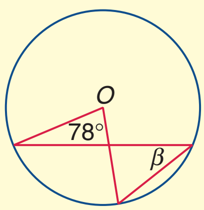

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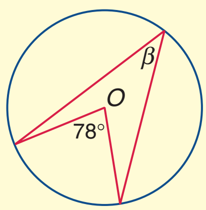

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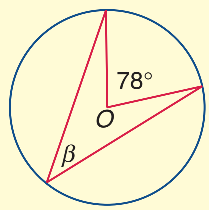
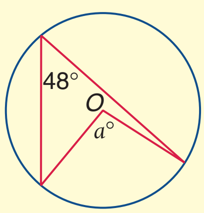
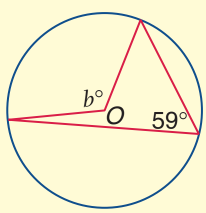


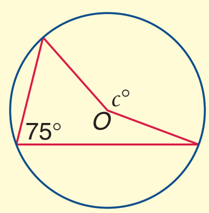
g



h

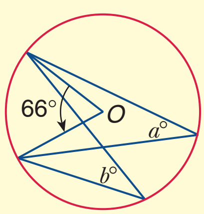


i

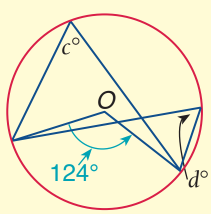


2 Find the value of each pronumeral.

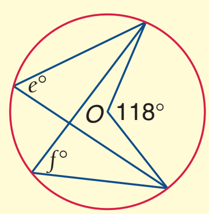
a



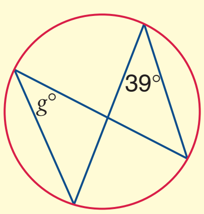
b



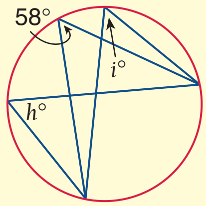
c



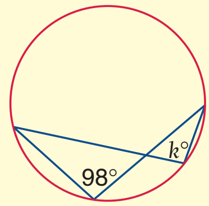
d



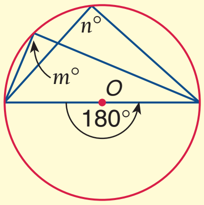
e



f



g



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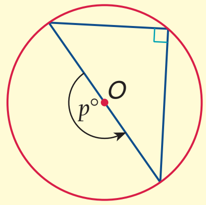
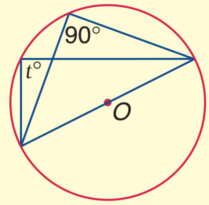
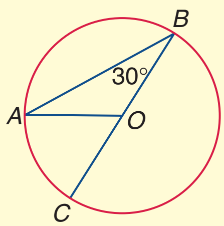
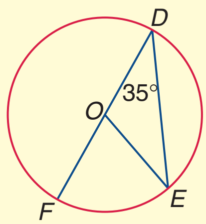


i



3 **a**

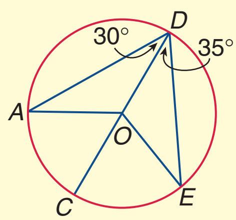




Using the figures to the left, find the size of:

i ∠*BAO** **ii** ∠*AOC*
iii ∠*DEO* **iv** ∠*EOF*
v ∠*ABO* + ∠*ODE*
vi ∠*AOC* + ∠*EOF*

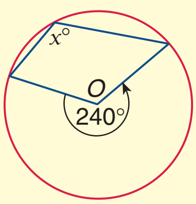
b



Using the figure to the left, find the size of:

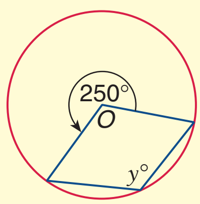
i ∠*DAO* **ii** ∠*AOC*
iii ∠*DEO* **iv** ∠*EOC*
v ∠*ADE* **vi** obtuse ∠*AOE*
vii Is the angle at the centre (∠*AOE*) twice the angle at the circumference (∠*ADE*)?

c



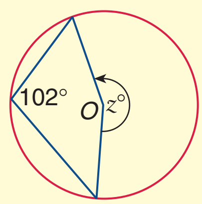
$x = \ldots$

d



$y = \ldots$

e

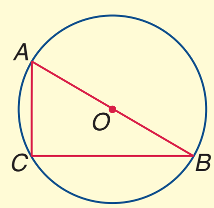


$z = \ldots$

13:05 | Angle Properties of Circles (2)

An angle subtended by a diameter at the circumference of a circle is called the angle in a semicircle.

The angle in a semicircle is a right angle.



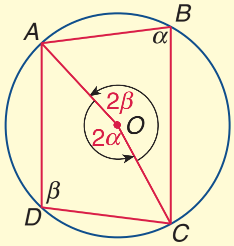
This result is easy to prove, as the diameter makes an angle of 180° at the centre.

Proof: ∠*AOB* = 180° (*AOB* is a straight line)

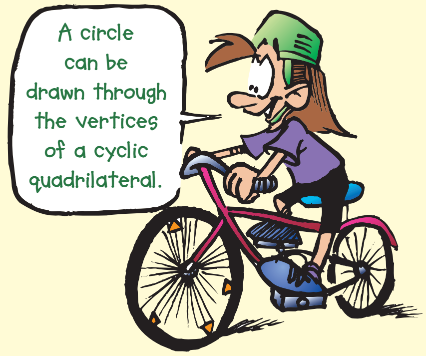
∴ ∠*ACB* = 90° (The angle at the centre is twice the angle at the circumference.)

∴ The angle in a semicircle is a right angle. Q.E.D.

Opposite angles of a cyclic quadrilateral are supplementary. (They add up to 180°.)



Data: *ABCD* is any cyclic quadrilateral.
Aim: To prove that opposite angles add up to 180°.
Construction: Draw in the radii *AO* and *OC*.



Proof: Let $\angle ABC$ be α and $\angle ADC$ be β.

Now, obtuse $\angle AOC = 2\alpha$ (angle at centre is twice $\angle ABC$)

and reflex $\angle AOC = 2\beta$ (angle at centre is twice $\angle ADC$)

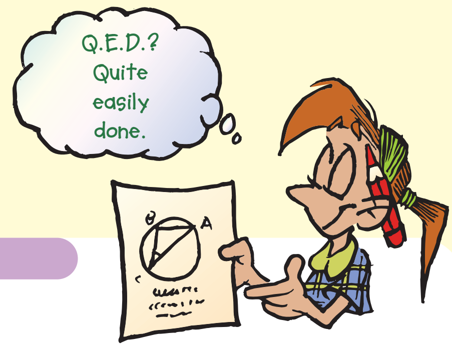
Now, $2\alpha + 2\beta = 360°$ (angles at a point make 1 revolution)

$\therefore\ \alpha + \beta = 180°$

$\therefore\ \angle ABC + \angle ADC = 180°$

Similarly, $\angle BAD + \angle BCD = 180°$

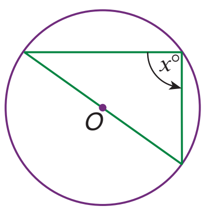
ie Opposite angles of a cyclic quadrilateral are supplementary. Q.E.D.



worked examples

Find the value of the pronumerals, giving reasons.

1



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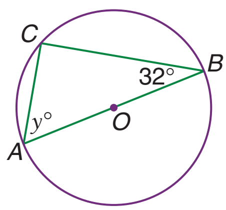

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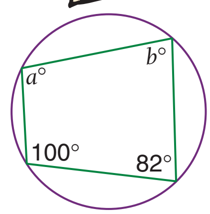


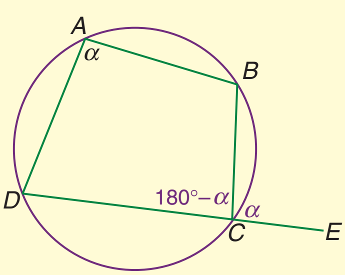
Solutions

1 $x = 90$
(angle in a semicircle)

2 $\angle ACB = 90°$
(angle in a semicircle)
$\therefore\ y = 58$
(angle sum of a Δ)

3 $a + 82 = 180$ $\quad b + 100 = 180$
$\therefore\ a = 98$ $\quad b = 80$
(opposite angles supplementary in a cyclic quadrilateral)

An exterior angle of a cyclic quadrilateral is equal to the interior opposite angle.



Data: *ABCD* is a cyclic quadrilateral, with *DC* produced to *E*, forming the exterior angle *BCE*.
Aim: To prove that $\angle BCE = \angle BAD$.
Proof: Let $\angle BAD = \alpha$

Then, $\angle BCD = 180° - \alpha$ (opp. $\angle$s in cyclic quad.)

Now, $\angle BCD + \angle BCE = 180°$ ($\angle$s on a straight line)

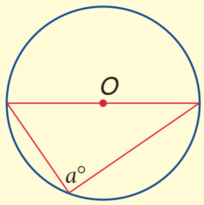
$\therefore\ \angle BCE = \alpha$

ie The exterior angle of a cyclic quadrilateral is equal to the interior opposite angle.

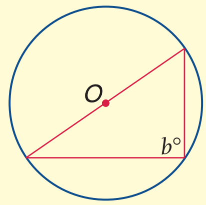
Exercise 13:05

1 Find the value of the pronumeral in each, giving reasons.

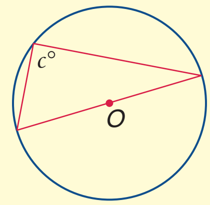
a



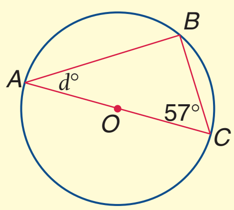
b



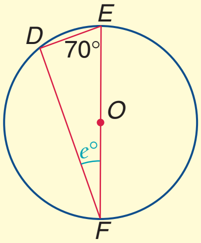
c



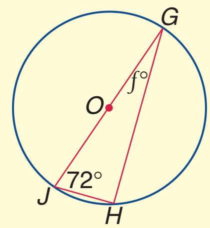
d



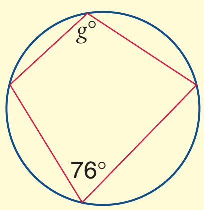
e



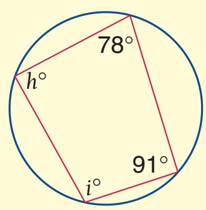
f



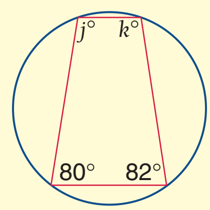
g



h

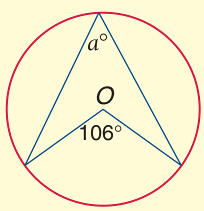


i

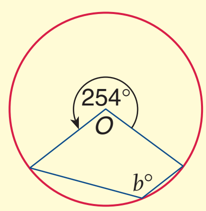


2 Find the value of the pronumerals in each part.

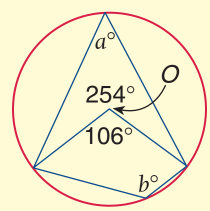
a



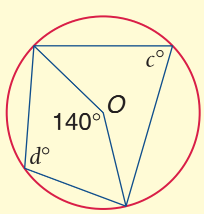
b



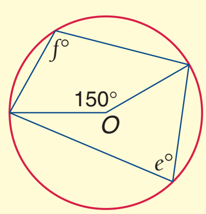
c



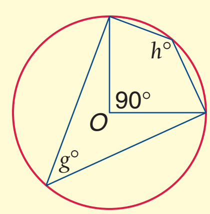
d



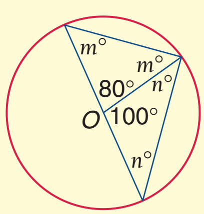
e



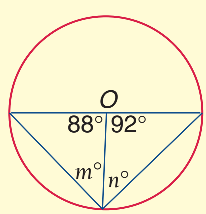
f



g



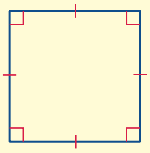
h



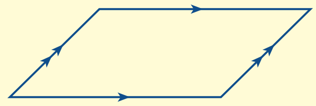
In **g** and **h**, $m + n = 90$.

3 Can each quadrilateral have a circle drawn through its vertices? Explain why it is possible in each such case.

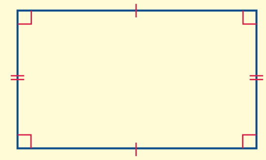
a



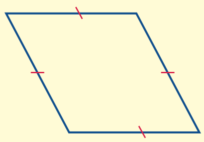
b



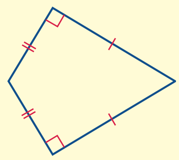
c



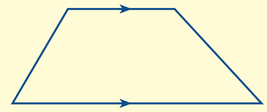
d



e

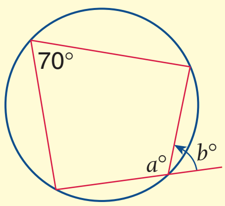


f

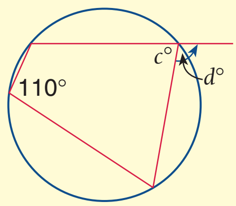


4 Find the value of the pronumeral, giving reasons.

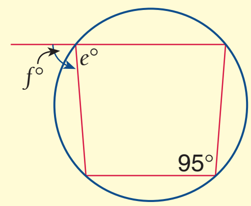
a



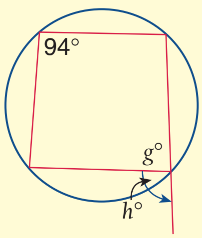
b



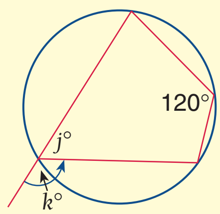
c



d

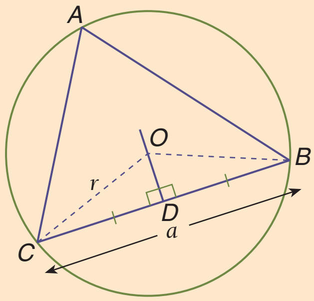


e



The exterior angle is always equal to the interior opposite angle.

Investigation 13:05 | The diameter of a circumcircle

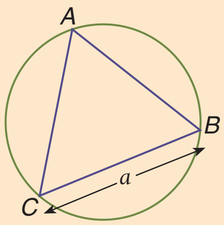


The circumcircle of a triangle is the circle that passes through all three vertices, as in the diagram. But for a particular triangle ABC, can we find a measure for the diameter of its circumcircle?

Follow carefully the steps of this investigation.

- Let $\angle A$ in triangle ABC be $A°$. What is the size of $\angle BOC$ and why?
- The centre O will lie on the perpendicular bisector of BC. What can be said about $\angle DOC$ and $\angle DOB$ and why?
- What is the size of $\angle DOC$?
- Noting that $DC = \frac{1}{2}a$ and letting $OC = r$ (radius), write down an expression for sin ($\angle DOC$).
- Rearrange this expression to make r the subject and hence find an expression for d, the diameter of the circumcircle.

Completing this investigation should lead you to the following result.

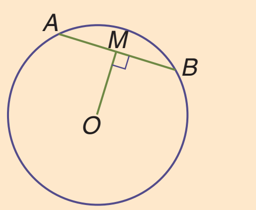


For a triangle ABC, the diameter d of its circumcircle will be given by:

$$d = \frac{a}{\sin A}$$

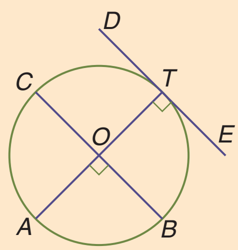
13:06 | Tangent Properties of Circles

1



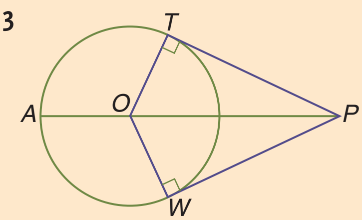
$AB = 12$ m, $MB = \ldots$

2



Name the axis of symmetry.

3

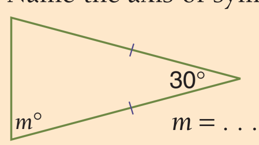


Name the axis of symmetry.

4 Name the tangent in question **2**.

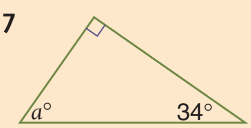
5 In question **2**, what name is given to interval OT?

6



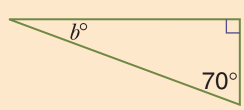
$m = \ldots$

7



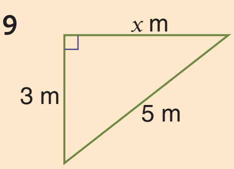
$a = \ldots$

8



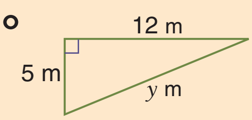
$b = \ldots$

9



$x = \ldots$

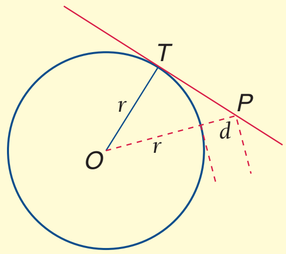
10



$y = \ldots$

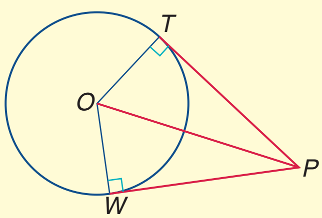
A tangent is a straight line that touches the circle at only one point. This point is called the **point of contact**.

The size of the angle between a tangent and the radius drawn to the point of contact is 90°.



Here, we aim to prove that $OT \perp TP$.
TP is a tangent and T is the point of contact.
Construction: Join O to any point P on the tangent (other than T).
Proof: **The shortest distance from a point to a line is the perpendicular distance.**
Let r be the radius of the circle.
$OT = r$
Since every point other than T is outside the circle,
$PO = r + d$ where d is a positive quantity.
$\therefore OT < OP$
$\therefore OT$ is the shortest distance to the line TP.
$\therefore OT \perp TP$
ie $\angle OTP = 90°$
Q.E.D.

From any external point, two equal tangents may be drawn to a circle. The line joining this point to the centre is an axis of symmetry.

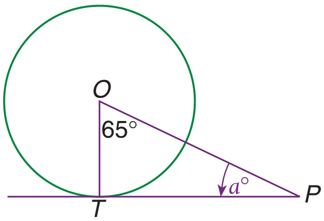


PT and PW are tangents drawn to the circle from the point P.
Our aim is to prove that $PT = PW$ and that OP is an axis of symmetry of the figure.
Construction: Draw in OT and OW, the radii drawn to the points of contact.
Proof: In the Δs OTP and OWP
1 $\angle OTP = \angle OWP = 90°$
(The angle between the tangent and radius is 90°.)
2 OP is common
3 $OT = OW$ (radii of the circle)
$\therefore \Delta OTP \equiv \Delta OWP$ (RHS)
$\therefore PT = PW$ (corresponding sides of congruent Δs)
Also OP is an axis of symmetry as ΔOTP is congruent to ΔOWP.

$\therefore$ The two tangents drawn from an external point are equal and the line joining this point to the centre is an axis of symmetry.   Q.E.D.

worked examples

Find the value of each pronumeral, giving reasons.

1


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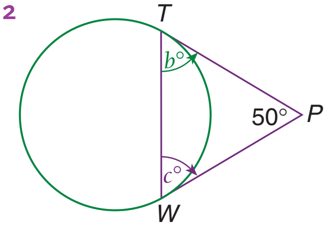

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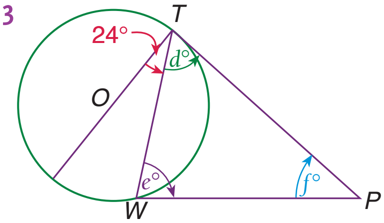


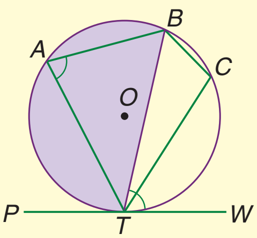
Solutions

1 $\angle OTP = 90°$
(radius $OT \perp$ tangent TP)
$a + 90 + 65 = 180$
$\therefore$ a = 25 (angle sum of a Δ)

2 $PT = PW$ (equal tangents from P)
$\therefore b = c$ (isosceles Δ)
$\therefore c + c + 50 = 180$
(angle sum of a Δ)
$\therefore c = 65$ and $b = 65$

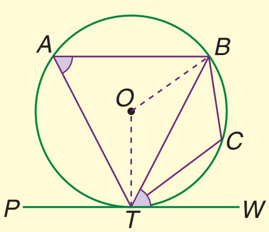
3 $d + 24 = 90$
(radius $OT \perp$ tangent TP)
$\therefore d = 66$
$PT = PW$ (equal tangents)
$\therefore e = 66$ (isosceles ΔPTW)
$\therefore f = 48$ (angle sum of a Δ)

The angle in the alternate segment



- $\angle BTW$ is the acute angle between the tangent PW and the chord BT.
- The shaded segment of the circle is called the alternate segment to $\angle BTW$, while $\angle BAT$ is an angle in the alternate segment.
- $\angle BCT$ is an angle in the alternate segment to $\angle BTP$.

An angle formed by a tangent to a circle with a chord drawn to the point of contact is equal to any angle in the alternate segment.



Data: The chord BT meets the tangent PW at the point of contact, T.
O is the centre of the circle.
$\angle BAT$ is any angle in the segment alternate to $\angle BTW$.
$\angle BCT$ is any angle in the segment alternate to $\angle BTP$.

Aim: To prove **1** that $\angle BTW = \angle BAT$
and **2** that $\angle BTP = \angle BCT$.

Construction: Draw OT and OB.

Proof: **1** Let $\angle BTW$ be $x°$.

Now $\angle OTW = 90°$ (radius $OT \perp$ tangent PW)
$\therefore \angle OTB = 90° - x°$
$\therefore \angle OBT = 90° - x°$ (ΔTOB is isosceles, OT and OB are radii.)
$\therefore \angle TOB = 180° - 2(90° - x°)$ (angle sum of ΔTOB)
$= 2x°$
$\therefore \angle BAT = x°$ (angle at circumference is half the angle at the centre on the same arc)
$\therefore \angle BTW = \angle BAT$ Q.E.D.

2 $\angle BTP = 180° - x°$ ($\angle PTW$ is a straight angle)

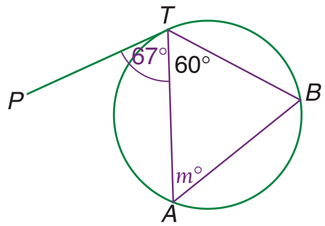
but $\angle BCT = 180° - x°$ (supplementary to $\angle BAT$, as quad. $ABCT$ is cyclic)

$\therefore$ $\angle BTP = \angle BCT$ Q.E.D.

$\therefore$ The angle between the tangent and a chord drawn to the point of contact is equal to any angle in the alternate segment.

worked example

Find the value of m.



Solution

$\angle ABT = 67°$ (angle in the alternate segment)

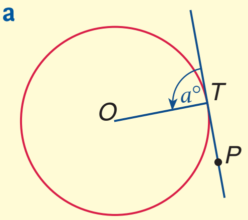
$m + 60 + 67 = 180$ (angle sum of Δ)

$\therefore m = 53$

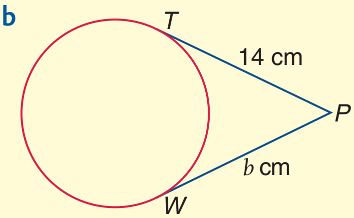
Exercise 13:06

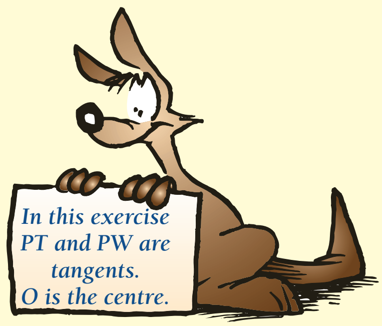
1 Find the value of the pronumerals in each, giving reasons.

a

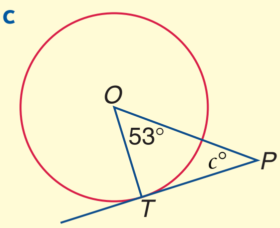


b

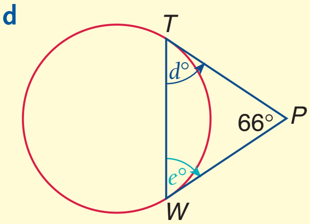




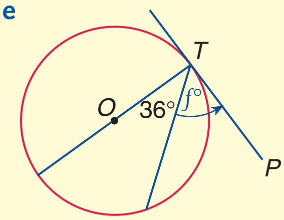
c



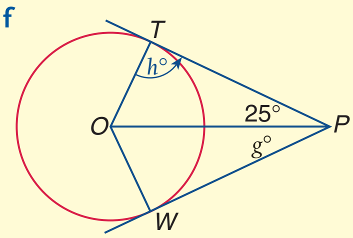
d



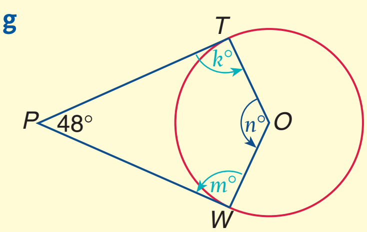
e



f



g



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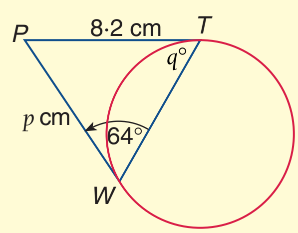
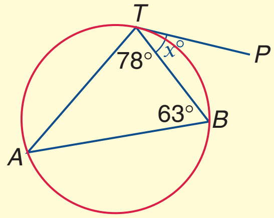


2 Find the value of each pronumeral. *PT* is a tangent in each diagram.

a



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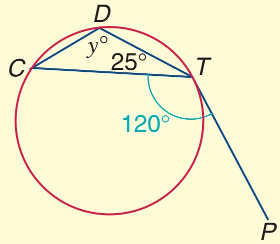

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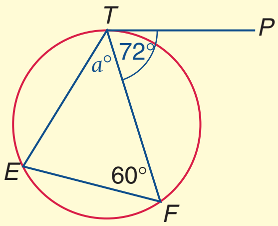

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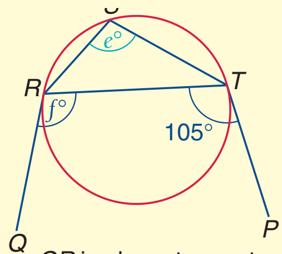
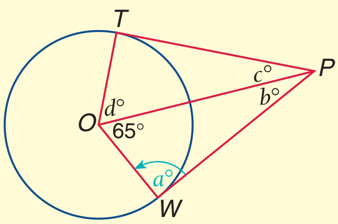


QR is also a tangent.

3 Find the value of the pronumerals in each.

a



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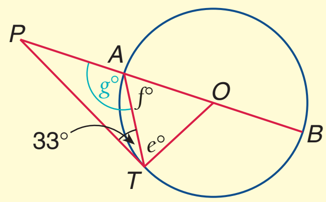

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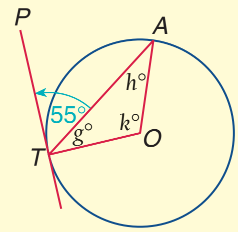

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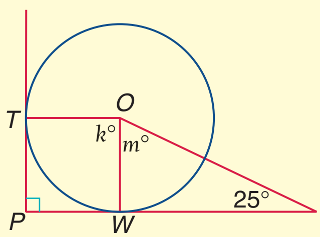

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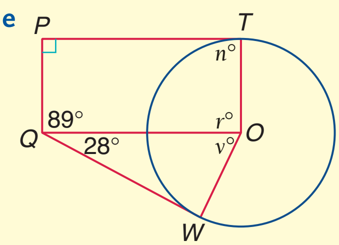

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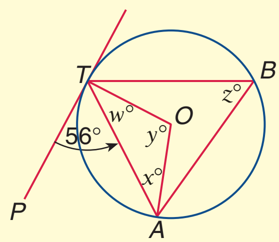

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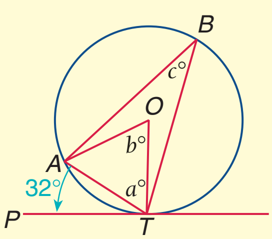

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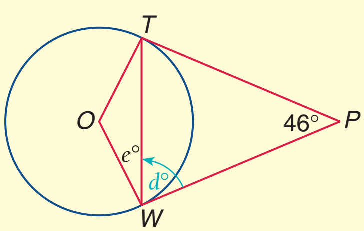
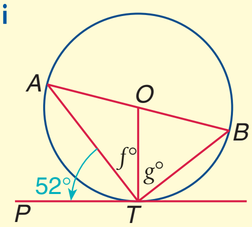
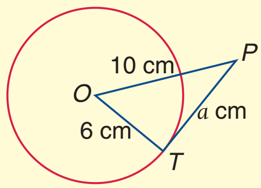


i

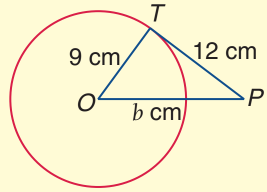


4 Use Pythagoras' theorem to find the value of the pronumerals.

a



b



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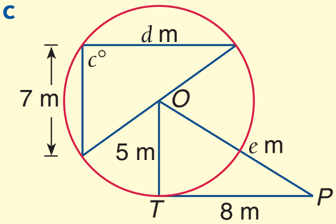

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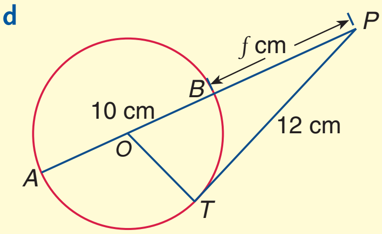
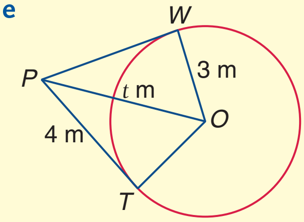
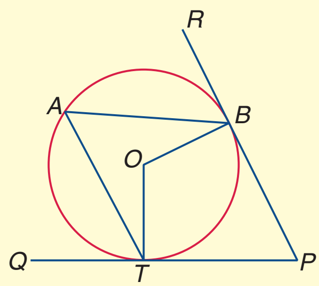


e



5



O is the centre of the circle. *BT* is a chord that subtends $\angle BAT$ at the circumference and $\angle TOB$ at the centre. *PT* and *PB* are tangents to the circle.

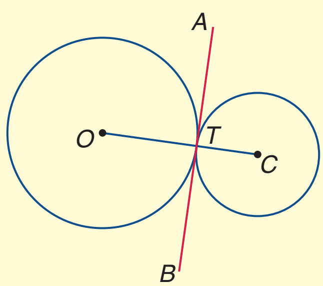
a Prove that $\angle BOT = 2\angle BTP$.

b Prove that $\angle ATQ + \angle RBA + \angle PBT = 180°$.

c Prove that $\angle BPT = 180° - 2\angle BAT$.

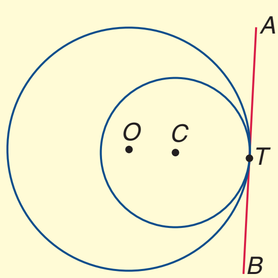
6 From a point *T* on a circle, chords of equal length are drawn to meet the circle at *A* and *B*. Prove that the tangent at *T* is parallel to the chord *AB*.

7 **a**



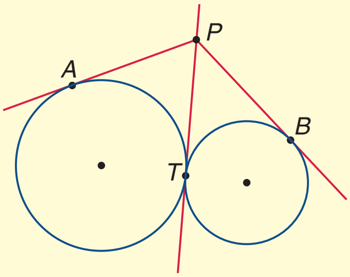
Two circles, with centres at *O* and *C*, touch externally, thus having a common tangent at *T*. Prove that the interval *OC* joining the centres is perpendicular to the common tangent.

b



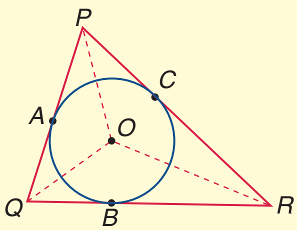
Two circles, with centres at *O* and *C*, touch internally, thus having a common tangent at *T*. Prove that the interval *OC* produced is perpendicular to the common tangent.

8



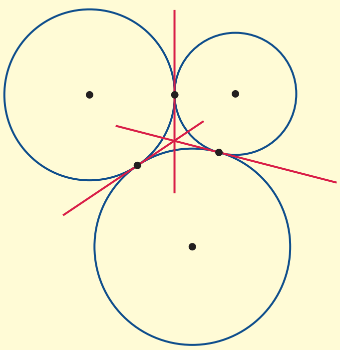
Two circles touch externally, having a common tangent at T. From a point P on this tangent, a second tangent is drawn to each of the circles as shown, touching the circles at A and B, respectively.
Prove that $PA = PB$.

9



Prove that the three angle bisectors of triangle PQR are concurrent and that their point of intersection is the centre of a circle which touches each side of the triangle.
(This is called the **incircle** of the triangle.)

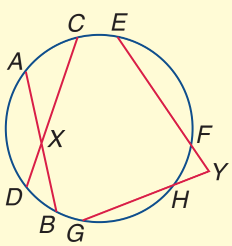
10



Three different sized circular discs touch each other.
Prove that the three common tangents are concurrent.

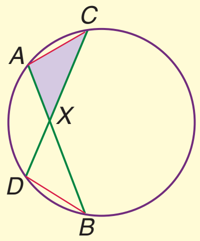
13:07 | Further Circle Properties

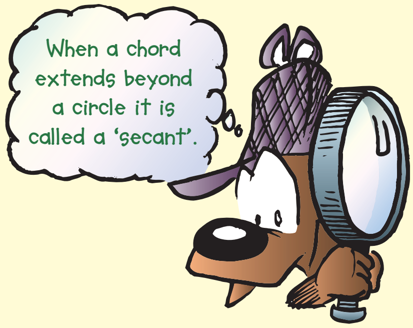
Intersecting chords and secants

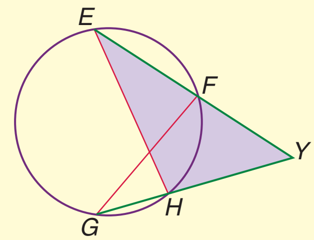


- AB and CD divide each other internally at X.
  AX and XB are called the intercepts of chord AB.
  CX and XD are called the intercepts of chord CD.
- EF and GH divide each other externally at Y.
  EY and YF are called the intercepts of secant EY.
  GY and YH are called the intercepts of secant GY.
- Note:
  (length AX) . (length XB) = (length CX) . (length XD)
  (length EY) . (length YF) = (length GY) . (length YH)

The products of intercepts of intersecting chords or secants are equal.







Data: Chords AB and CD meet at X.
Aim: To prove that $AX \,.\, XB = CX \,.\, XD$.
Construction: Draw AC and DB.
Proof: In Δs ACX and DBX

1	$\angle ACX = \angle DBX$	(angles subtended by the same arc AD)
2	$\angle AXC = \angle DXB$	(vert. opp. angles)
$\therefore$	$\Delta ACX \;\vert\vert\vert\; \Delta DBX$	(equiangular)
$\therefore$	$\dfrac{AX}{XD} = \dfrac{CX}{XB}$	(corresponding sides are in same ratio)

$\therefore$ $AX \,.\, XB = CX \,.\, XD$ (Q.E.D.)

Data: The secants EY and GY cut the circle at F and H, respectively.
Aim: To prove that $EY \,.\, YF = GY \,.\, YH$.
Construction: Draw FG and EH.
Proof: In Δs EYH and GYF

1	$\angle EYH = \angle GYF$	(same angle)
2	$\angle YEH = \angle YGF$	(angles subtended by the same arc FH)
$\therefore$	$\Delta EYH \;\vert\vert\vert\; \Delta GYF$	(equiangular)
$\therefore$	$\dfrac{EY}{GY} = \dfrac{YH}{YF}$	(corresponding sides are in same ratio)

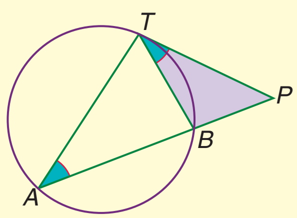
$\therefore$ $EY \,.\, YF = GY \,.\, YH$ (Q.E.D.)

$\therefore$ The products of intercepts of intersecting chords or secants are equal.

A dot can be used to show multiplication.

The square of the length of a tangent is equal to the product of the intercepts of a secant drawn from an external point.

ie $(PT)^2 = AP \,.\, PB$



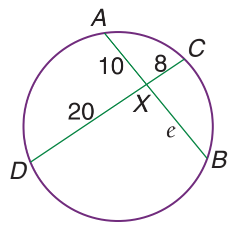
Data: PT is a tangent to the circle. PA is a secant that cuts the circle at A and B. (The chord AB is divided externally at P.)
Aim: To prove that $(PT)^2 = AP \,.\, PB$.
Construction: Draw AT and BT.
Proof: In Δs PTB and PAT

1	$\angle PTB = \angle PAT$	(angle in alternate segment)
2	$\angle TPB = \angle APT$	(same angle)
$\therefore$	$\Delta PTB \;\vert\vert\vert\; \Delta PAT$	(equiangular)
$\therefore$	$\dfrac{PT}{AP} = \dfrac{PB}{PT}$	(corresponding sides of similar Δs are proportional)
$\therefore$	$(PT)^2 = AP \,.\, PB$	(Q.E.D.)

worked examples

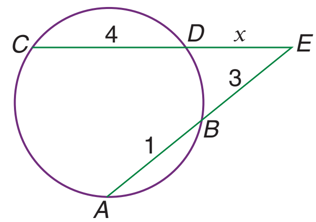
Find the value of each pronumeral, giving reasons.

1



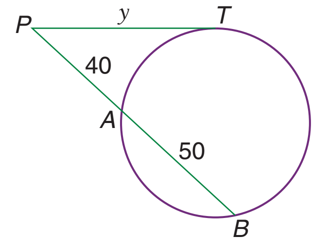
$AX = 10$ mm, $CX = 8$ mm, $DX = 20$ mm, $XB = e$ mm.

2



$CD = 4$ m, $AB = 1$ m $BE = 3$ m, $DE = x$ m.

3



$PT = y$ cm, $AP = 40$ cm $AB = 50$ cm.

Solutions

1 $AX \,.\, XB = CX \,.\, XD$ (products of intercepts of intersecting chords)

$10e = 8 \times 20$

$\therefore\ e = 16$

3 $(PT)^2 = BP \,.\, PA$ (square of the tangent equals the product of the intercepts)

$\therefore\ y^2 = (50 + 40) \,.\, 40$

$y^2 = 3600$

$\therefore\ y = \pm\sqrt{3600}$

$= \pm 60$

Since a length must be positive,

$y = 60$

2 $CE \,.\, ED = AE \,.\, EB$ (products of intercepts of intersecting secants)

$\therefore\ (4 + x) \,.\, x = (1 + 3) \,.\, 3$

$4x + x^2 = 4 \times 3$

$x^2 + 4x - 12 = 0$

$(x + 6)(x - 2) = 0$

$\therefore\ x = -6$ or 2

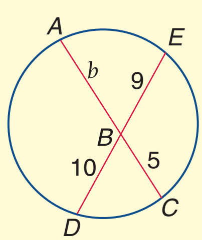
Since a length must be positive,

$x = 2$

Exercise 13:07

1 Find the value of each pronumeral. All lengths are in centimetres and PT is a tangent wherever it is used.

a



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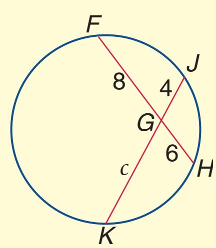

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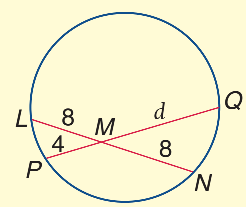

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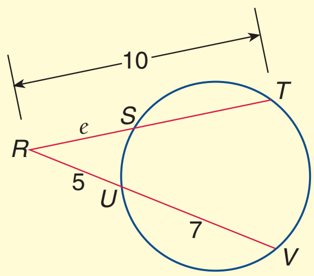

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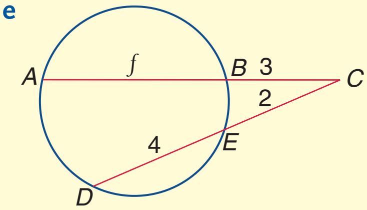

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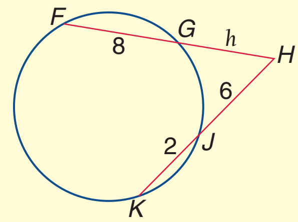

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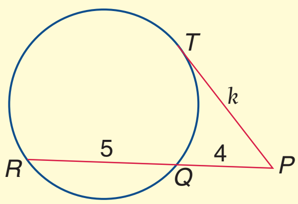

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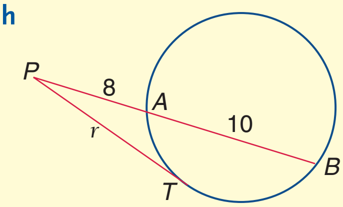
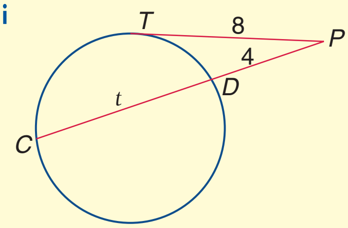
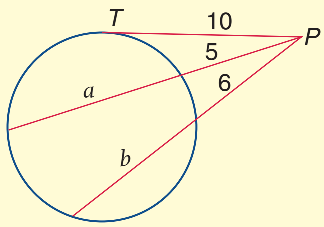


i



j



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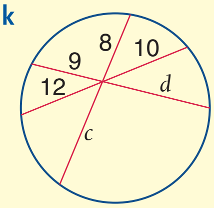
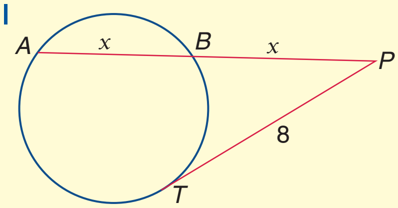
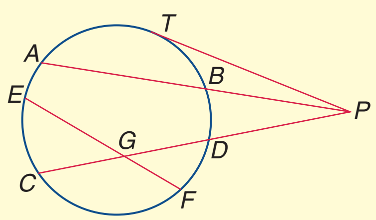


l



2



PT is a tangent

a Find *PT* if *AB* = 9 m and *BP* = 3 m.

b Find *AB* if *BP* = 10 cm and *PT* = 13 cm.

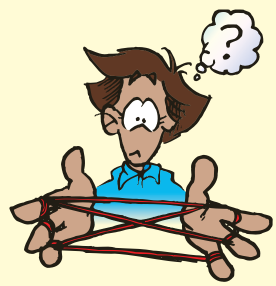
c Find *CD* if *DP* = 5 m, *AB* = 8 m, *BP* = 6 m.

d Find *EG* if *GF* = 20 m, *CG* = 30 m, *GD* = 25 m.

e Find *CD* if *CG* = 15 m, *EF* = 35 m, *EG* = 22 m.

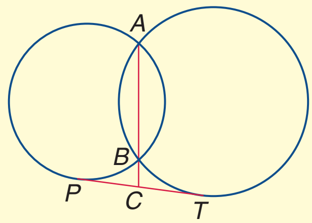
f Find *CD* if *TP* = 9 cm and *DP* = 5 cm.

g Find *PT* correct to one decimal place if *CD* = 8 cm and *DP* = 10 cm.



3 *AB* is the diameter of a circle. *AB* bisects a chord *CD* at the point *E*. Find the length of *CE* if *AE* = 3 m and *BE* = 9 m.

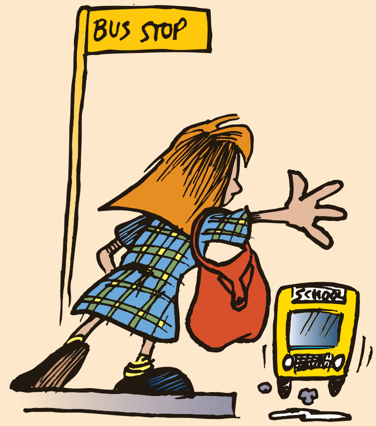
4



PT is a direct common tangent of the circles drawn. *AB* is a common chord that has been produced to meet the common tangent at *C*. Use the 'square of the tangent' result to prove that *CP* = *CT*.

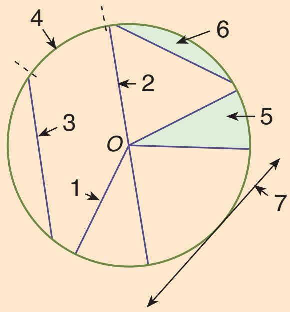
Fun Spot 13:07 | How do you make a bus stop?

Answer each question and put the letter for that question in the box above the correct answer.

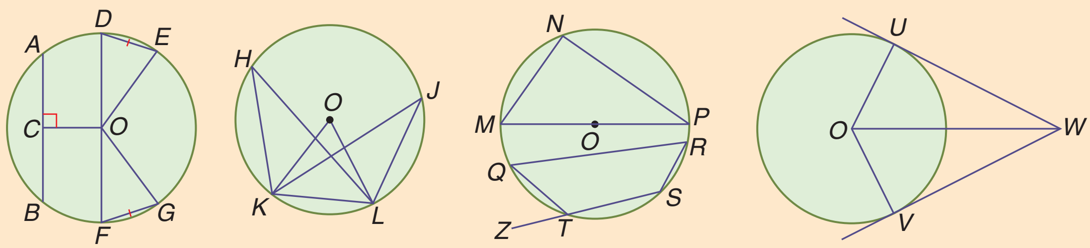


From the circle below, give the number of the:

A sector
B radius
D tangent
E chord
A diameter
C arc
E segment



In all these circles, O is the centre.



From the figures above, what is equal to:

G *AC*?
I ∠*UOW*?
O $\frac{1}{2}$∠*MOP*?
T What is the size of ∠*OUW*?
H ∠*DOE*?
N ∠*VWO*?
S 90° − ∠*NMP*?
H ∠*KHL*?
N *UW*?
S ∠*QTZ*?
I 2(∠*KJL*)?
N 180° − ∠*RQT*?

If ∠KHL = 40°, what is the size of:

T ∠*KJL*?
W ∠*KOL*?
W ∠*OKL*?

50°	∠*KOL*	40°	∠*KJL*	90°	80°	∠*MNP*	1	3	*VW*	4	∠*FOG*	6	∠*NPM*

5	∠*RST*	7	2	∠*QRS*	∠*VOW*	*CB*	∠*UWO*

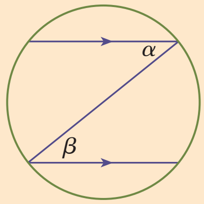
13:08 | Deductive Exercises Involving the Circle

Use ID Card 6 on page xxi to identify number:

1 13 **2** 14 **3** 15 **4** 16 **5** 18 **6** 19 **7** 20

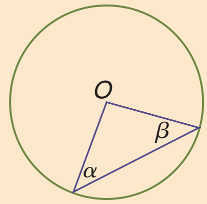
Give reasons why the fact under each figure is true.

8



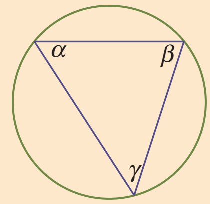
$\alpha = \beta$

9



$\alpha = \beta$

10



$\alpha + \beta + \gamma = 180°$

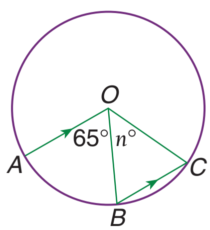
prep quiz 13:08

In this section, reasons must be given for any claim made in numerical exercises.

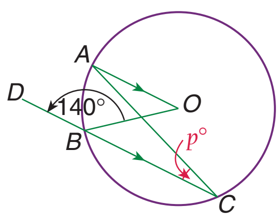
worked examples

Find the value of each pronumeral, giving reasons.

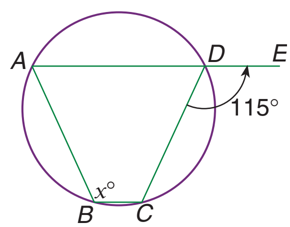
1



2



3



Solutions

1 $\angle OBC = 65°$ (alternate to $\angle AOB$, $AO \parallel BC$)
$\angle OCB = 65°$ (ΔOBC is isosceles)
$n + 130 = 180$ (angle sum of ΔOBC)
$\therefore\ n = 50$

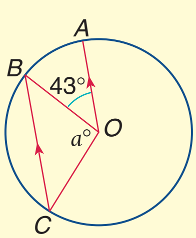
2 $\angle AOB = 40°$ (cointerior to $\angle DBO$, $AO \parallel DC$)
$\angle ACB = 20°$ (half the angle at the centre)
$\therefore\ p = 20$

3 $\angle ADC = 65°$ ($\angle ADE$ is a straight angle)
$\angle ABC = 115°$ (opposite angles of a cyclic quadrilateral are supplementary)
$\therefore\ x = 115$

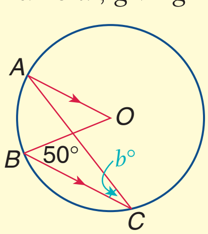
Exercise 13:08

1 Find the value of each pronumeral, giving reasons.

a

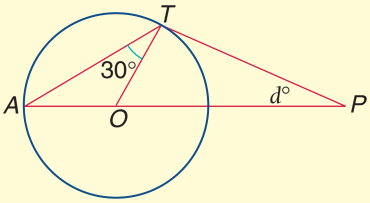


b

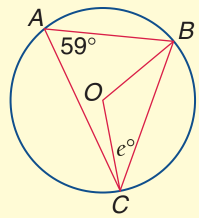


O is the centre. PT and PW are tangents.

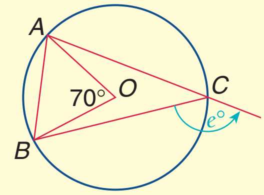
c



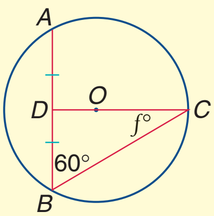
d



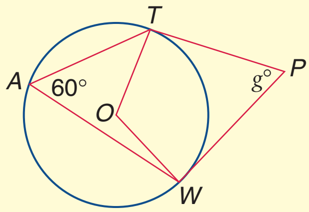
e



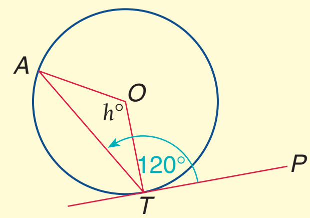
f



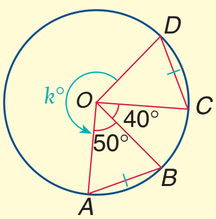
g



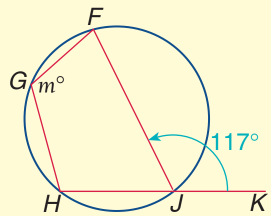
h



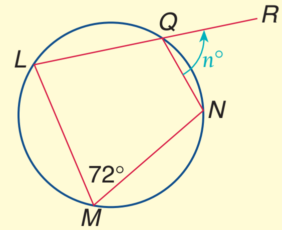
i



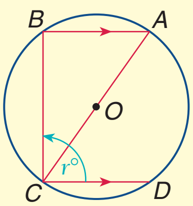
j



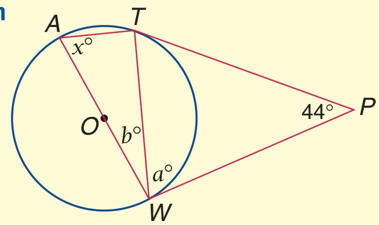
k



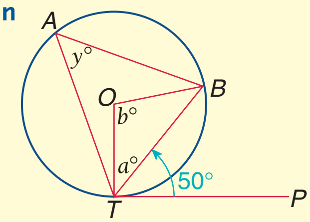
l



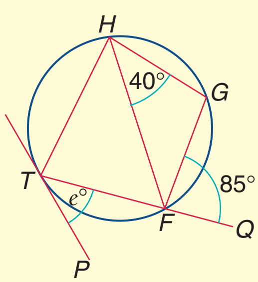
m



n

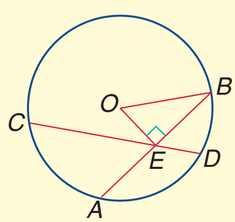


2 **a**



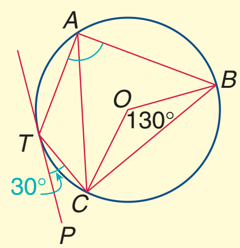
∠*FHG* = 40°, ∠*GFQ* = 85°, ∠*PTF* = *e*°.

b



CE = 8 cm, *ED* = 2 cm, *OB* = 5 cm, *BE* = *x* cm, *OE* = *y* cm.

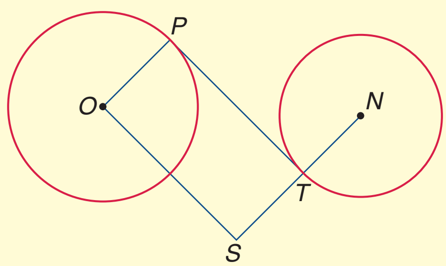
c



∠*PTC* = 30°, ∠*COB* = 130°, ∠*BAT* = *x*°.

3 *PT* is a tangent to a circle, centre *O*, and the tangent touches the circle at *T*. *A* is a point on the circle and *AP* cuts the circle at *B* such that *AB* = *BP*. Find the length of *AB* if *PT* is 8 cm.

4



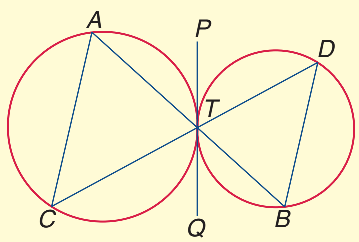
PT is an indirect common tangent of the two circles that have centres *O* and *N*. $OP = 6$ cm, $NT = 5$ cm and $ON = 15$ cm. OS ∥ *PT*.

a Show that $\angle OSN = 90°$.

b Show that *OPTS* is a rectangle.

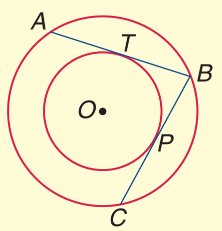
c Find the length of *PT*.

5



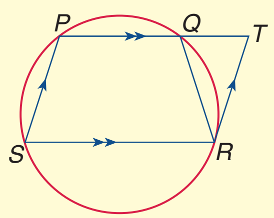
PQ is a common tangent. *AB* and *CD* intersect at the point where the tangent meets the circles. Prove that *AC* ∥ *DB*.

6



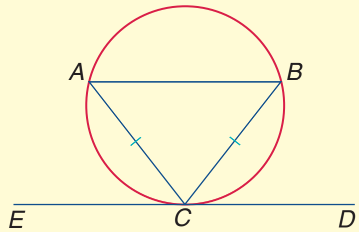
The two circles are concentric. *AB* and *CB* are chords of the larger circle and tangents to the smaller circle. Prove that $AB = CB$.

7



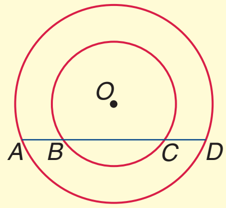
PQRS is a cyclic quadrilateral. Side *PQ* has been produced to *T* so that *PTRS* is a parallelogram. Prove that *RQT* is an isosceles triangle.

8

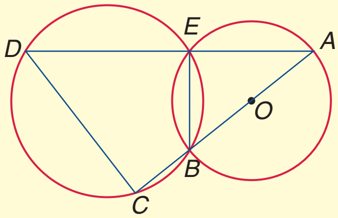


A, *B* and *C* are three points on a circle where $AC = BC$. *ED* is a tangent to the circle at *C*. Prove that *AB* ∥ *ED*.

9



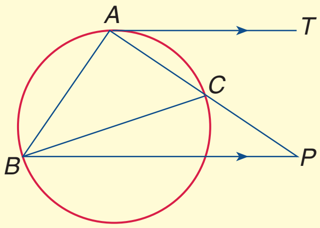
In the diagram, *O* is the centre of two concentric circles. *ABCD* is a straight line. Prove that $AB = CD$.

10 

EB is the common chord of the intersecting circles. *AB* is a diameter of the smaller circle which is produced to meet the larger circle at *C*. *DA* passes through *E*.

a Prove that ΔAEB is similar to ΔACD.

b Hence or otherwise prove that $\angle ACD = 90°$.

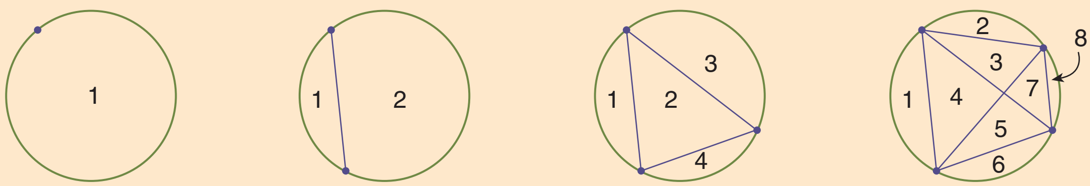
11 

AT is a tangent and is parallel to *BP*. *ACP* is a straight line. Prove that $\angle ABP = \angle ACB$.

fun spot 13:08

Fun Spot 13:08 | How many sections?

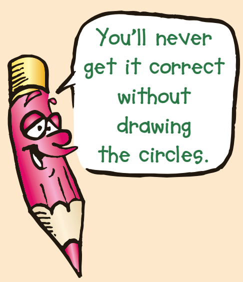
Below are a series of circles. On them have been placed 1 dot, 2 dots, 3 dots and 4 dots. Chords have been drawn connecting every possible pair of dots. The number of chords and the number of divided sections of each circle were counted and recorded in the table.



1 dot 2 dots 3 dots 4 dots

Dots	*Chords*	*Sections*
1	0	1
2	1	2
3	3	4
4	6	8
5	?	?
6	?	?

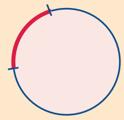
The problem is of course to find the next two lines of the table.



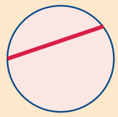
Make sure your dots are spread **unevenly** around the circle.

Mathematical Terms 13

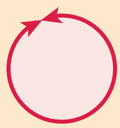
arc
- Part of the circumference of a circle.



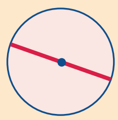
chord
- An interval joining two points on the circumference of a circle.



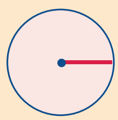
circumference
- The perimeter of a circle.



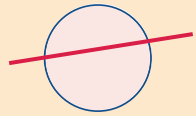
diameter
- A chord that passes through the centre of a circle.
- The width of a circle.



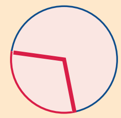
radius (plural: radii)
- An interval joining the centre of a circle to its circumference.
- A radius is half the length of a diameter.



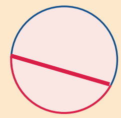
secant
- A line that intersects a circle in two places.



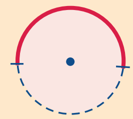
sector
- Part of the area of a circle cut off by two radii.



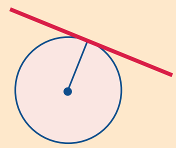
segment
- Part of the area of a circle cut off by a chord.



semicircle
- Half a circle.



tangent
- A line that touches a circle at one point.
- A tangent and a radius are perpendicular at the point of contact.



Mathematical terms 13

- A famous circle in history, this replica of King Arthur's Round Table is in Winchester, England.

diagnostic test 13

Diagnostic Test 13 | Circle Geometry

- These questions reflect the important skills introduced in this chapter.
- Errors made will indicate areas of weakness.
- Each weakness should be treated by going back to the section listed.

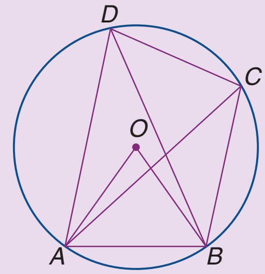
Section

1 From the figure shown, name: (Section 13:01)

a an angle subtended at the centre *O*

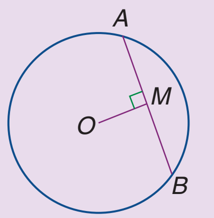
b an angle standing on the arc *AB*, subtended at the circumference

c an angle in the same segment as ∠*BAC*, standing on *BC*



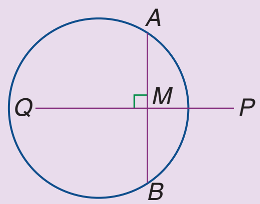
2 (Section 13:02)

a



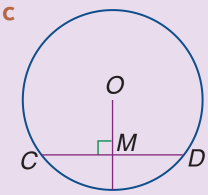
AB = 9 cm. Find the length of *AM*, giving reasons.

b



AM = *MB*. Give reasons why *PQ* must pass through the centre of the circle.

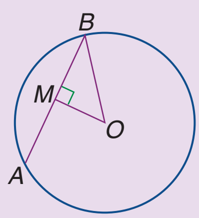
c



CM = 18 m. Find the length of *CD*, giving reasons.

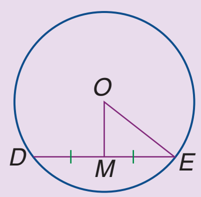
3 (Section 13:02)

a



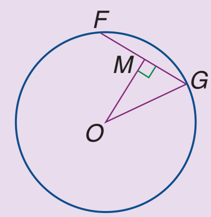
BO = 26 m, *AB* = 48 m. Find the length of *OM*.

b



OM = 9 cm, *ME* = 12 cm. Find the length of *OE*.

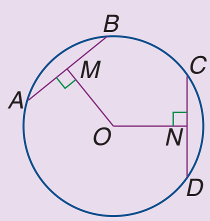
c



OM = 40 m, *OG* = 50 m. Find the length of *FG*.

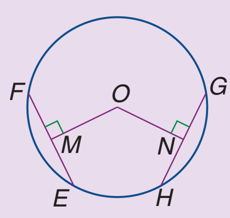
4 (Section 13:03)

a



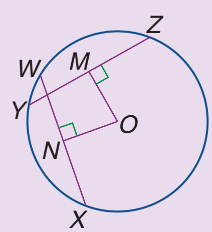
AB = *CD*, *OM* = 8 m. Find the length of *ON*, giving reasons.

b



OM = *ON*, *EF* = 11 km. Find the length of *GN*, giving reasons.

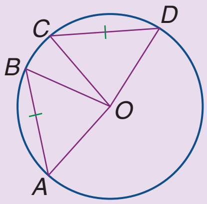
c



OM = *ON*, *WX* = 14 m. Find the length of *YZ*, giving reasons.

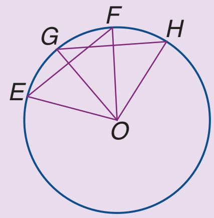
Section

5 **a** 13:03



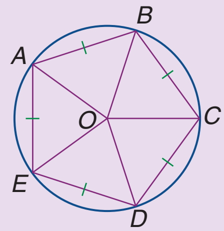
$AB = CD$, $\angle COD = 69°$. Find the size of $\angle AOB$, giving reasons.

b



$\angle GOH = \angle EOF$, $GH = 1{\cdot}1$ m. Find the length of EF, giving reasons.

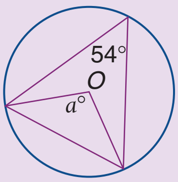
c



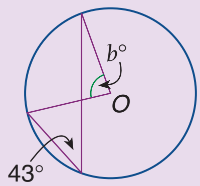
In this regular polygon, find the size of $\angle BOC$ and obtuse $\angle BOD$.

Find the value of the pronumerals in the following questions.

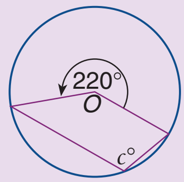
6 **a** 13:04



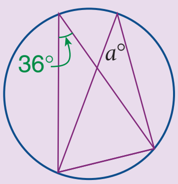
b



c



7 **a** 13:04



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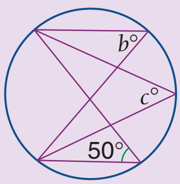

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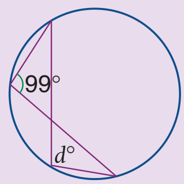
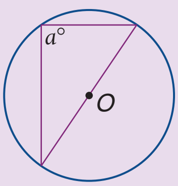


8 **a** 13:05



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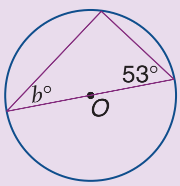

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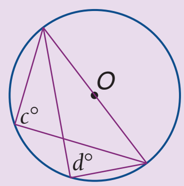
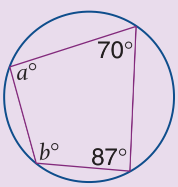


9 **a** 13:05



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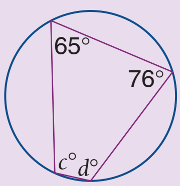

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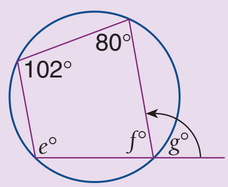
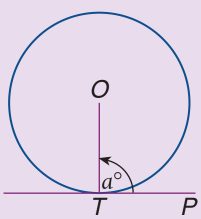


In questions **10** to **13**, PT and PW are tangents.

10 **a** 13:06



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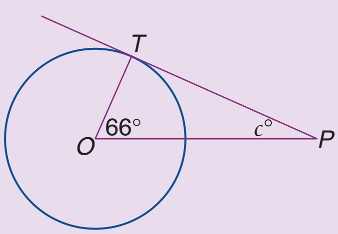

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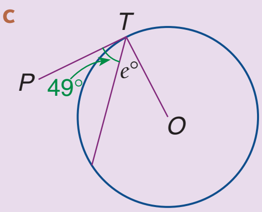
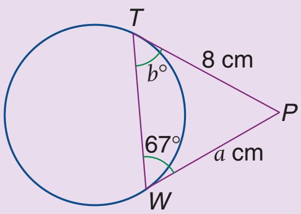


Section

11 **a**

13:06



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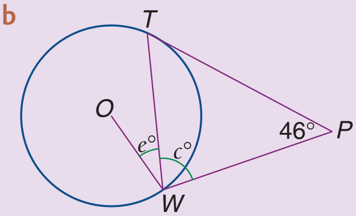

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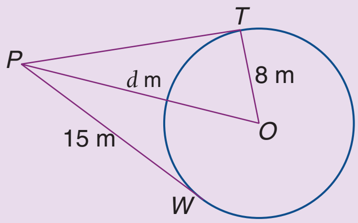
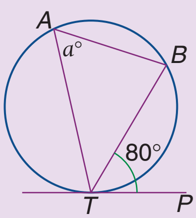


12 **a**

13:06



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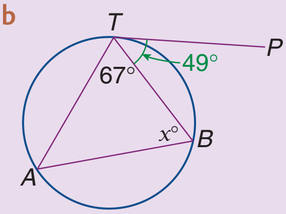

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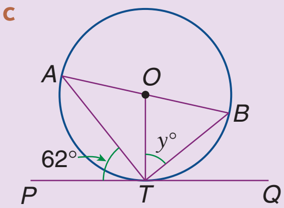
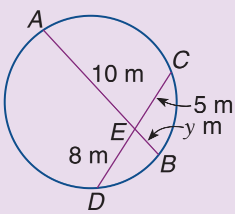


13 **a**

13:07



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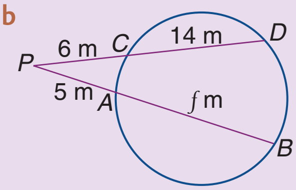

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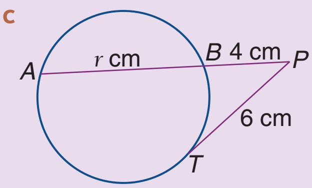


- The circle in the sky seen throughout the world.

assignment

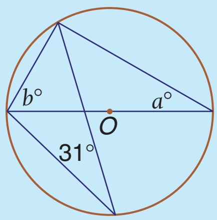
13A

Chapter 13 | Revision Assignment

O is the centre of each circle. *PT* and *PW* are tangents.

1 Find the size of the pronumerals in each diagram.

a



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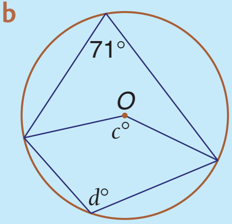

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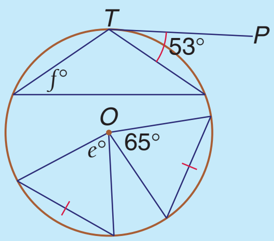

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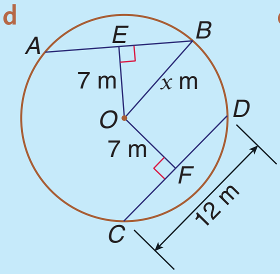

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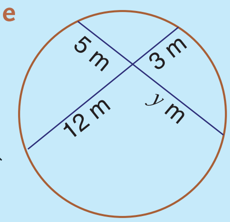

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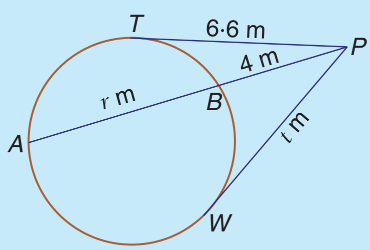

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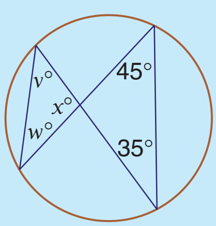

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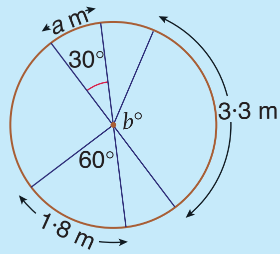

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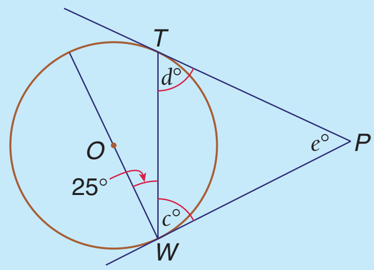

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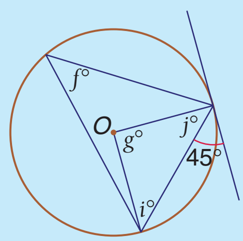

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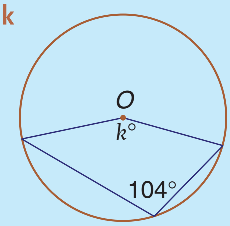

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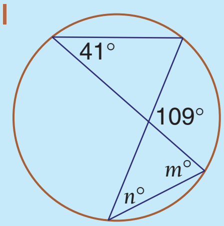

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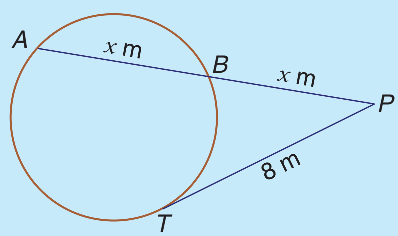

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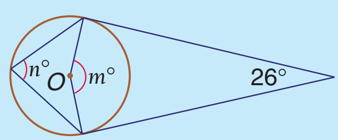
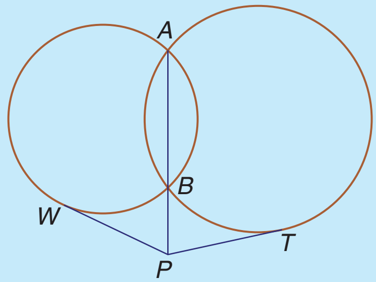


2 **a** *AB* is the common chord of the circles and has been produced to *P*. From *P*, tangents *PT* and *PW* have been drawn to the circles.

i Prove that $WP = PT$.

ii If $BP = 14{\cdot}6$ cm and $PT = 19{\cdot}4$ cm, find the length of *AB* correct to three significant figures.



b Prove that the bisector of the angle between the tangents drawn to a circle from an external point passes through the centre.

c If *AB*, *AC* are two tangents to a circle and $\angle BAC = 84°$, what are the sizes of the angles in the two segments into which *BC* divides the circle?

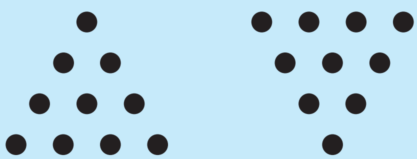
d *P* is a point within a circle of radius 13 cm and *XY* is any chord drawn through *P* so that $XP \,.\, PY = 25$. Find the length of *OP* if *O* is the centre of the circle.

assignment

13B

Chapter 13 | Working Mathematically

1



Move 3 dots in the diagram on the left to obtain the diagram on the right.

2 A ladder hangs over the side of a ship. The rungs in the ladder are each 2·5 cm thick and are 18 cm apart. The fifth rung from the bottom of the ladder is just above the water level. If the tide is rising at a rate of 15·5 cm per hour, how many rungs will be under water in 3 hours?

3 Three circles are touching each other so that the distance between each pair of centres is 8 cm, 9 cm and 13 cm. What is the radius of each circle?

4 Four people are to be accommodated in two rooms. In how many ways can the people be arranged if there is at least one person in each room?

5 What is the smallest whole number that, if you multiply by 7, will give you an answer consisting entirely of 8s?

6 **a** How many digits are needed to number the pages of a 50-page book?

b How many digits are needed to number the pages of a 500-page book?

c If a book had 5000 pages, how many digits would be needed?

d Can you determine an expression that will give the number of digits needed for a book with *n* pages if $100 < n < 1000$?

Chapter Review Questions

Technology Applications Activities — Circles

Technology Applications Drag and Drops — 1 Parts of a circle, 2 Circle geometry

Technology Applications Animations — Spin graphs

14

Transformations and Matrices

Chapter Contents

Learning Outcomes

Students will be able to:

- Transform objects through enlargement, translation, reflection and rotation.
- Recognise which transformation has been performed between an object and its image.
- Describe a transformation in words and by using matrices.
- Perform transformations through given matrices.
- Recognise the transformation which will result from a given matrix.

Areas of Interaction

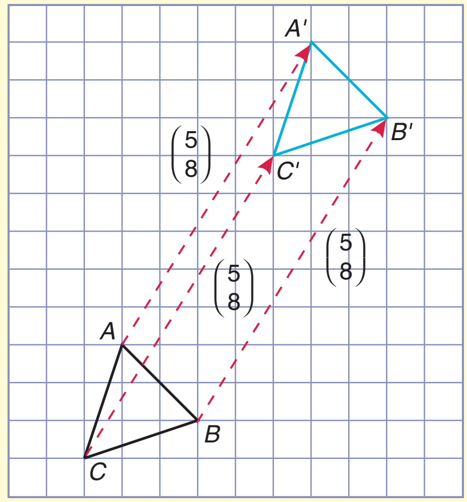
Approaches to Learning (Knowledge Acquisition, Reflection, Organisation, Technology), Human Ingenuity – the manipulation of shapes to form patterns

14:01 | Translations

When a shape is changed in some way so that it becomes a different shape, called the **image**, either in size or orientation, it has been **transformed**. This chapter looks at the different ways shapes can be transformed and mathematics behind them.

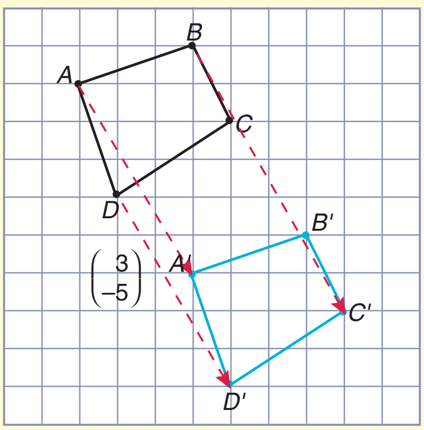
When a shape has been moved, but its size and orientation remain the same, it has been **translated**. A shape and its image after translation are congruent.

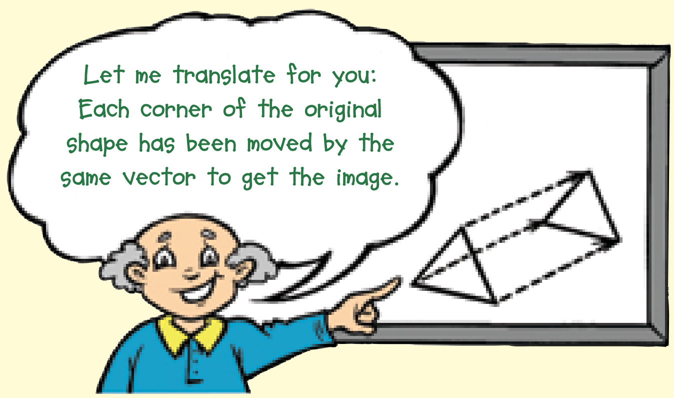
In the diagram shown, the original triangle ΔABC has been moved 5 units to the right and 8 units up to get the image $\Delta A'B'C'$.



We say ΔABC has been translated 5 units horizontally and 8 units vertically to obtain $\Delta A'B'C'$.

This can be described using a vector representation as $\begin{pmatrix}5\\8\end{pmatrix}$





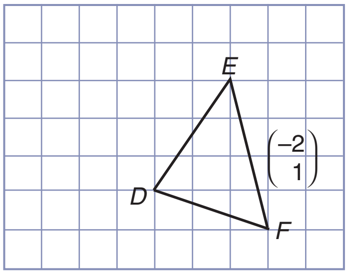
In this diagram the figure $ABCD$ has been translated 3 units horizontally and −5 units vertically to obtain figure $A'B'C'D'$.

This can be described using a vector representation as $\begin{pmatrix}3\\-5\end{pmatrix}$

Worked examples

1 Translate the following shapes through the given vectors.

a



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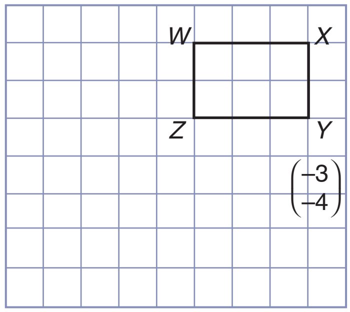
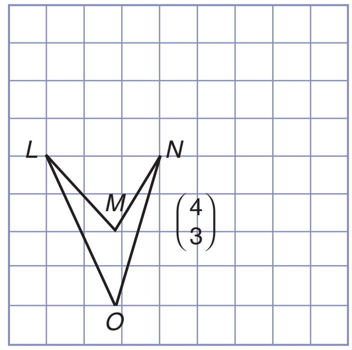
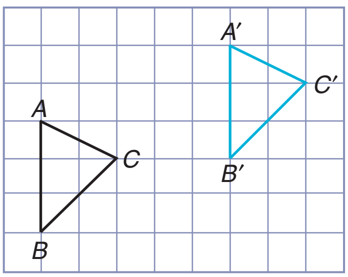


c

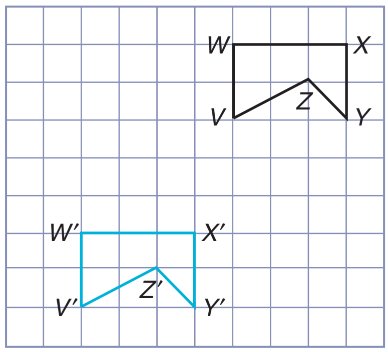


2 Describe the following translations using both words and vector representation. The image is shown in blue with ′ after the vertex.

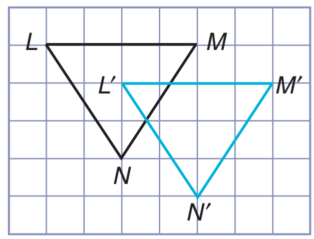
a



b

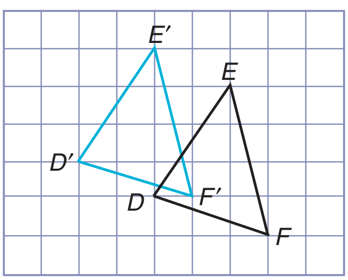


c

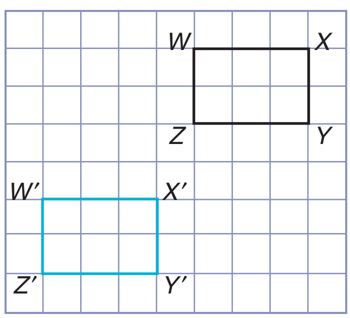


Solutions

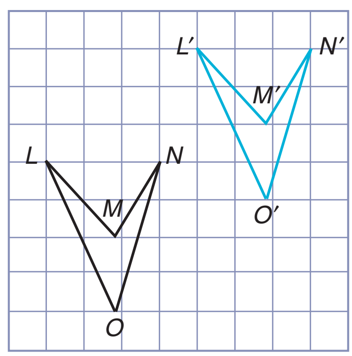
1 **a**



b



c



2 **a** Horizontal translation of –2 units and vertical translation of 1 unit, $\begin{pmatrix} -2 \\ 1 \end{pmatrix}$

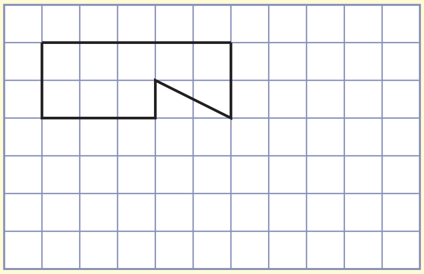
b Horizontal translation of –3 units and vertical translation of –4 units, $\begin{pmatrix} -3 \\ -4 \end{pmatrix}$

c Horizontal translation of 4 units and vertical translation of 3 units, $\begin{pmatrix} 4 \\ 3 \end{pmatrix}$

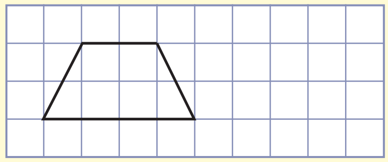
Exercise 14:01

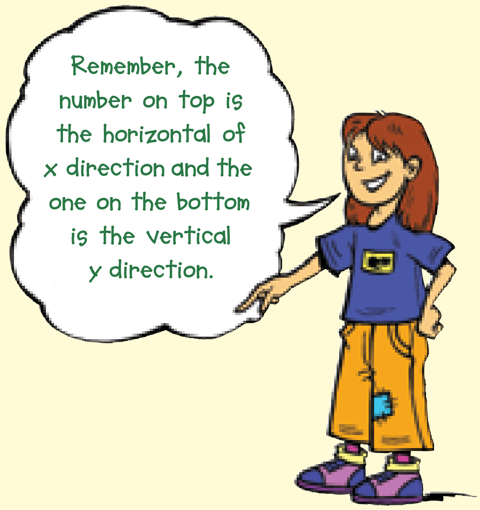
1 Translate the following shapes by the given vector:

a $\begin{pmatrix} 4 \\ -3 \end{pmatrix}$

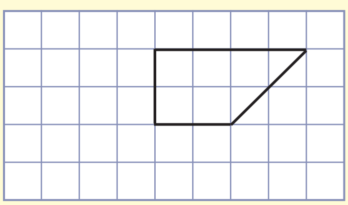


b $\begin{pmatrix} 4 \\ 0 \end{pmatrix}$

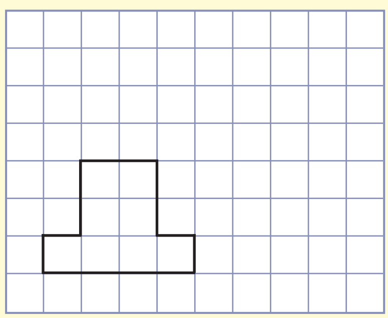




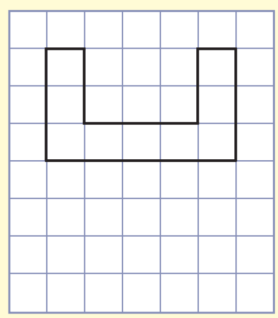
c $\begin{pmatrix} -3 \\ -1 \end{pmatrix}$



d $\begin{pmatrix} 4 \\ 2 \end{pmatrix}$

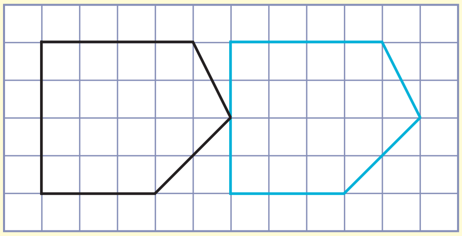


e $\begin{pmatrix} 0 \\ -3 \end{pmatrix}$

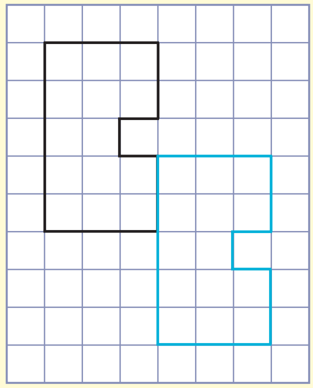


2 Describe the following translations in words and in vector form.
In each question the image is shown in blue.

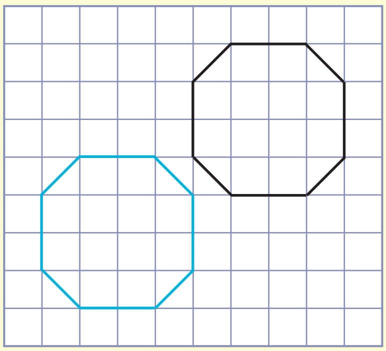
a



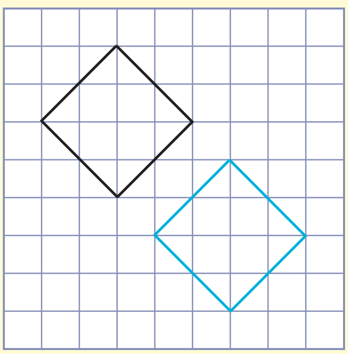
b



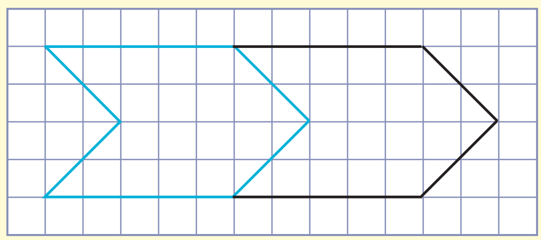
c



d



e



3 The quadrilateral $A(1, 2)$, $B(1, 5)$, $C(3, 7)$, $D(4, 0)$ is translated by the vector $\begin{pmatrix} -2 \\ 4 \end{pmatrix}$
What are the coordinates of the image $A'B'C'D'$?

4 The polygon $PQRST$ has been translated twice: once by the vector $\begin{pmatrix} -3 \\ 6 \end{pmatrix}$ and then by the vector $\begin{pmatrix} 2 \\ -1 \end{pmatrix}$.
The coordinates of the image are $P'(-2, 5)$, $Q'(0, 1)$, $R'(1, -2)$, $S'(3, 3)$ and $T'(2, 7)$.

a Find the coordinates of P, Q, R, S and T.

b What single vector could be applied to obtain the same image?

5 A shape has been translated by successive vectors: $\begin{pmatrix} 3 \\ -4 \end{pmatrix}$, $\begin{pmatrix} 1 \\ -2 \end{pmatrix}$ and $\begin{pmatrix} 5 \\ 8 \end{pmatrix}$.
What single vector would translate the shape back to its original position?

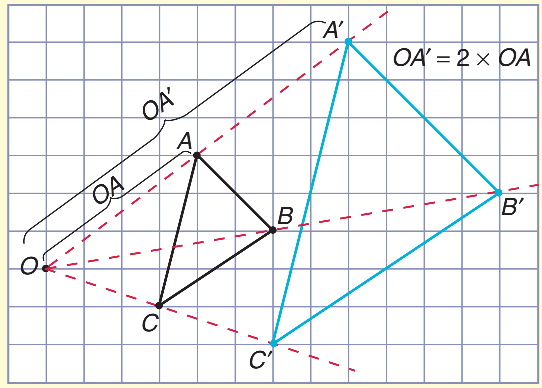
14:02 | Enlargement

When a shape is transformed so that it retains the same basic shape but the image is bigger or smaller than the original then it has been **enlarged**.

A shape and its image after enlargement are similar.

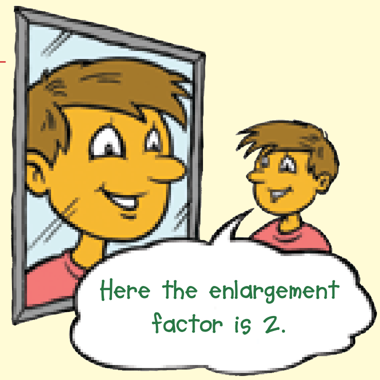
A shape is usually enlarged by first using a centre of enlargement as in this example.

Here the centre of enlargement is O.



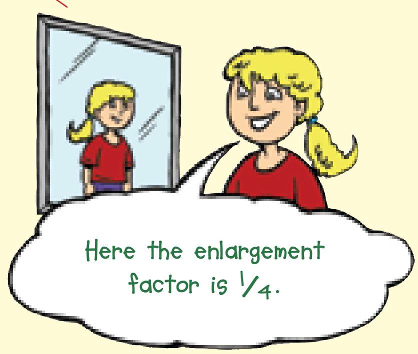
To create an image twice as big as the original ΔABC, lines are drawn from the centre of enlargement, through each of the vertices. The distance from O to each vertex in the image is twice the distance from O to the vertex in the original.

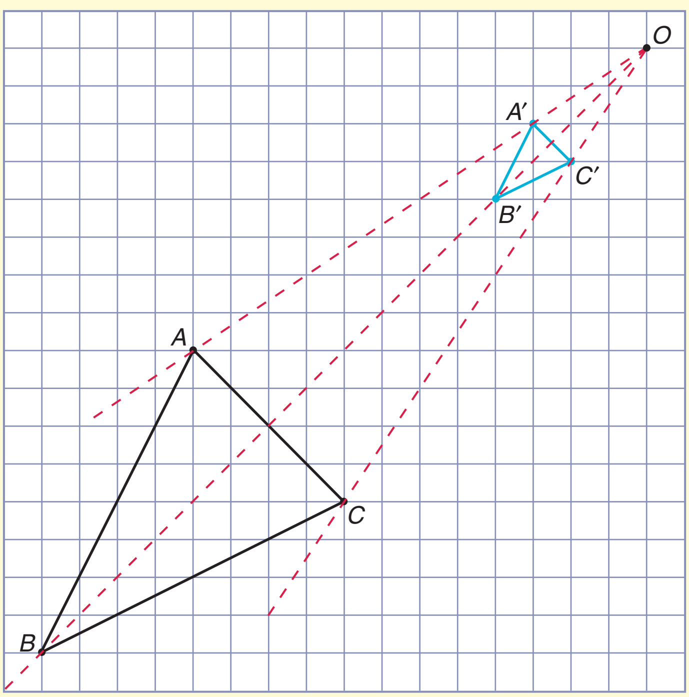
So $OA' = 2 \times OA$, $OB' = 2 \times OB$, and $OC' = 2 \times OC$



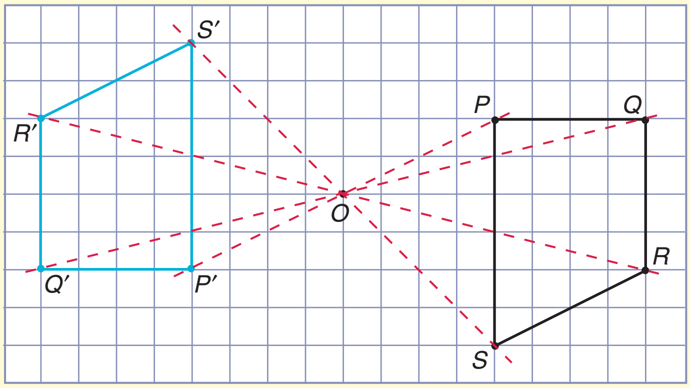
If we wanted the image to be 3 times the size then the distance from O to the image would be 3 times the distance to the original.

In the diagram below, the distance from O to each vertex in the image is a quarter of the distance to the original vertex, so the image is a quarter of the size of the original.





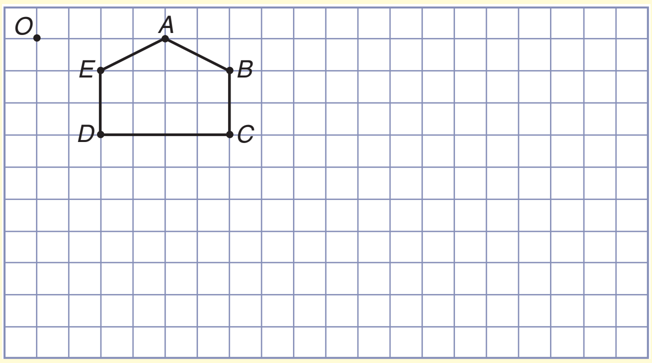
If the enlargement factor is a negative, then the image is on the other side of the centre of enlargement. This diagram shows an enlargement factor of –1.



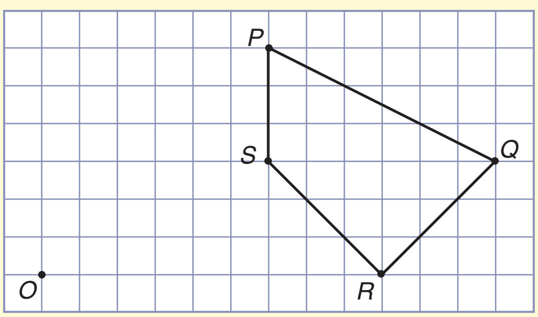
Exercise 14:02

1 Enlarge each of the following by the enlargement factor given.

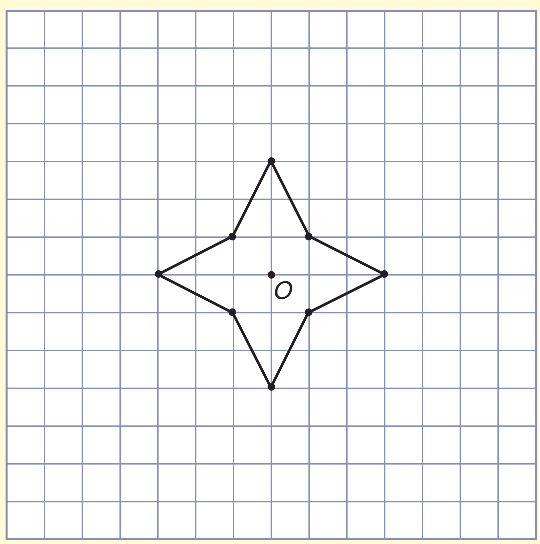
a Scale factor 3



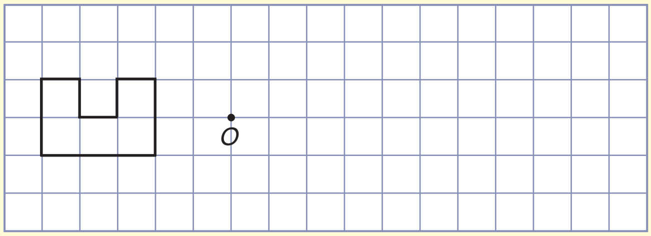
b Scale factor $\frac{1}{3}$



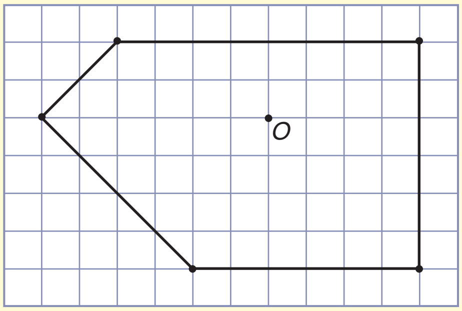
c Scale factor 2



d Scale factor −2

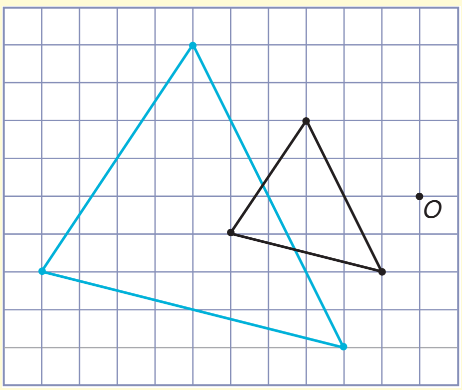


e Scale factor $-\frac{1}{2}$

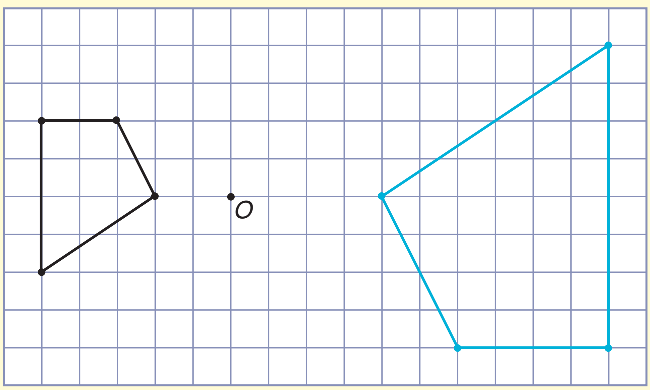


2 Find the scale factor in each of the following enlargements. The image is in blue:

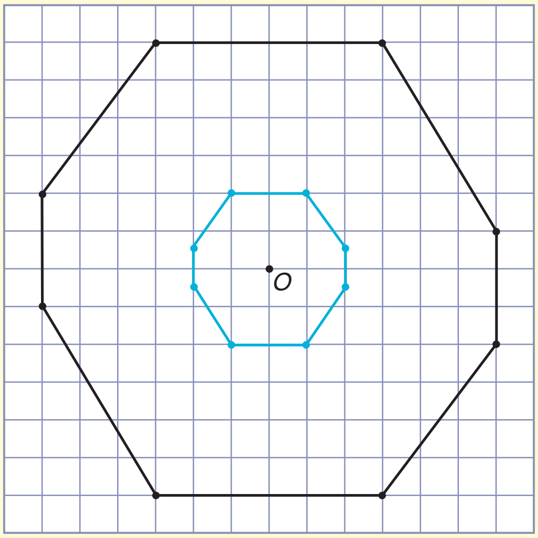
a



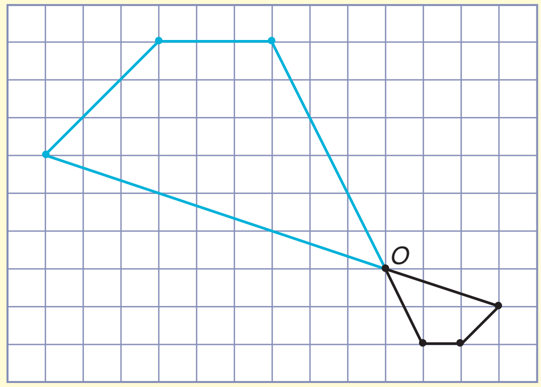
b



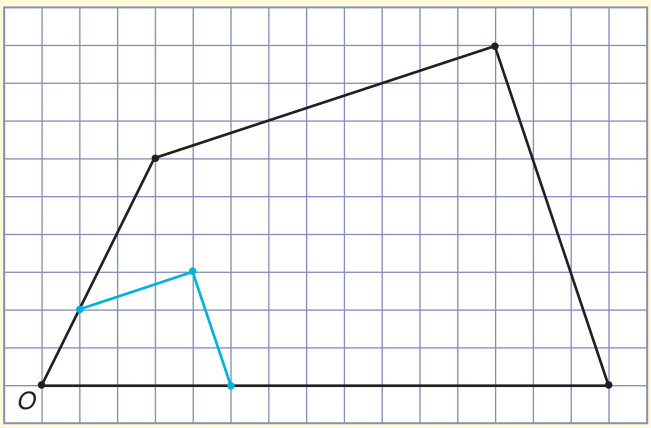
c



d

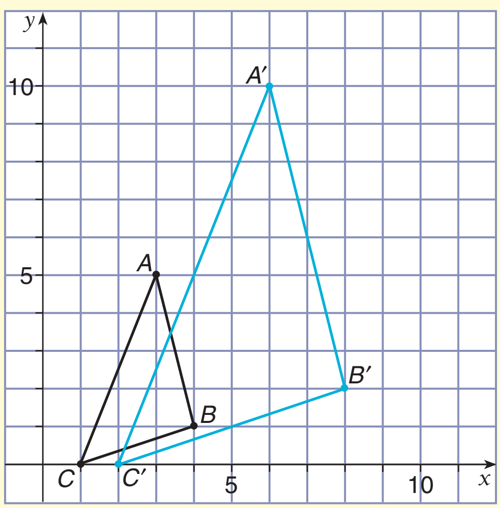


e

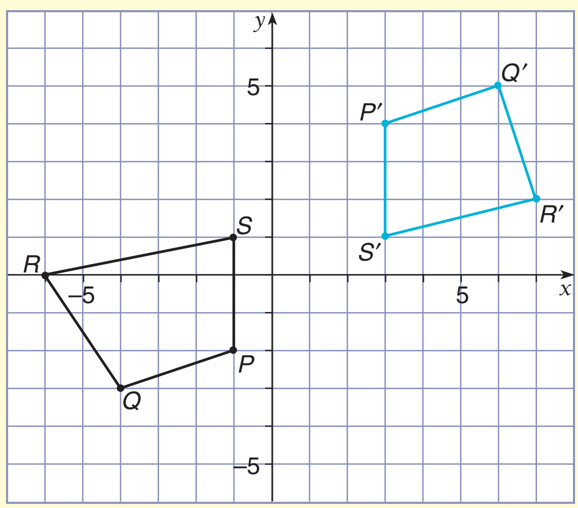


3 Find the centre of enlargement and the enlargement factor in each the following. The image is in blue.

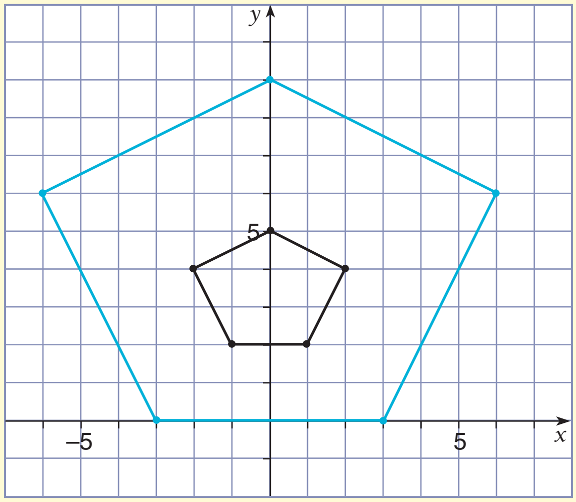
a



b

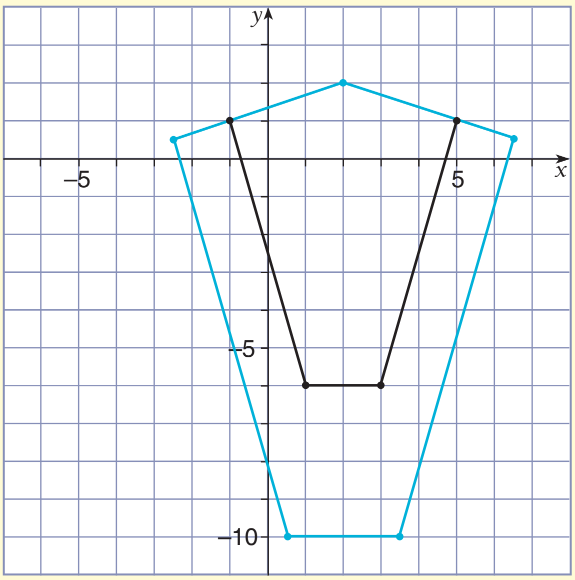


c



d $\Delta A(3, 3)$ $B(4, 7)$ $C(7, 2)$ is transformed under enlargement so that the image is $\Delta A'(-3, 3)$ $B'(-5, -5)$ $C'(-11, 5)$

e



4 The quadrilateral $P(-8, 2)$ $Q(-4, 2)$ $R(-3, -2)$ $S(-10, -2)$ is enlarged by a factor of 3 using P as the centre of enlargement. Find the coordinates of the image $P'Q'R'S'$.

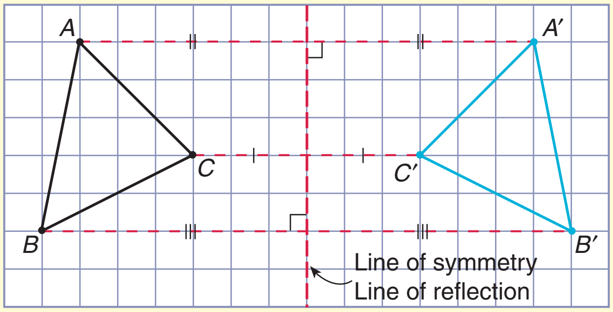
5 Under enlargement, the image of a point $P(-3, 5)$ is $P'(3, -7)$. If the enlargement factor is 3, find the centre of enlargement.

14:03 | Reflection

When a shape is transformed so that its image is the reverse of the original but the size and shape remains unchanged, it has been **reflected**.

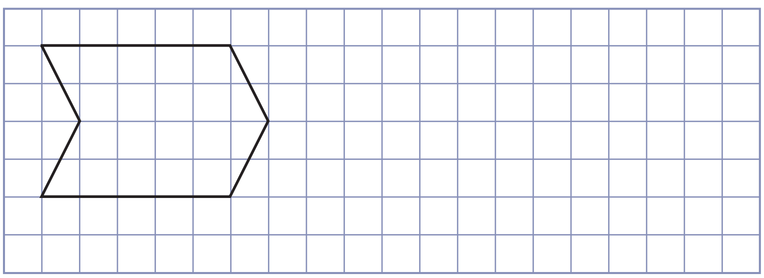
A shape and its image under reflection are congruent.

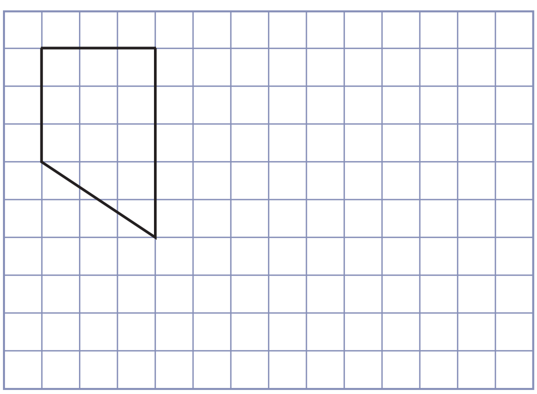
A shape is reflected through a **line of symmetry** or **line of reflection**. Lines joining a shape and its image are perpendicular to the line of symmetry and are bisected by it. See the example below:

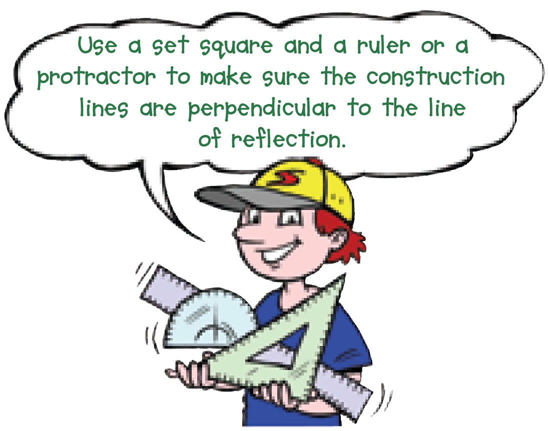


worked examples

1 Reflect the given shapes in the line of reflection shown.

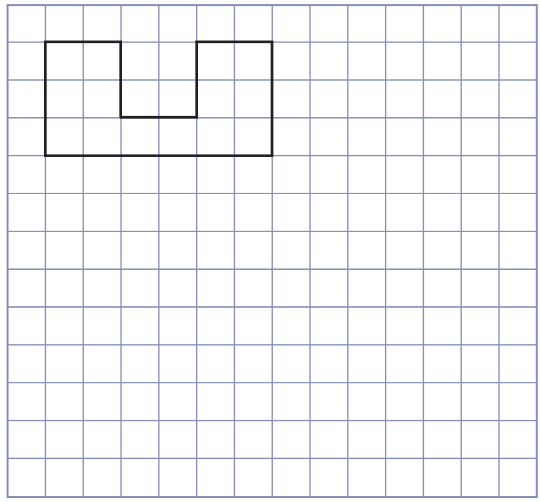






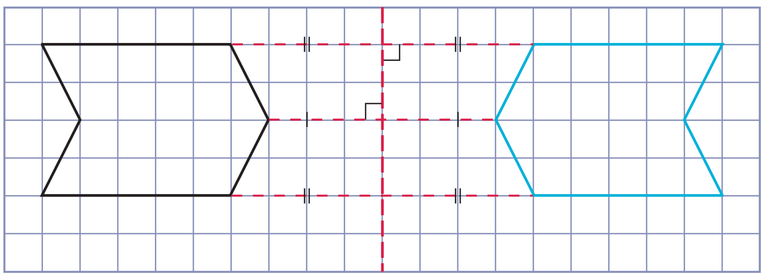
continued →→→

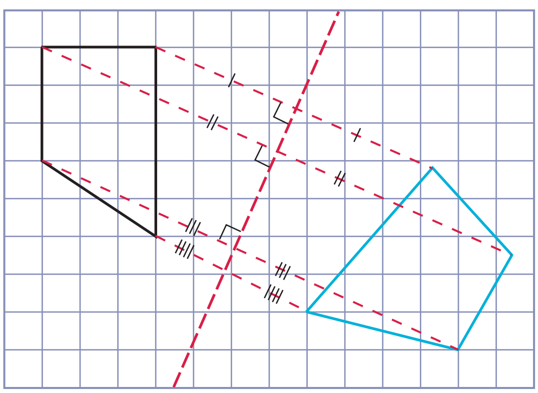
2 Find the line of reflection in the following:



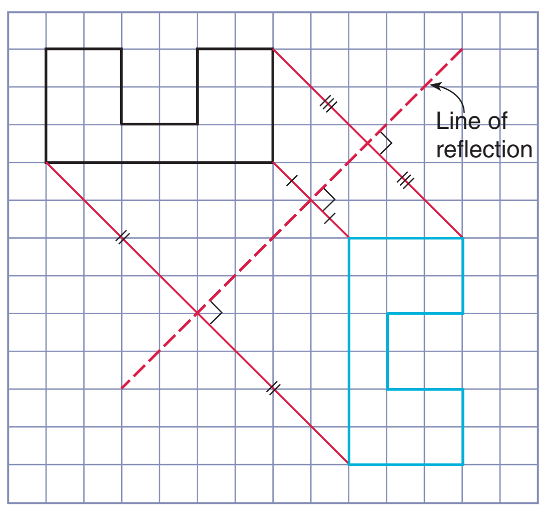
Solutions

1





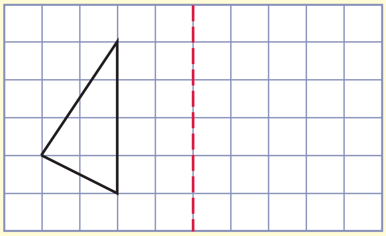
2 Join together two or more corresponding vertices and bisect them either with a ruler or pair of compasses. The line of reflection lies along the midpoints.



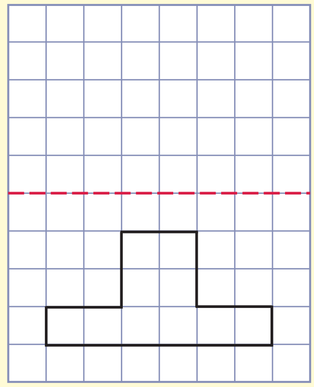
Exercise 14:03

1 Reflect the following shapes in the line of reflection given.

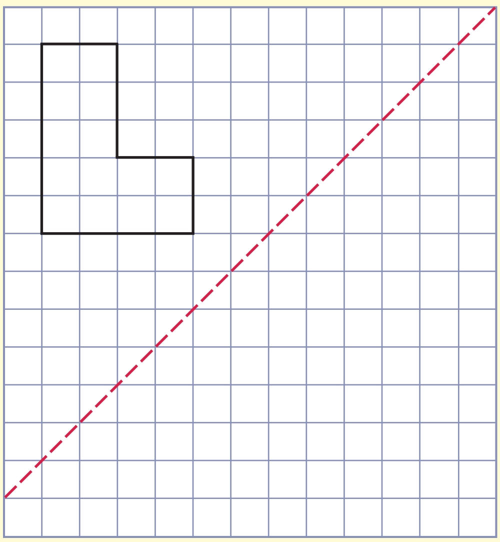
a



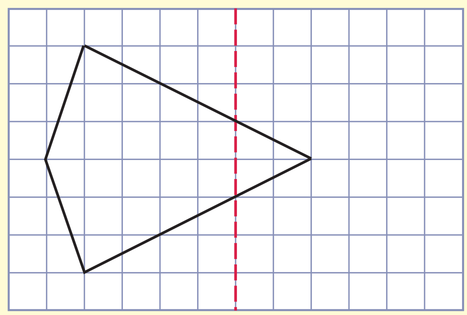
b



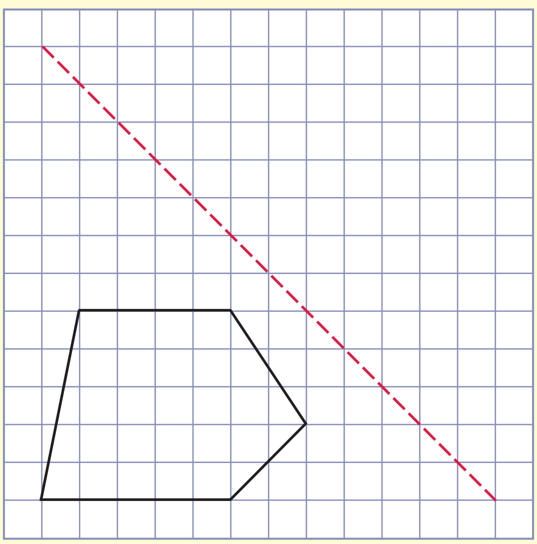
c



d

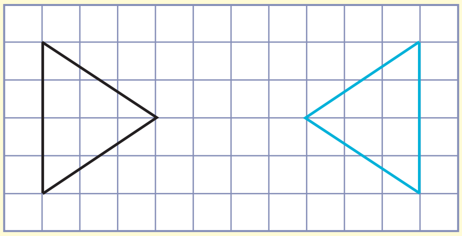


e

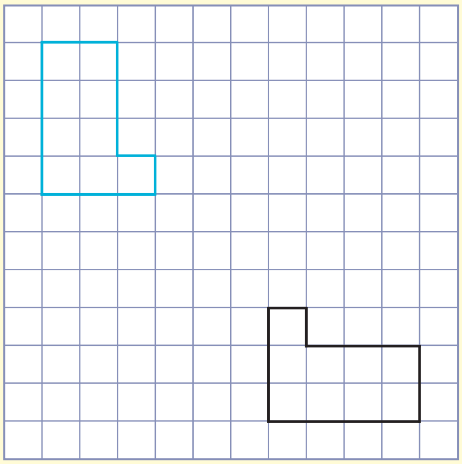


2 Find the line of reflection between the following shapes and their image shown in blue.

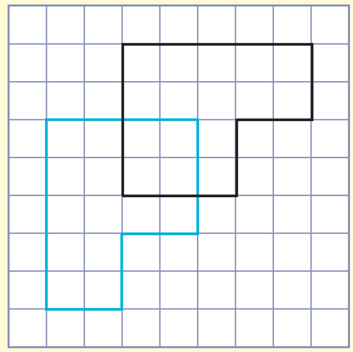
a



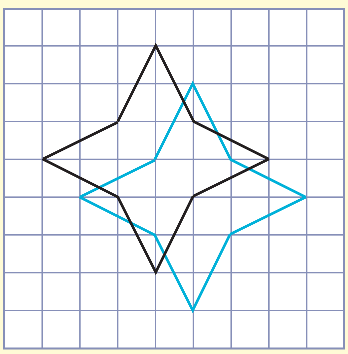
b



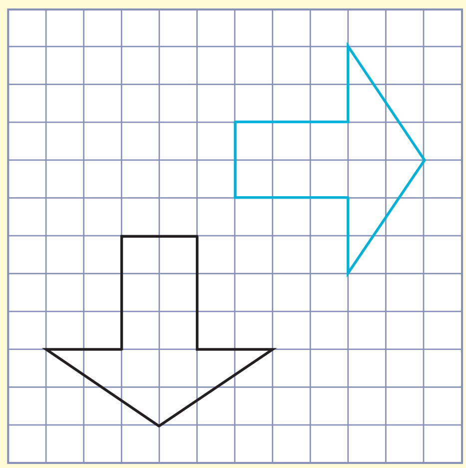
c



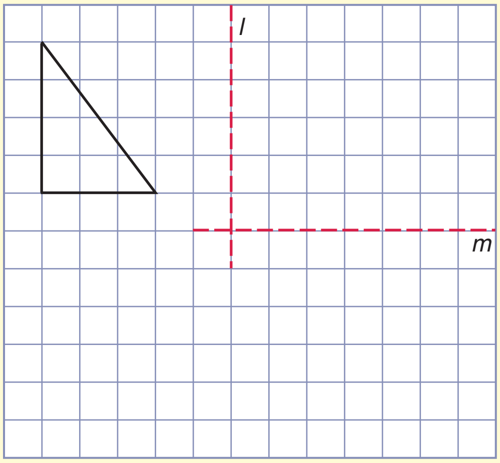
d



e

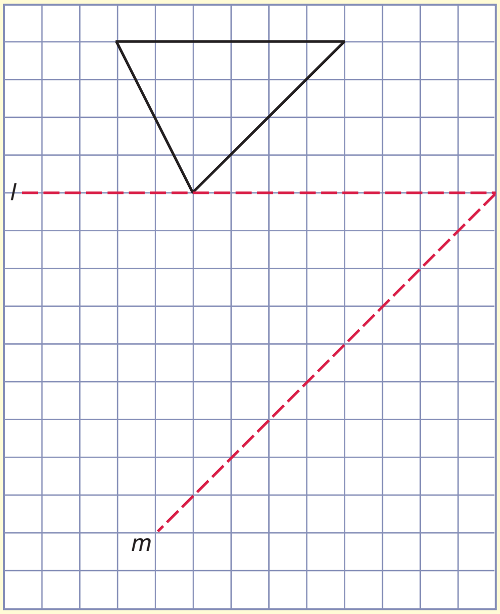


3 The shape given is first reflected in the line l and then in the line m.



a Show the image after each transformation.
b What would be the result if the final image was again reflected in the line l and then in the line m?

4 The shape given is first reflected in the line l and then in the line m.



a Show the image after each transformation.
b If the final shape in (a) was again reflected in the line l and then in the line m, would you expect the same result as in question 3?

5 Triangle ABC has coordinates $A(2, 5)$, $B(6, 3)$ and $C(3, 2)$. It is reflected in the line $y = 2 - x$. Find the coordinates of the image, $\Delta A'B'C''$.

6 Triangle PQR has coordinates $P(6, 0)$, $Q(6, -5)$ and $R(2, -4)$. It has been reflected so that its image, $P'Q'R'$ has coordinates $P'(3, 3)$, $Q'(-1, -1)$ and $R'(-2, 3)$.
Find the equation of the line of reflection.

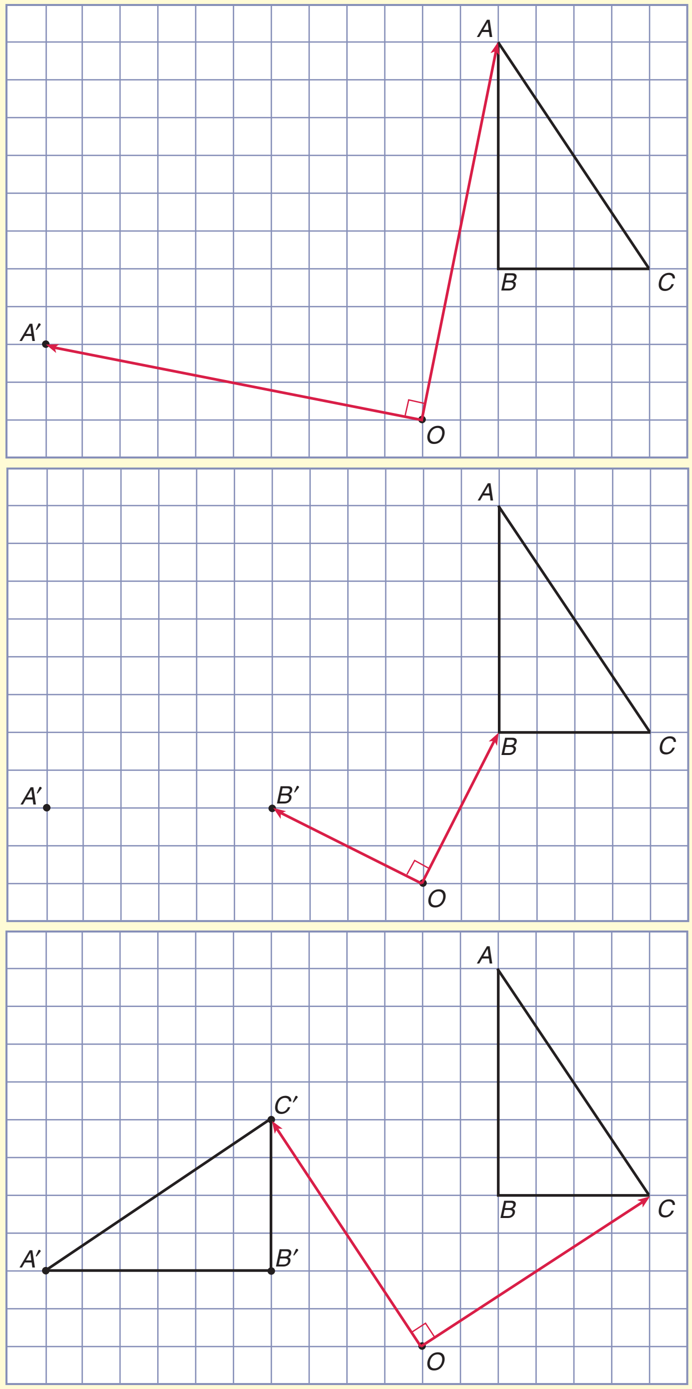
14:04 | Rotation

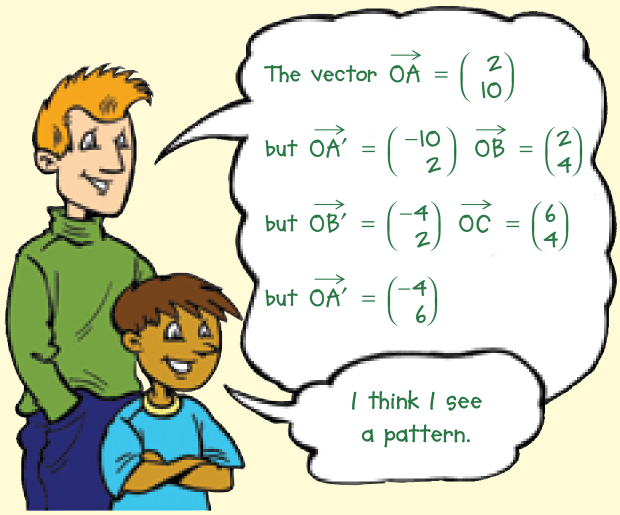
When a shape is transformed so that it is rotated about a point to obtain the image it has been **rotated**. An object and its image under rotation are congruent.

A shape is always rotated through a given number of degrees about a **centre of rotation**. Rotation in an anticlockwise direction is a **positive rotation** and through a clockwise direction is a **negative rotation**.

In the diagram below the ΔABC is to be rotated about the point O through 90°.

To do this, the image of each vertex is found by rotating the vector from O to the vertex through 90°.





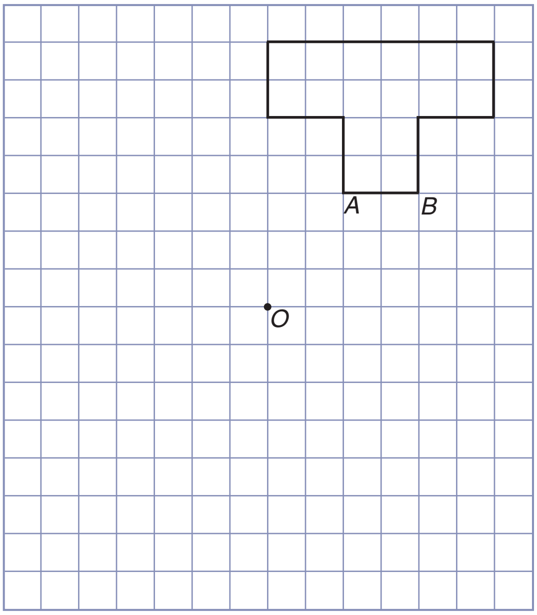
You can see that the vector $\overrightarrow{OA}$ is at right angles to vector $\overrightarrow{OA'}$ and the same length, the vector $\overrightarrow{OB}$ is at right angles to vector $\overrightarrow{OB'}$ and the same length and the vector $\overrightarrow{OC}$ is at right angles to vector $\overrightarrow{OC'}$ and the same length.

Once the new vertices are located, the image $\Delta A'B'C'$ can be drawn.

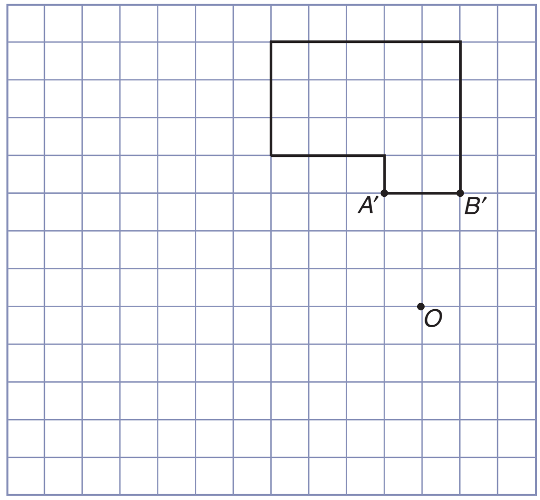
worked examples

1 Rotate the following shapes using the directions given:

a 180° about O



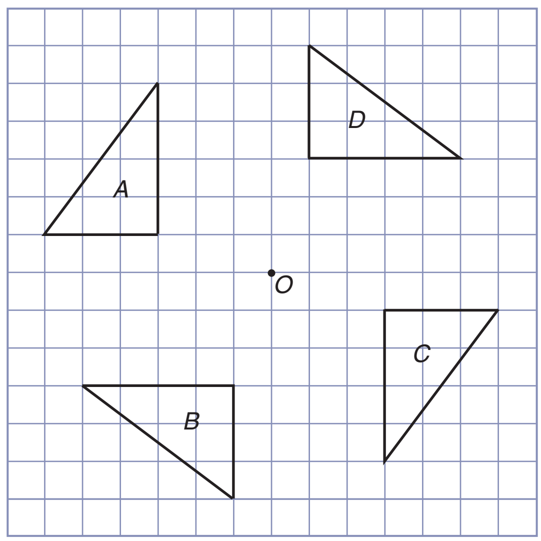
b −90° about O



2 Describe in words the following translations:

a ΔA to ΔC **b** ΔA to ΔD

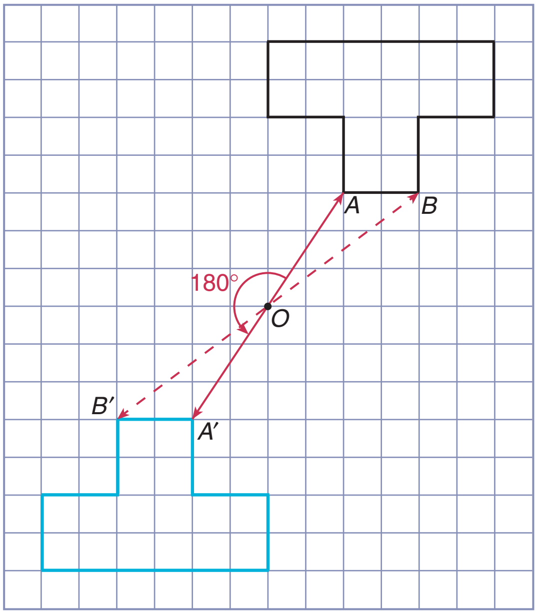
c ΔD to ΔA **d** ΔB to ΔC



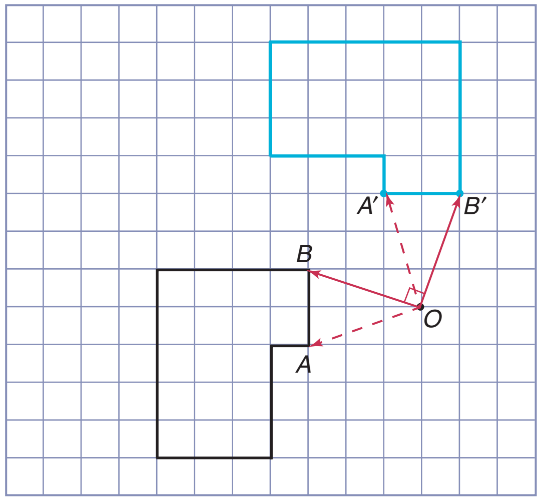
continued →→→

Solutions

1 a



b



2 a Rotation of 180° about *O*, or –180° about *O*

b Rotation of –90° about *O*

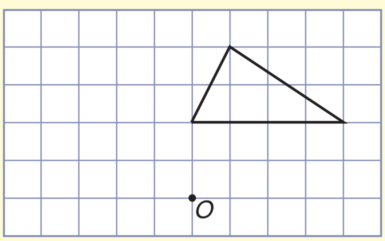
c Rotation of 90° about *O*

d Rotation of 90° about *O*

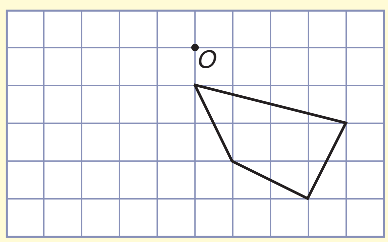
Exercise 14:04

1 Rotate the shapes shown about O through the angles given.

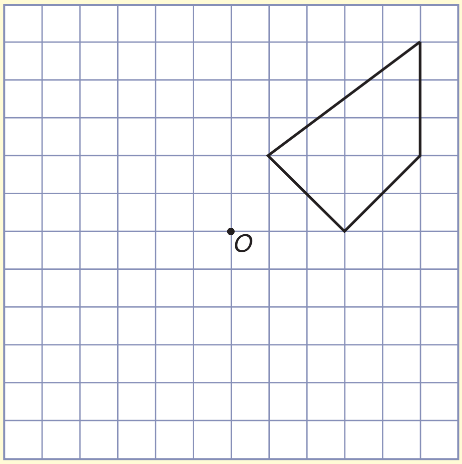
a 90°



b –90°

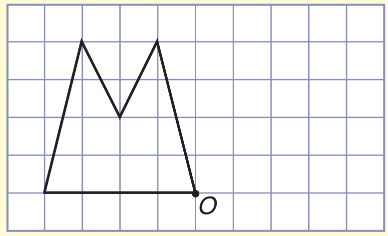


c



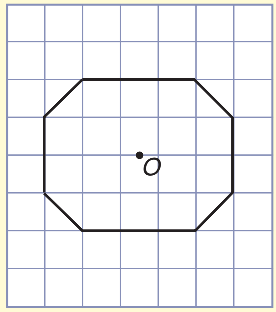
180°

d



270°

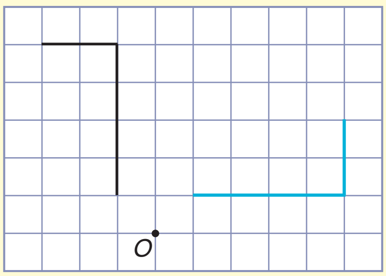
e



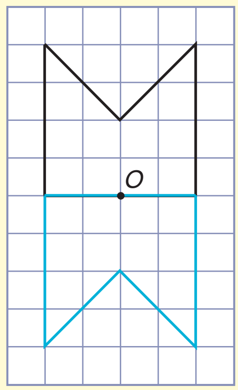
90°

2 Describe the following rotations in words. In each case the image is shown in blue.

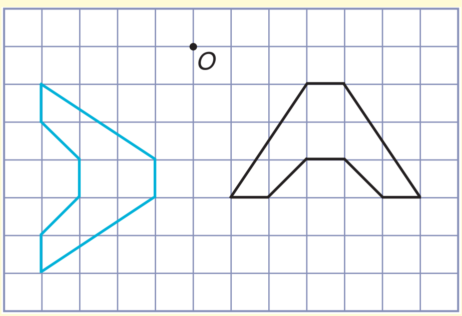
a



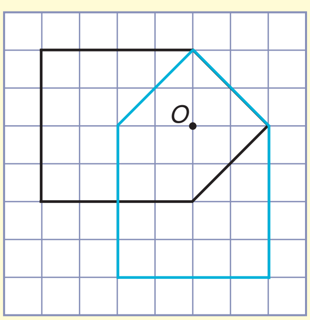
b



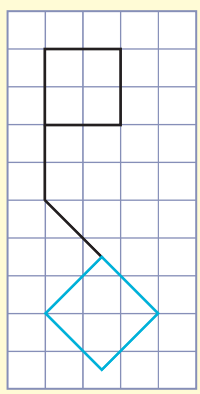
c



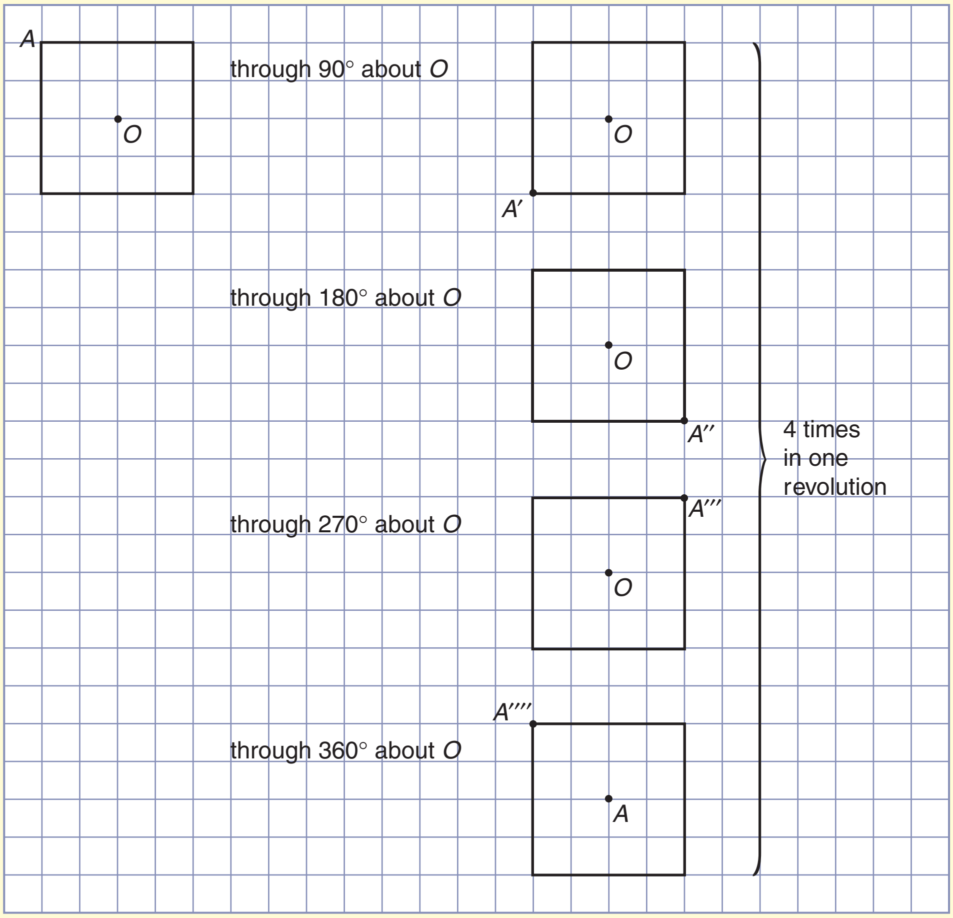
d



e

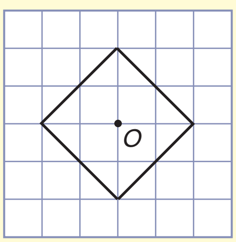


3 If an object can be rotated about any point so that it remains unchanged, it is said to have **rotational symmetry**. The number of times the shape maps onto itself in a revolution is the **order of symmetry**.

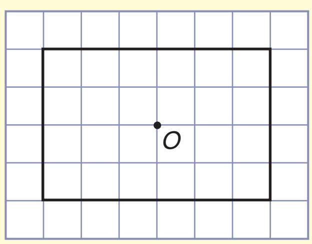


Find the order of the rotational symmetry in the following:

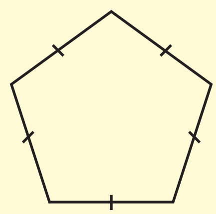
a



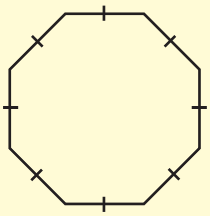
b



c

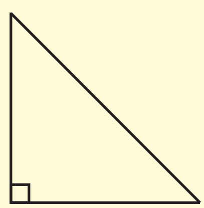


d



e An equilateral triangle

f



4 Triangle ABC has vertices $A(4, 1)$, $B(3, 4)$ and $C(1, 2)$. Find the coordinates of the image of ΔABC after a rotation of 90° about:

a the origin　　**b** the point $(0, -1)$

5 Triangle PQR has vertices $P(5, 4)$, $Q(5, 1)$ and $R(3, 1)$.
The image of ΔPQR after a rotation has vertices $P'(4, -3)$, $Q'(1, -3)$ and $R'(1, -1)$.
Using a scale of 1 cm = 1 unit, plot ΔPQR and $\Delta P'Q'R'$ on a number plane.

a What has been the angle of rotation?

b Draw the intervals QQ' and RR' and construct the perpendicular bisectors of these intervals. What are the coordinates of their point of intersection and what is the significance of this point?

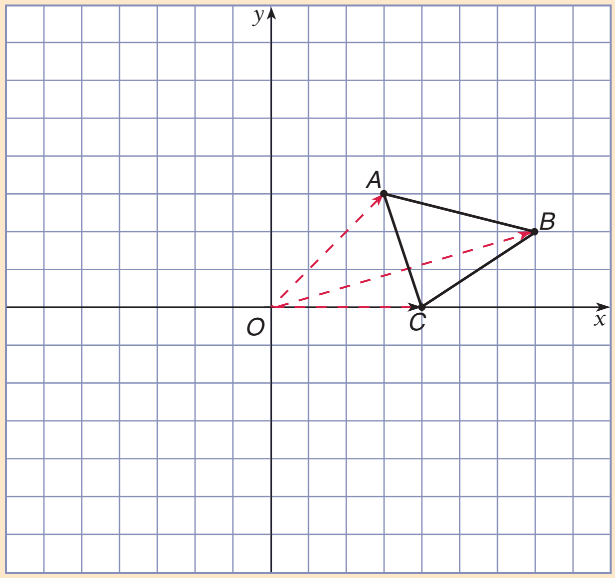
6 Repeat question 5 for triangle $L(2, 4)$ $M(6, 5)$ $N(6, 2)$ and its image $L'(0, 0)$ $M'(-1, 4)$ $N'(2, 4)$.

Investigation 14:04 | Matrix methods

Please use the Assessment Grid on page 424 to help you understand what is required for this Investigation.

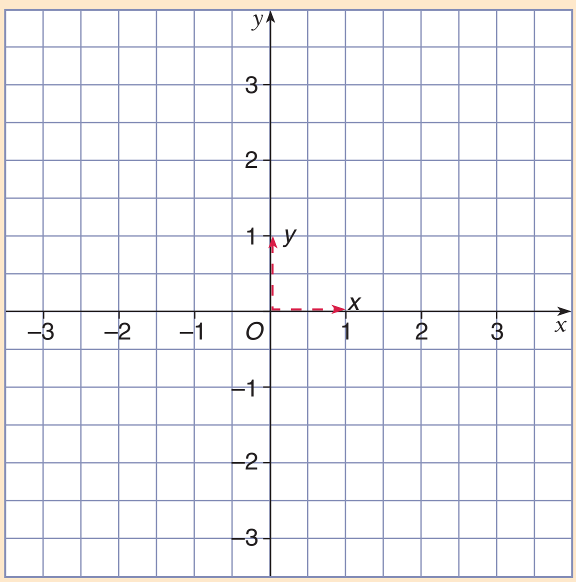
Consider the triangle *ABC* which has vertices *A*(3, 3), *B*(7, 2) and *C*(4, 0). Each of the vertices can be written as a position vector from the origin

$\vec{OA} = \begin{pmatrix} 3 \\ 3 \end{pmatrix}$, $\vec{OB} = \begin{pmatrix} 7 \\ 2 \end{pmatrix}$ and $\vec{OC} = \begin{pmatrix} 4 \\ 0 \end{pmatrix}$



As we have noted, rotating the vectors through the required degrees gives the image of ΔABC under a rotation transformation centre (0, 0).

Suppose we now consider the unit vectors $\vec{OX} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\vec{OY} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ in the *x* and *y* directions as shown in the diagram below and these are rotated through 90° about *O*. Now the image of these are $\vec{OX}' = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ and $\vec{OY}' = \begin{pmatrix} -1 \\ 0 \end{pmatrix}$



If we write these as a single matrix we obtain $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$

1 Multiply this matrix by each of the position vectors for ΔABC as shown below:

$\overrightarrow{OA'} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 3 \\ 3 \end{pmatrix}$ $\quad \overrightarrow{OB'} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 7 \\ 2 \end{pmatrix}$ $\quad \overrightarrow{OC'} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 4 \\ 0 \end{pmatrix}$

2 What do you notice about the positions of the point A', B' and C' and the resulting $\Delta A'B'C'$?

3 Now write the matrix obtained when the unit vectors $\overrightarrow{OX} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\overrightarrow{OY} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ are reflected through the x axis.

4 What would you expect to happen if you multiplied this matrix by each of the position vectors for ΔABC? Show that your conjecture is true.

5 Complete the table below. Justify all of your answers with diagrams and full working out.

6 Explain why two of the matrices are identical.

7 Predict what the unit vector matrix would be after a reflection in the line $y = x$. Justify your answer.

8 Can you think of any limitations on the use of these matrices for transformation purposes?

Transformation	Unit vector matrix	Resulting vectors for $\Delta A'B'C'$	How has ΔABC been transformed
Reflection in the x axis			
Reflection in the y axis			
Enlargement by a factor of 3, centre O			
Enlargement by a factor of −3, centre O			
Rotation of 90° about O	$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$		
Rotation of −90° about O			
Rotation of 180° about O			
Rotation of 270° about O			

Assessment Grid for Investigation 14:04 | Matrix methods

The following is a sample assessment grid for this investigation. You should carefully read the criteria *before* beginning the investigation so that you know what is required.

Assessment Criteria (B, C, D) for this investigation				Achieved ✓
Criterion B Investigating Patterns	a	None of the standards below has been reached.	0	
	b	Some help was needed to complete the tables and recognise patterns.	1	
			2	
	c	The tables have been completed independently. Patterns have been recognised and described.	3	
			4	
	d	Tables have been completed and patterns recognised and described. Conclusions consistent with the results have been made.	5	
			6	
	e	All of the above has been completed with the addition that patterns are described in full using words and symbols, and a logical response is given to 6, 7 and 8.	7	
			8	
Criterion C Communication in Mathematics	a	None of the standards below has been reached.	0	
	b	There has been a basic use of mathematical language. Lines of reasoning are hard to follow. There are some diagrams.	1	
			2	
	c	The use of mathematical language is sufficient. Movement between the table, diagrams and discussion has been carried out with some success. Lines of reasoning are clear but not always logical.	3	
			4	
	d	There is a good use of mathematical language and an effective movement between the table, diagrams and discussion. Lines of reasoning are logical and complete.	5	
			6	
Criterion D Reflection in Mathematics	a	None of the standards below has been reached.	0	
	b	There has been an attempt to connect the table with diagrams and to explain the results in context.	1	
			2	
	c	The connection between the table and diagrams has been made correctly and the explanation is correct but brief.	3	
			4	
	d	A full explanation of the connections is given and answers for 6, 7 and 8 are given in full.	5	
			6	

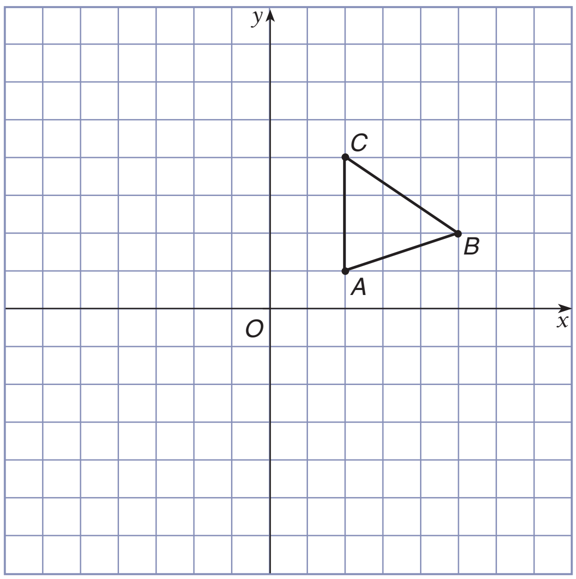
14:05 | Transformations and Matrices

As was discovered in Investigation 14:04, we can use matrices to transform each vertex on a shape to get the image of that vertex.

By considering the unit vectors $\overrightarrow{OX} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\overrightarrow{OY} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ and their images under a transformation we can determine the matrix by which to multiply the position vector of each vertex.

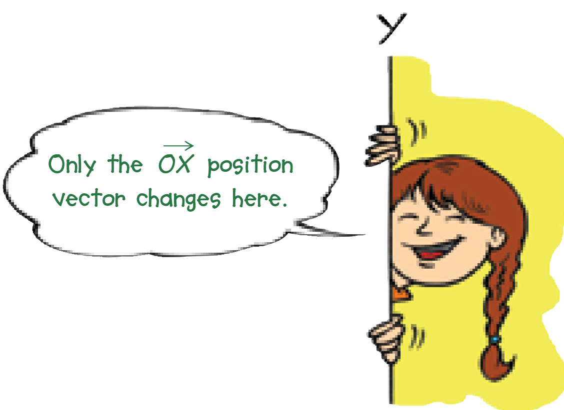
worked example

Consider ΔABC shown in the diagram.



a Use matrix multiplication to find its image after reflection in the y axis.

b Show that your answer is correct by plotting the image on a diagram.



continued →→→

Solution

a The position vectors of its vertices are $\overrightarrow{OA} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$, $\overrightarrow{OB} = \begin{pmatrix} 5 \\ 2 \end{pmatrix}$ and $\overrightarrow{OC} = \begin{pmatrix} 2 \\ 4 \end{pmatrix}$

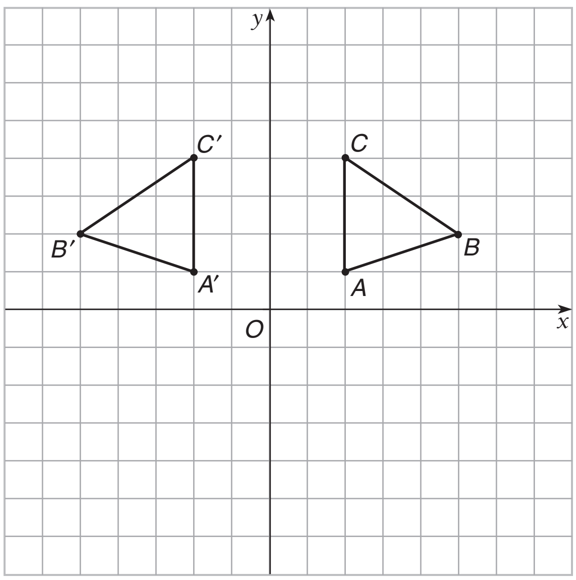
The images of unit vectors $\overrightarrow{OX} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\overrightarrow{OY} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ under reflection in the y axis are $\overrightarrow{OX}' = \begin{pmatrix} -1 \\ 0 \end{pmatrix}$ and $\overrightarrow{OY} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ respectively, giving the transformation matrix $\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$

So to find the position vectors of the image of each vertex we multiply as shown below

$$\overrightarrow{OA} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} -2 \\ 1 \end{pmatrix}, \quad \overrightarrow{OB}' = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 5 \\ 2 \end{pmatrix} = \begin{pmatrix} -5 \\ 2 \end{pmatrix} \text{ and } \overrightarrow{OC}' = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 2 \\ 4 \end{pmatrix} = \begin{pmatrix} -2 \\ 4 \end{pmatrix}$$

So the image is triangle $A'(-2, 1)$, $B'(-5, 2)$ and $C'(-2, 4)$.

b



Exercise 14:05

1 **a** Find the image of the unit vectors $\overrightarrow{OX} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\overrightarrow{OY} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ under reflection in the x axis.

b Write the transformation matrix that corresponds to a reflection in the x axis.

c Use matrix multiplication to find the image of ΔABC after being reflected in the x axis given the vertices $A(2, 0)$ $B(3, 1)$ and $C(1, 3)$.

d Show that your image is correct by plotting the information on a diagram.

2 **a** Find the image of the unit vectors $\overrightarrow{OX} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\overrightarrow{OY} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ under a rotation of –90° about O.

b Write the transformation matrix that corresponds to a rotation of –90° about O.

c Use matrix multiplication to find the image of the quadrilateral $PQRS$ given the vertices $P(-1, -1)$, $Q(-2, 3)$, $R(-5, 2)$, $S(-6, -1)$ after a rotation of –90° about O.

d Show that your image is correct by plotting the information on a diagram.

3 **a** Find the image of the unit vectors $\overrightarrow{OX} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\overrightarrow{OY} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ under an enlargement by factor 3, centre of enlargement O.

b Write the transformation matrix that corresponds to an enlargement by factor 3, centre of enlargement O.

c Use matrix multiplication to find the image of the ΔLMN given the vertices $L(-1, -2)$, $M(-2, 3)$, $N(3, 2)$ after an enlargement by factor 3, centre of enlargement O.

d Show that your image is correct by plotting the information on a diagram.

4 **a** By considering the unit vectors $\overrightarrow{OX} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\overrightarrow{OY} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, write the transformation matrix for:

- **i** A reflection in the y axis.
- **ii** A reflection in the x axis.
- **iii** A rotation of 180° about O.

b A triangle has vertices $D(-7, 0)$, $E(-3, -2)$ and $F(-6, -7)$.

- **i** Find the vertices of $\Delta D'E'F'$, the image of ΔDEF after reflection in the y axis.
- **ii** Find the vertices of $\Delta D''E''F''$, the image of $\Delta D'E'F'$ after reflection in the x axis.
- **iii** Find the vertices of $\Delta D'''E'''F'''$, the image of ΔDEF after rotation of 180° about O.

c Complete this statement:
'The image of a shape after reflection in the y axis and then the x axis is the same as

______________________________ about the origin.'

d Confirm your statement by plotting ΔDEF, $\Delta D'E'F'$, $\Delta D''E''F''$ and $\Delta D'''E'''F'''$ on a number plane.

5 **a** By considering the unit vectors $\overrightarrow{OX} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\overrightarrow{OY} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, write the transformation matrix for a reflection in the line $y = x$.

b Check that your transformation matrix is correct by applying it to ΔPQR which has vertices $P(2, 2)$, $Q(0, -2)$ and $R(4, -2)$.

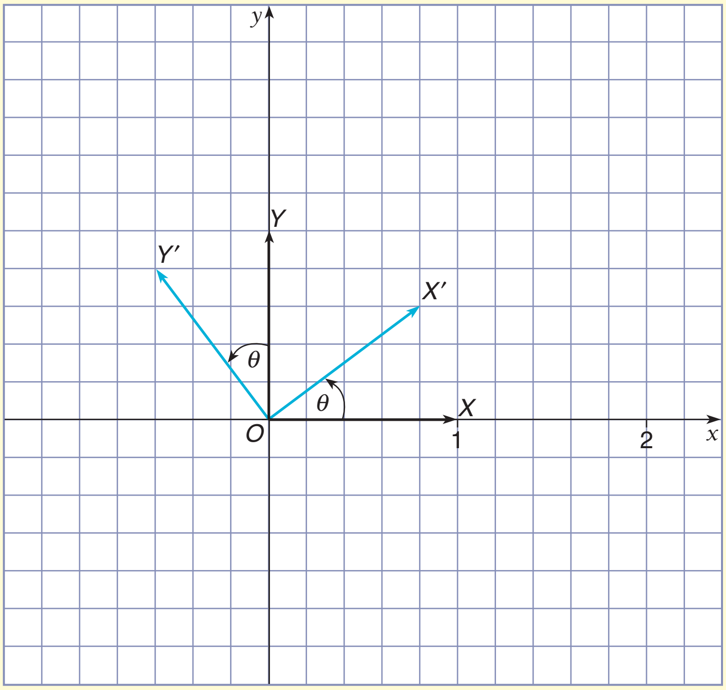
c Confirm your result with a diagram.

6 A shape is transformed by using the transformation matrix $\begin{pmatrix} 0 & \frac{1}{2} \\ -\frac{1}{2} & 0 \end{pmatrix}$. This represents two successive transformations.

a What are they?

b Prove this by the matrix multiplication of a point using the matrix given, and then by the two successive matrices identified in (a).

7 Consider the unit vectors $\overrightarrow{OX} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\overrightarrow{OY} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$. Suppose they are rotated through $\theta°$ about O as shown in the diagram.



a Show that the vector $\overrightarrow{OX'} = \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}$ and that the vector $\overrightarrow{OY'} = \begin{pmatrix} -\sin\theta \\ \cos\theta \end{pmatrix}$

b Hence, write the transformation matrix for rotation about O of 90° and find the image of ΔDEF with vertices $D(2, 1)$ $E(7, 1)$ $F(7, 6)$.

c Sketch ΔDEF and its image $\Delta D'E'F'$.

d Determine, by substitution into the matrix found in (b), the transformation matrix for a rotation of 90° about O.

Mathematical Terms 14

mathematical terms 14

enlargement
- When a shape is transformed so that it retains the same basic shape but the image is bigger or smaller than the original.

enlargement factor
- By how many times a shape is enlarged.

image
- The shape that is arrived at after a transformation.

line of symmetry or line of reflection
- The line through which a shape is reflected.

reflection
- When a shape is transformed so that its image is the reverse of the original but the size and shape remains unchanged.

rotation
- When a shape is transformed so that it is rotated about a point to obtain the image.

transformation
- When a shape is changed in some way so that it becomes a different shape either in size or orientation.

transformation matrix
- A matrix which when multiplied by a vertex, gives its image under a transformation.

translation
- When a shape has been moved, but its size and orientation remain the same.

Technology Applications

Mathematical terms 14

Drag and Drops

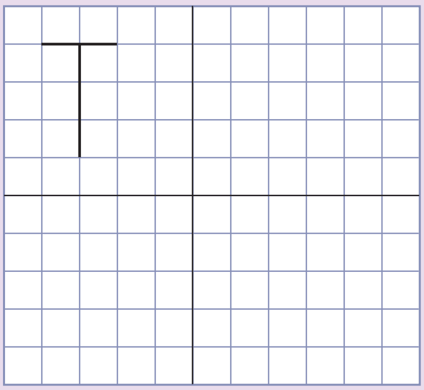
Diagnostic Test 14: | Transformation and matrices

- These questions reflect the important skills introduced in this chapter.
- Errors made will indicate areas of weakness.
- Each weakness should be treated by going back to the section listed.

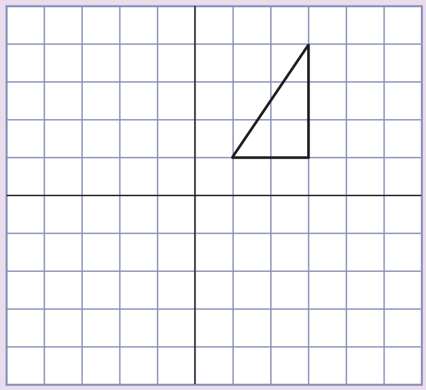
Section

1 Translate the following shapes by the given vector. — 14:01

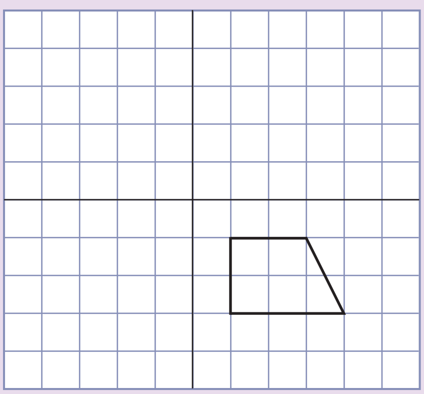
a $\begin{pmatrix} 3 \\ -2 \end{pmatrix}$



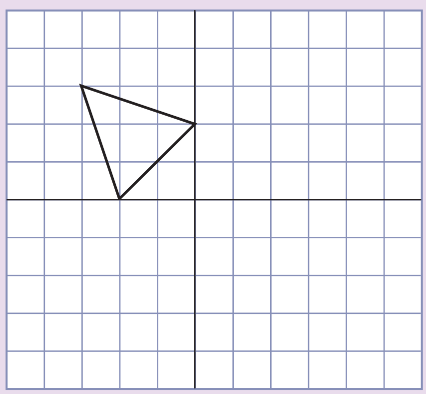
b $\begin{pmatrix} 4 \\ -3 \end{pmatrix}$



c $\begin{pmatrix} -4 \\ 4 \end{pmatrix}$ — 14:01

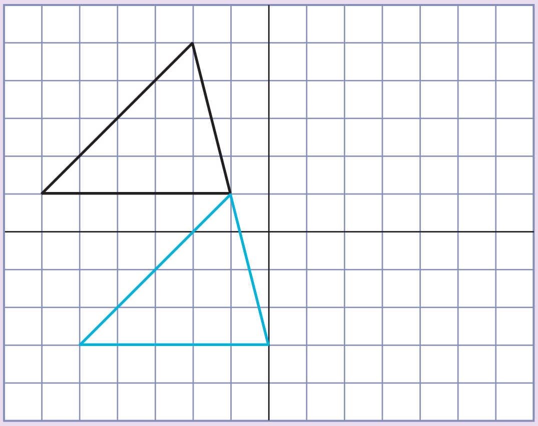


d $\begin{pmatrix} 5 \\ 0 \end{pmatrix}$

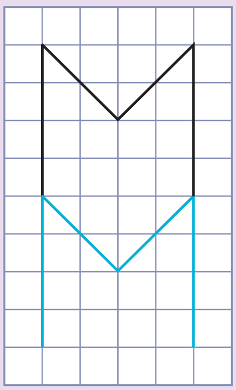


2 Describe the following transformations in words and in vector form. The images are shown in blue.

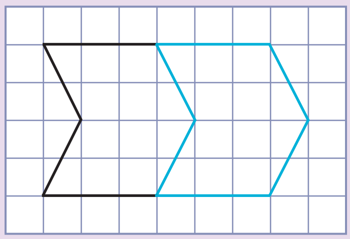
a



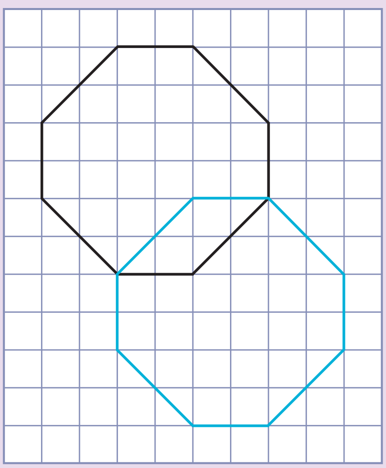
b



c



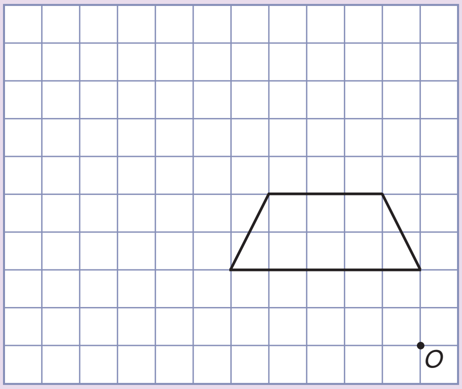
d



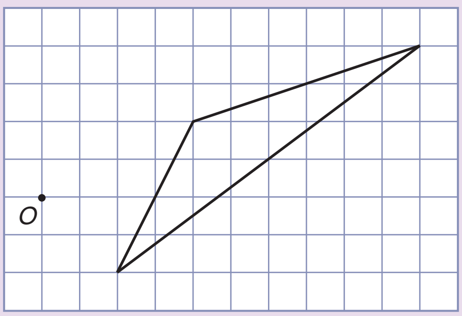
Section 14:02

3 Enlarge the following shapes by the enlargement factor given with centre of enlargement O.

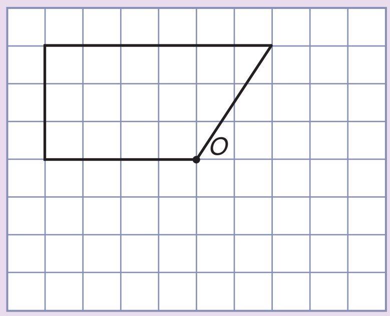
a Enlargement factor 2



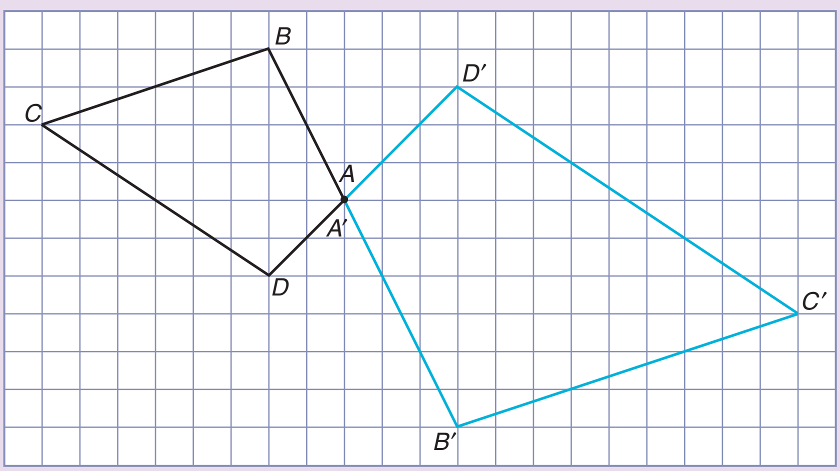
b Enlargement factor $\frac{1}{2}$



c Enlargement factor -1

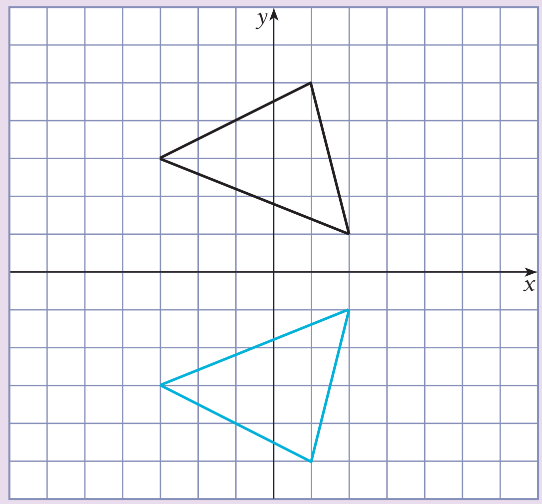


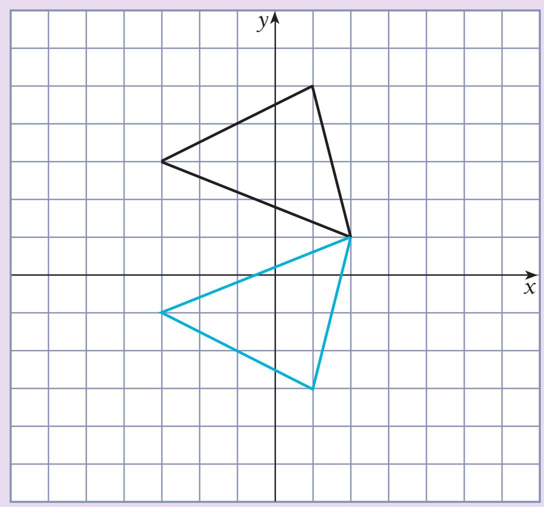
4 Describe the following transformation in words. The image is shown in blue. 14:02

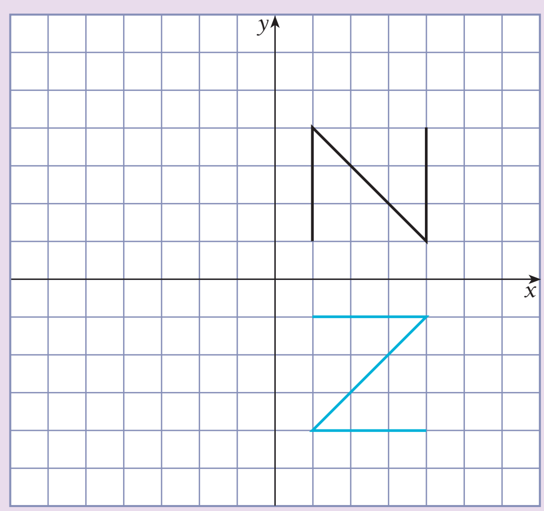


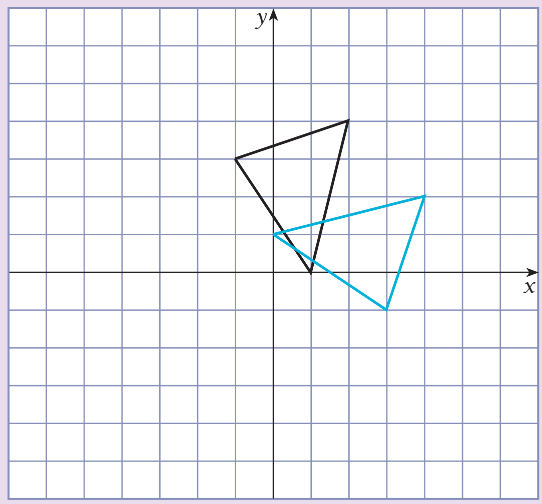
Section
14:03, 14:04

5 Describe the following transformations in words. The images are shown in blue.

a


b


c


	Section
d 	14:03, 14:04
6 By considering the unit vectors $\overrightarrow{OX} = \begin{pmatrix}1\\0\end{pmatrix}$ and $\overrightarrow{OY} = \begin{pmatrix}0\\1\end{pmatrix}$ find the transformation matrices for the following: **a** Reflection in the y axis. **b** Reflection in the line $y = x$. **c** Rotation of 90° about O. **d** Enlargement by a factor of −4, centre O.	14:05
7 Find the vertices of the image of ΔABC after a rotation of 60° about O if $A(8, -1)$, $B(0, 1)$, $C(5, 3)$	

Chapter 14 | Revision Assignment

assignment

14A

1 A shape has been translated by successive vectors: $\begin{pmatrix}-2\\1\end{pmatrix}$ and $\begin{pmatrix}-4\\3\end{pmatrix}$.
What single vector would translate the shape back to its original position?

2 Triangle $A(6, 4)$ $B(5, 8)$ $C(8, 3)$ is enlarged by a factor of 3 using A as the centre of enlargement. Find the coordinates of the image $A'B'C'$.

3 Line MN has endpoints $M(3, 4)$ and $N(-1, 1)$ and is reflected in the line $y = 1 + x$. Find the coordinates of the image $M'N'$.

4 Triangle PQR has vertices $P(2, 3)$, $Q(1, 0)$ and $R(4, 1)$. Find the coordinates of the image of triangle ABC after a rotation of 90° about the point $(-1, 1)$.

assignment

14B

Chapter 14 | Working Mathematically

1 A point $(a, -5)$ has been translated by the vector $\begin{pmatrix} 4 \\ -3 \end{pmatrix}$ to the point $(1, b)$ find a and b.

2 Under enlargement, the image of a point $P(3, -7)$ is $P'(-3, 5)$. If the enlargement factor is 3, find the centre of the enlargement.

3 Triangle ABC has coordinates $A(3, 3)$ $B(-1, -1)$ and $C(-2, 3)$ and is reflected so its image $A'B'C'$ has coordinates $A'(6, 0)$ $B'(6, -5)$ and $C'(2, -4)$.

4 a Find the image of the unit vectors $\overrightarrow{OX} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\overrightarrow{OY} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ under reflection in the y axis.

b Write the transformation matrix that corresponds to a reflection in the y axis.

c Use matrix multiplication to find the image of triangle ABC after being reflected in the y xis given the vertices $A(2, 0)$ $B(3, 1)$ $C(1, 3)$.

15 Statistics

Chapter Contents

Learning Outcomes

Students will be able to:

- Work with data arranged in unequal intervals.
- Use standard deviation and the mean to compare sets of data.
- Understand the normal distribution.
- Find a line of best fit for a set of data.
- Use correlation to compare sets of data.

Areas of Interaction

Approaches to Learning (Knowledge Acquisition, Logical Thinking, Communicating, Reflection), Human Ingenuity, Environments

15:01A | Review: Representing Data

In Book 4 you learnt how to represent data in a number of ways:

Frequency Distribution Tables

worked example

The percentage results for sixty students in an examination were:

78	63	89	55	92	74	62	69	43	90	91	83	49	37	58
73	78	65	62	87	95	77	69	82	71	60	61	53	59	42
43	33	98	88	73	82	75	63	67	59	57	48	50	51	66
73	68	46	69	70	91	83	62	47	39	63	67	74	52	78

To organise this data into a table we use **class intervals** or **groups**: 29–37, 38–46 etc.

Class	*Class centre (c.c.)*	*Tally*	*Frequency (f)*	*Cumulative frequency*
29–37	33	||	2	2
38–46	42	||||̸	5	7
47–55	51	||||̸ |||	8	15
56–64	60	||||̸ ||||̸ ||	12	27
65–73	69	||||̸ ||||̸ ||||	14	41
74–82	78	||||̸ ||||	9	50
83–91	87	||||̸ ||	7	57
92–100	96	|||	3	60
		Totals:	60	

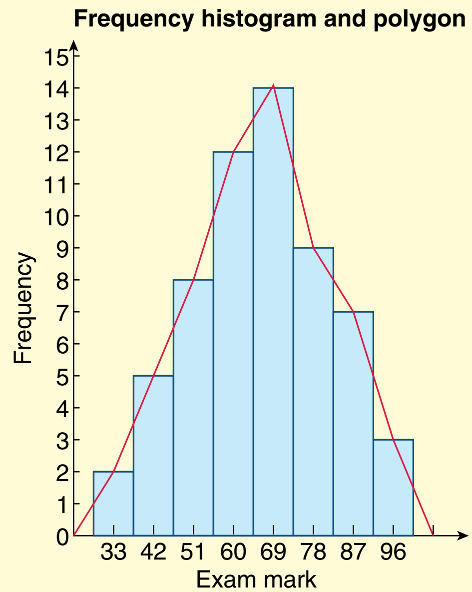
Frequency Histograms and Polygons

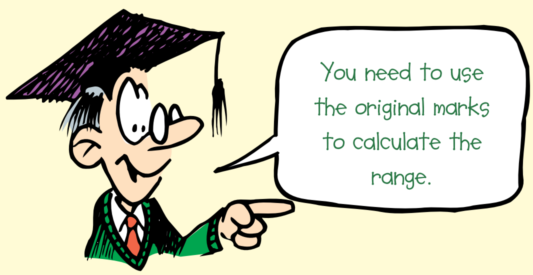
To give a visual representation of the data we can arrange it into a column graph called a **frequency histogram**, and a line graph called a **frequency polygon**.

In these graphs the class centres are used for the middle of the column and to plot the line.

When constructing frequency diagrams for grouped data, the only point to note is that the columns are indicated on the horizontal axis by the class centres. The diagrams for the worked example above would look like these.

- The modal class 65–73 is represented by the class centre 69.
- The **frequency polygon** can be drawn by joining the midpoints of the tops of columns. To complete the polygon, assume that the classes on either side of the columns have zero members.



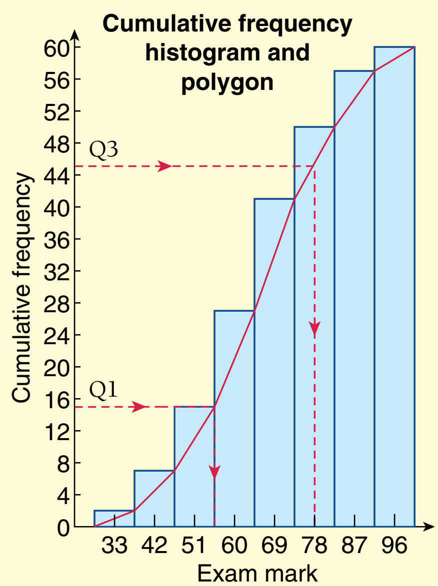


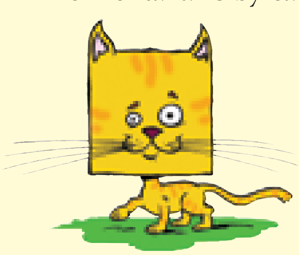
Note: the area under the columns is the same as the area under the line — this is important as it represents the total number of pieces of data — in this case 60.

Cumulative Frequency Histograms and Polygon

The cumulative frequency histogram is plotted in the same way using the respective numbers from the table. The polygon in this case, however, is a little different.

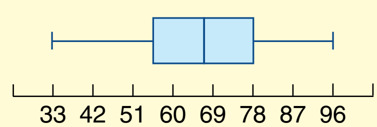
- The **cumulative frequency polygon** can be drawn by joining the top right corners of each column.
- There are 60 scores altogether so to find the median class we come across from 30 until we meet the polygon and then down to the horizontal axis.
- Clearly the median class is 65–73.
- An estimate of the median mark can be read from the horizontal axis, ie 67.
- The inter-quartile range can be calculated from the horizontal axis by calculating Q3 – Q1 = 78 – 56 = 22.





Box and Whisker Plots

A box and whisker plot gives a visual representation of the spread of the scores by showing the median and the inter-quartile range as shown:



Exercise 15:01A

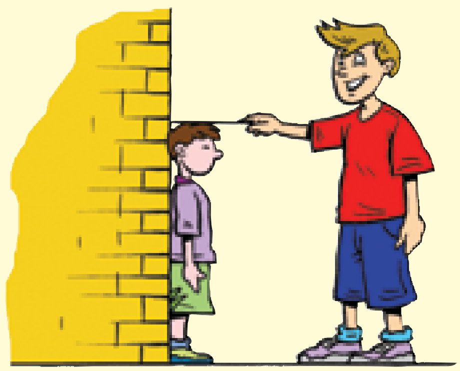
1 The following table shows the distribution of marks in a mathematics test.

Class	*Class centre*	*Frequency*	*Cumulative frequency*
30–34		1	
35–39		3	
40–44		6	
45–49		7	
50–54		6	
55–59		5	
60–64		2	

a Complete the table.
b Construct a frequency histogram and polygon.
c Construct a cumulative frequency histogram and polygon.
d Use your cumulative frequency polygon to estimate:
  i the median **ii** the first quartile Q1
  iii the third quartile Q3 **iv** the inter-quartile range.
e Construct a box and whisker plot of this data.

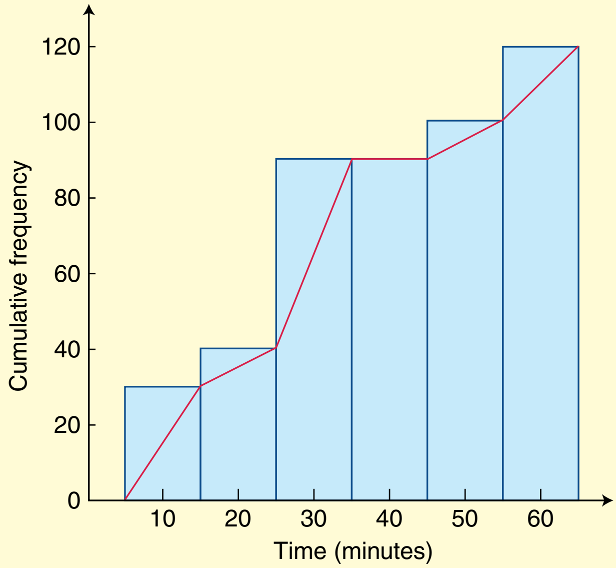
2 The following table shows the heights of 50 Grade 10 boys when measured to the nearest centimetre.

Height (cm)	*Class centre*	*Frequency*	*Cumulative frequency*
146–150			2
151–155			9
156–160			18
161–165			36
166–170			44
171–175			49
176–180			50



a Complete the table.
b Construct a frequency histogram and polygon.
c Construct a cumulative frequency histogram and polygon.
d Use your cumulative frequency polygon to estimate:
  i the median **ii** the first quartile Q1
  iii the third quartile Q3 **iv** the inter-quartile range
e Construct a box and whisker plot of this data.

3 The cumulative frequency histogram below shows the time taken for students in Keishi's grade to travel to school in the morning.



a How many students are in Keishi's grade?

b Complete the following table from the information in the histogram.

Time (min)	*Class centre*	*Frequency*	*Cumulative frequency*
$5 \leqslant x < 15$	10		
$15 \leqslant x < 25$			

c This table has described the groups differently than in the previous two questions. Explain why this might be the case.

d Using the graph, estimate

- **i** the median
- **ii** the first quartile Q1
- **iii** the third quartile Q3
- **iv** the inter-quartile range

e Construct a box and whisker plot of the data.

4 The table below shows the distance from Bangkok airport to a random sample of international destinations.

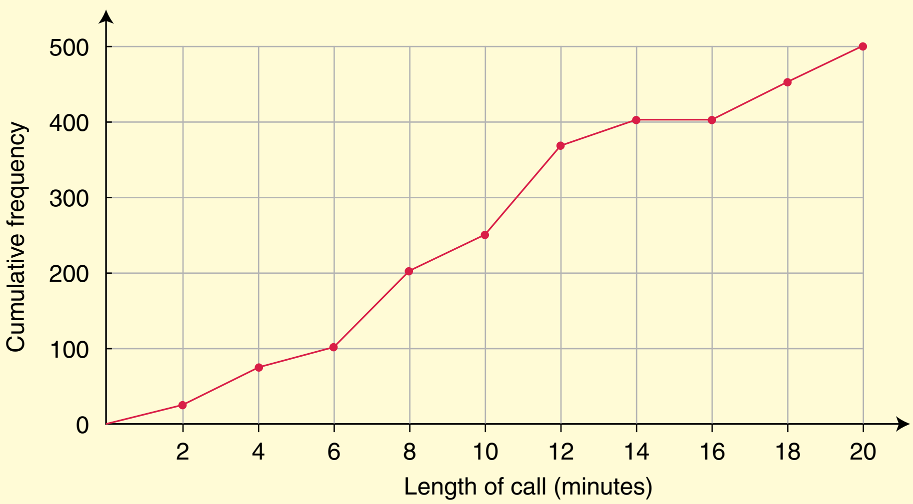
Destination	*Distance (km)*	*Destination*	*Distance (km)*
Auckland	5944	Johannesburg	5574
Athens	4920	Kunming	790
B.S. Begawan	1154	Los Angeles	8260
Brisbane	4522	Madrid	6314
Busan	2305	Muscat	2833
Chengdu	1188	Nagoya	2701
Chittagong	823	New York	8656
Copenhagen	5350	Phnom Penh	329
Dhaka	960	Seoul	2304
Frankfurt	5574	Sydney	4679
Ho Chi Minh	461	Tokyo	2879
Hong Kong	1076	Yangon	363
Isalamabad	2197	Zurich	5608

a Complete this frequency distribution table to summarise the distances.

Distance (km)	*Class centre*	*Frequency*	*Cumulative frequency*
0–			
1000–			
2000–			
3000–			
4000–			
5000–			
6000–			
7000–			
8000–			

b Draw a cumulative frequency polygon to estimate:
- **i** the median.
- **ii** the lower quartile Q1
- **iii** the upper quartile Q3
- **iv** the inter-quartile range.

5 A telephone exchange records the length (in minutes) of all international phone calls. The cumulative frequency polygon below shows the length of 500 international calls made from Singapore on Christmas Day.



a Arrange this information into a frequency distribution table.

b From the polygon, estimate values for
- **i** the median.
- **ii** The lower quartile Q1
- **iii** The upper quartile Q3
- **iv** The inter-quartile range.

c Construct a box and whisker plot of the data.

15:01B | Review: Analysing Data

As covered in Book 4, there are a number of measures which help us analyse and interpret data.

Some of these give us information about the location of the **middle** of the data. These are called **measures of central tendency**. These are:

Mean ($\bar{x}$) $= \dfrac{\text{Sum of the scores}}{\text{The number of scores}} = \dfrac{\sum fx}{n}$

Median = middle score when they are arranged in ascending order

Mode = the score (or group of scores) with the highest frequency

Other measures tell us about the spread of the scores. Some of these are better than others:

Range = highest score − lowest score
This only uses 2 scores and does not take into account outlying scores

Inter-quartile range = 3rd quartile − 1st quartile
Although this only measures the spread of the middle 50 per cent of the scores, it still only uses two scores in its calculation.

Standard deviation (σ_n) = average distance of the scores from the mean.
This is the best measure of spread since it uses every score in its calculation.

Measures of central tendency

worked example

Consider the following sets of scores:

Set A: 16, 12, 13, 11, 13, 14, 9, 15, 15, 12

Set B:

Score	38	39	40	41	42	43	44
Frequency	1	4	8	9	4	1	1

To find the mean of the scores we need to find the sum of the scores ($\sum x$) and the number of scores (n)

For set A

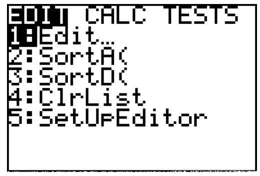
$$\bar{x} = \frac{\sum n}{n} = \frac{16 + 12 + 13 + 11 + 13 + 14 + 9 + 15 + 15 + 12}{10}$$
$$= 13$$

For set B

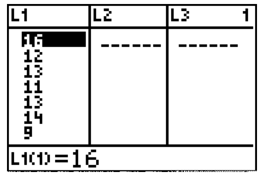
$$\bar{x} = \frac{\sum fx}{n} = \frac{1 \times 38 + 4 \times 39 + 8 \times 40 + 9 \times 41 + 4 \times 42 + 1 \times 43 + 1 \times 44}{28}$$
$$= 40{\cdot}6 \text{ (3 significant figures)}$$

These can also be done on your calculator. The steps required depend on the calculator you use. Here are the steps for the TI-83 or the TI-84.

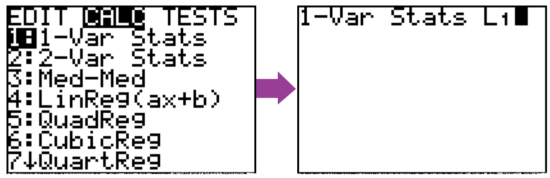
1 Choose statistics by pressing STAT and choosing 1:Edit



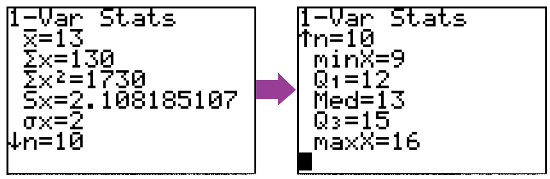
2 This will take you to a table where you need to enter the individual scores. These are for set A.



3 Now go back to STAT and highlight CALC and choose 1: 1-var stat as there is only 1 column of information. You then need to input the column where the information is located. In this case it is in column L1 – or list 1.



4 By scrolling through the next window we can read off the following data:



the mean $\bar{x}$ = 13

the sum of the scores $\sum x$ = 130

the sum of the scores squared $\sum x^2$ = 1730

the standard deviation of the sample $Sx = 2{\cdot}11$

the standard deviation of the population $\sigma x = 2$

the number of scores $n = 10$

the minimum score $min\ X = 9$

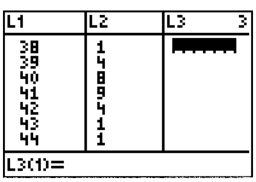
the first quartile $Q_1 = 12$

the median $Med = 13$

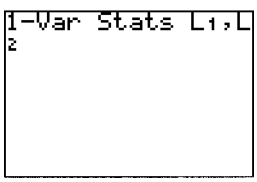
the third quartile $Q_3 = 15$

the maximum score $max\ X = 16$

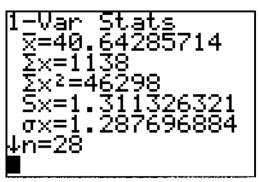
We can do the same for set B by using 2 columns.

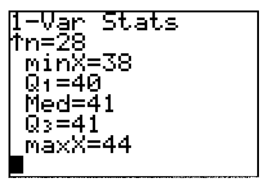


Input scores and frequency in two columns.



This time you need to input two columns, the first the score and the second the frequency.





The output is in the same format as before.

continued →→→

This also works for grouped scores; however, the centres must be put in the score column.

To find the mode: This is the score with the highest frequency.

In the case of set A, the mode = 12, 13 and 15 as they all have a frequency of 2.

In the case of set B, the mode = 41 as it has a frequency of 9.

To find the median: This is the middle score when the scores are arranged in ascending order.

In the case of set A: 16, 12, 13, 11, 13, 14, 9, 15, 15, 12

Becomes 9, 11, 12, 12, (13, 13,) 14, 15, 15, 16

As there are 10 scores, there are two middle scores: the 5th and 6th. The median is the mean of these two scores: 13.

In the case of set B the scores are already arranged in order.

By adding the cumulative frequency column we get

Score	38	39	40	41	42	43	44
Frequency	1	4	8	9	4	1	1
C. frequency	1	5	13	22	26	27	28

The 14th to the 22nd scores are here.

As there are 28 scores, the 14th and 15th are the middle scores. Both of these are 41. So the median = 41

The median can also be obtained from the calculator as shown above.

For grouped scores, the median must be found from the cumulative frequency polygon as shown in the previous section.

Measures of spread

Suppose we take the same two sets of data:

Set A: 16, 12, 13, 11, 13, 14, 9, 15, 15, 12

Set B:

Score	38	39	40	41	42	43	44
Frequency	1	4	8	9	4	1	1

To find the range: Subtract the lowest score from the highest score

For set A the range = 16 – 9
= 7

For set B the range = 44 – 38
= 14

To find the inter-quartile range: Subtract the lower quartile (Q_1) from the upper quartile (Q_3).

For set A: we must find the scores which mark the quarters

9, 11, (12,) 12, 13, 13, 14, (15,) 15, 16

Q_1 Q_2 or median Q_3

Inter-quartile range = 15 – 12
= 3

For set B: we again use the cumulative frequency table

Score	38	39	40	41	42	43	44
Frequency	1	4	8	9	4	1	1
C. frequency	1	5	13	22	26	27	28

There are 28 scores, which can be divided into 4 equal lots of 7 scores.

$\therefore Q_1$ = the mean 7th and 8th scores
= 40

$\therefore Q_3$ = the mean 21st and 22nd scores
= 41

Inter-quartile range = 41 – 40
= 1

> ***Note*: the median is another name for the second quartile Q_2**
> **Q_2 = the mean 14th and 15th scores**
> **= 41**

These can also be obtained from your calculator in the same way as the mean.

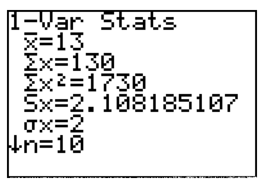
For grouped scores, the quartiles must be found from the cumulative frequency polygon as shown in the previous section.

To find the standard deviation:

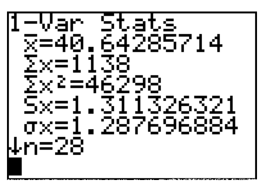
This must be obtained from your calculator by first entering the scores as shown above.

As mentioned before, there are two standard deviations. Since our data is not a sample, but represents the whole population, we use σx for standard deviation.

For set A the standard deviation $\sigma x = 2$



For set B the standard deviation $\sigma x = 1{\cdot}28$



Exercise 15:01B

For each of the questions in exercise 15:01A find

a the mean **b** the mode **c** the range **d** the standard deviation.

investigation 15:01

Investigation 15:01 | Comparing sets of data

Please use the Assessment Grid on page 451 to help you understand what is required for this Investigation.

The object of this investigation is to use different measures to compare sets of scores with each other.

1 Gerui and Maher have a holiday job picking apples. The lists below show how many buckets of apples picked over a 17-day period.

Gerui: 65, 73, 86, 90, 99, 106, 45, 92, 94, 102, 97, 107, 107, 99, 83, 101, 91

Maher: 49, 84, 95, 99, 103, 102, 95, 103, 100, 99, 108, 0, 96, 105, 102, 97, 95

Either by entering these sets of data into your calculator, or by constructing a frequency table, complete the following table.

	Gerui	*Maher*
Mean		
Median		
Range		
Q1		
Q2		
IQR		

What does this information tell you about the data?

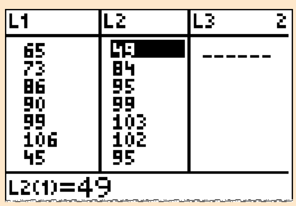
Make a comparison about

i the middles of the sets of data

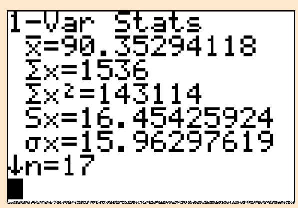
ii the spread of the data in each set.

TI Calculator Instructions

Using STAT EDIT enter the data.



Calculate the statistics on each set of data.

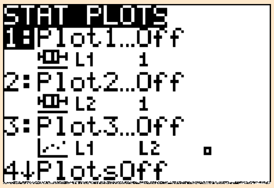


Construct box and whisker plots of both sets of data. This can be done on the TI calculator following the steps below:

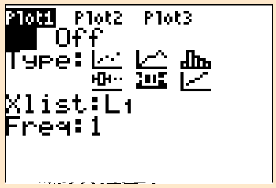
TI Calculator Instructions

You have the data entered into two lists.

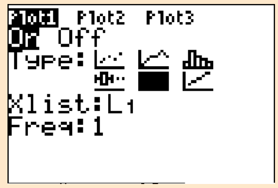
Use STAT PLOT



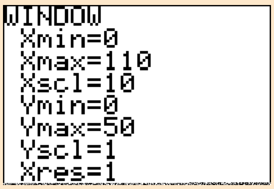
Turn the plot ON



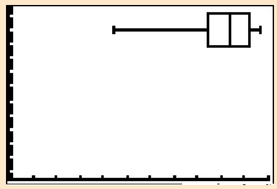
Choose Box and Whisker



Choose an appropriate window



Graph

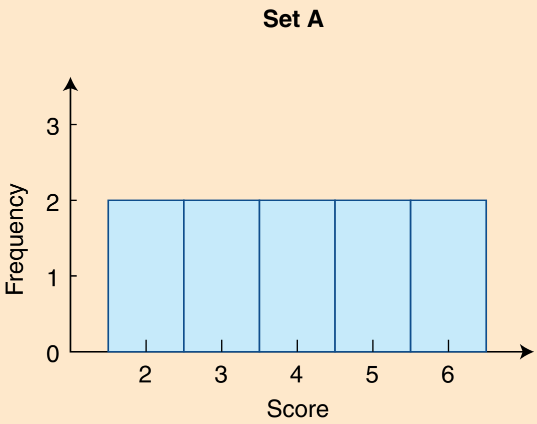


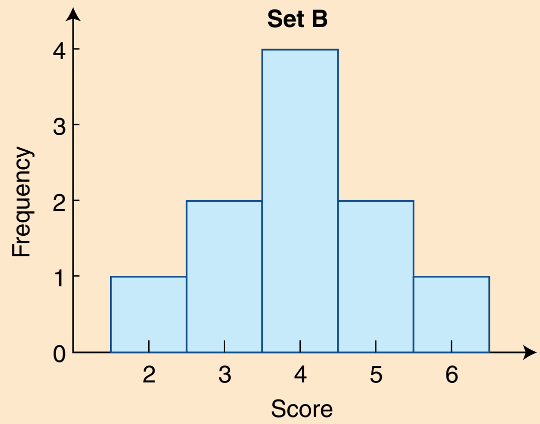
Repeat for the other set of data and you will have two box plots, one on top of the other.

Explain how the box and whisker plots together give a good visual comparison of the sets of data.

Which is a better measure of the spread of data in this case, the range or the inter-quartile range? Why?

2 Consider the following sets of data.





i Describe the spread of the scores from the histograms provided.
Complete the following table using your calculator or otherwise and construct box and whisker plots to represent this information visually.

	Set A	*Set B*
Mean		
Median		
Range		
Q1		
Q2		
IQR		

ii Describe the spread of the scores from the box and whisker plots you constructed.
iii Do you think the range or the inter-quartile are a good measure of the spread of the scores in this case?

Now calculate the standard deviation of each set of scores using your calculator as shown in section 15:01B.

Do you think the standard deviation is a better measure of the spread of these sets of scores? Why?

You might want to refer back to section 15:01B to help with your answer.

3 Suppose the height of an object dropped from a tall building is modelled by the quadratic function $y = 180 - 9{\cdot}8x^2$, where y is the height after x seconds. Find the equation of the inverse of this function and use it to find:
a when the height is 100 m.
b how long the object is in the air.

Assessment Grid for Investigation 15:01 | Comparing sets of data

The following is a sample assessment grid for this investigation. You should carefully read the criteria *before* beginning the investigation so that you know what is required.

Assessment Criteria (C, D) for this investigation				Achieved ✓
Criterion C Communication in Mathematics	a	None of the following descriptors have been achieved.	0	
	b	There is a basic use of mathematical language and representation. Lines of reasoning are insufficient.	1	
			2	
	c	There is satisfactory use of mathematical language and representation. Graphs, tables and explanations are clear but not always logical or complete. Calculations are easy to follow.	3	
			4	
	d	A good use of mathematical language and representation. Graphs and tables are accurate, to scale and fully labeled. Explanations are complete and concise. Mathematical arguments are well written with explicit terminology and support.	5	
			6	
Criterion D Reflection in Mathematics	a	None of the following descriptors have been achieved.	0	
	b	An attempt has been made to explain whether the results make sense and are consistent, and to use them to make comparisons between data sets.	1	
			2	
	c	There is a correct but brief explanation of whether results make sense and how they can be used to make meaningful comparisons between different groups.	3	
			4	
	d	All comparisons of data sets are well written and detailed, showing reflection on the statistical results from analyses. Measures of centre and spread are calculated to an appropriate degree of accuracy and critically compared to each other, considering pros and cons. All statistics are combined used effectively to draw meaning and relevance to the groups being compared.	5	
			6	

15:02 | Using the Standard Deviation

Investigation 15:01 demonstrated that the range and inter-quartile range, although useful in some cases, do not always give a good indication of the spread of the data.

- The standard deviation is better because it measures the average distance of scores from the mean, or the middle of the data.

To find the standard deviation we first of all work out the distance between each score and the mean:
- Since some of these are negative, each is squared.
- These are now added.
- The mean of these is now calculated.
- Now we need to find the square root as the distances were squared in the first place.

This is the formula for standard deviation.

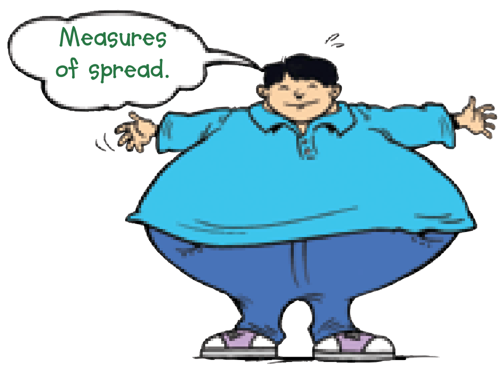
- The greater the standard deviation, the more spread out the scores are from the mean.

However, we rely on the calculator to calculate the standard deviation by entering the scores.

worked examples

The last 10 assignment results for Thilo and Sabrina are shown below.

Thilo: 40, 60, 60, 58, 50, 55, 61, 90, 71, 75
Sabrina: 60, 40, 58, 68, 58, 59, 61, 65, 90, 57



a Complete the table:

	Thilo	*Sabrina*
Mean		
Maximum score		
Minimum score		
Standard deviation		

b Whose scores were spread least from the mean?
c Who performed most consistently?

Solution

a

	Thilo	*Sabrina*
Mean	62	61·4
Maximum score	90	90
Minimum score	40	40
Standard deviation	13·2	11·4

b Sabrina, since the standard deviation of her scores is less.
c Sabrina. Her scores are less spread out.

Exercise 15:02

1 Ted and Anna have been keeping a record of the time (in seconds) it takes them to swim one length of the school's 25-metre pool. Their last eight times are shown below:

Ted: 25·0, 26·1, 21·4, 22·8, 24·7, 25·6, 27·0, 28·9

Anna: 28·3, 30·3, 23·2, 26·0, 27·5, 27·9, 28·0, 28·2

a Complete the table

	Ted	*Anna*
Mean		
Maximum time		
Minimum time		
Standard deviation		

b What is the range of scores for each swimmer?

c Who is the most consistent swimmer based on these statistics?

d Explain the last two answers.

2 The school's soccer coach has kept records of the number of shots at goal players have made per game. Below are the results for the team's top two players in the first 14 games of the season:

Mattheus: 1, 1, 3, 2, 3, 2, 5, 4, 6, 7, 4, 5, 6, 7

Emil: 5, 2, 5, 4, 3, 7, 6, 4, 2, 6, 1, 4, 3, 4

a Calculate the mean and standard deviation for each player.

b Which player is more likely to have close to 4 shots at goal in the next game?

c If the coach had to pick just one of these two players, why might he pick Mattheus? What allows him to make this decision?

3 Two mini marts on opposite sides of the street record how many customers they have per hour over a 10-day period. The results are shown below:

Max's mini-mart: 5, 6, 5, 8, 9, 8, 7, 6, 7, 9

Sam's mini-mart: 7, 8, 5, 6, 7, 8, 9, 6, 7, 7

a Calculate the mean and inter-quartile range for each mini-mart.

b Is it possible to tell from these statistics which mini-mart has the most regular number of customers? Why?

c Use the standard deviation to decide which has the most regular number of customers.

4 Hillary and Bron are comparing their end-of-semester reports They score grades out of 7 in each of their six subjects (a grade of 7 is the highest). Their scores are shown below.

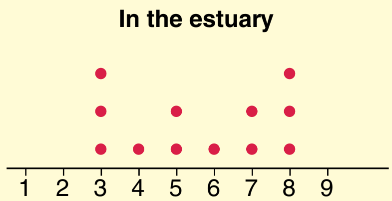
	Hillary	*Bron*
English	4	3
Mathematics	5	4
History	3	4
IT	4	5
Science	3	6
Mandarin	7	4

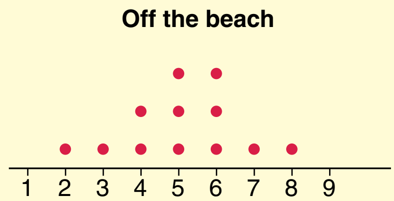
a Calculate their grade point average (the mean of their grades).

b By looking at their scores, who do you think is most consistent between subjects?

c Show this using standard deviation.

5 Huy has two places where he sets traps to catch crabs. To find out which produces more consistent results, he graphed the results of each place over 12 days. The results are shown in the dot plots below. Each day Huy put a dot above the number of crabs in each trap.

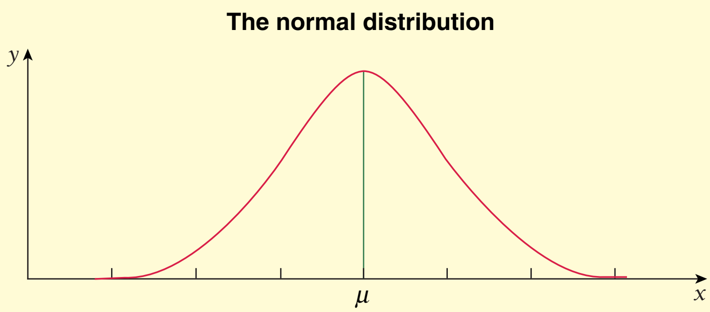




a Calculate the mean and standard deviation for each place.
b Which place gives the most consistent results?
c Why might Huy choose to use the other place regardless of the statistics?

15:03 | The Normal Distribution

Usually, when a large sample is taken and the results graphed, a *normal* or *bell-shaped* curve is obtained.



Examples of statistics which might produce a normal curve come from populations or samples:

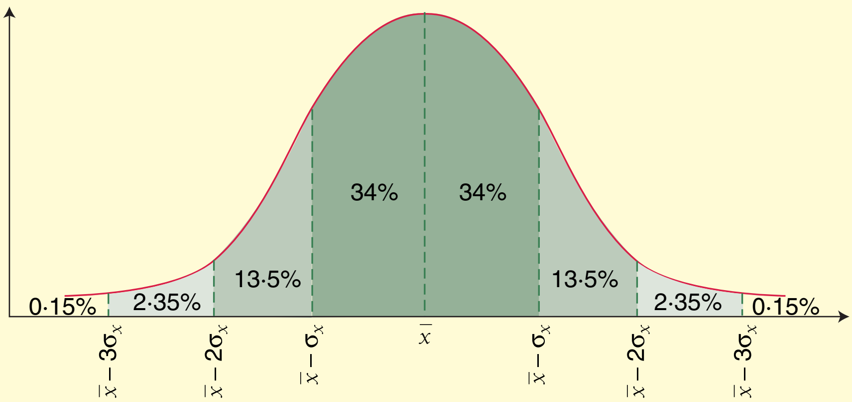
- The mass of Grade 10 boys in the country
- The number of matches in a box produced by a factory
- The length of 10 cm nails produced by a machine
- The results of a statewide spelling test.

Although some will be less than the mean and some will be more than the mean, the distribution of the population should be symmetrical about the mean.

> ■ Definition: In a normal distribution:
> Approximately 68% of the scores lie within one standard deviation of the mean.
> Approximately 95% of the scores lie within two standard deviations of the mean.
> Approximately 99·7% of the scores lie within three standard deviations of the mean.

This suggests that the standard deviation is important when using, measuring and defining a normal distribution.

Percent of normal distribution scores in each interval



worked examples

A machine is used to fill 2-litre tins with paint. It has been found that the amount of paint in the tins has a mean of 2 litres, with a standard deviation of 10 mL.

Approximately what proportion of the tins contain

a between 1·990 litres and 2·010 litres of paint?

b between 1·980 litres and 2·020 litres of paint?

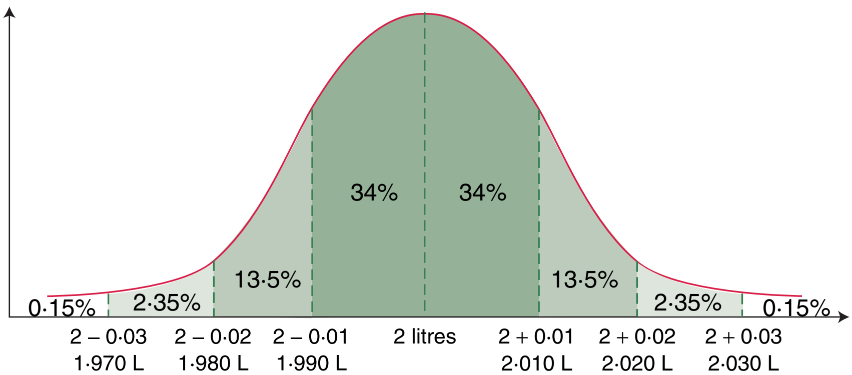
c more than 2·010 litres of paint?

d less than 1·970 litres of paint?

continued →→→

Solution

Using the diagram of a normal distribution on which the percentages are marked:



a between 1·990 litres and 2·010 litres = 34% + 34% = 68% of the tins

b between 1·980 litres and 2·020 litres = 13·5% + 34% + 34% + 13·5% = 95% of tins

c more than 2·010 litres = 100% − (50% + 34%) = 16%

d less than 1·970 litres = 0·15%

When considering the probability of choosing a score at random, the following terminology is usually used.

> If a score is chosen at random from a normal distribution:
> It will *most probably* lie within one standard deviation of the mean.
> It will be *very likely* that it lies within two standard deviations of the mean.
> It will *almost certainly* lie within three standard deviations of the mean.

Exercise 15:03

1 A machine is producing 5 cm screws. It is found that the screws it produces are in normal distribution with a mean of 5 cm and a standard deviation of 0.8 mm.

i Approximately what percentage of the screws produced have a length:

a between 4·84 cm and 5·16 cm?

b between 47·6 mm and 50 mm?

c greater than 50·8 mm?

ii All screws with length greater than 51·6 mm are rejected. What percentage of screws are rejected?

2 In a Grade 10 test, the results were in normal distribution with a mean score is 60% with a standard deviation of 12·5%.

i What proportion of Grade 10 students scored between

a 60% and 85%?

b 22·5% and 35%?

c 72·5% and 85%?

ii If 400 students sat the test and Jeremy scored 85%, how many students scored higher than Jeremy?

3 The heights of one-month-old saplings in a new park form a normal distribution. It is known that the middle 68% of the saplings have heights between 0·75 metres and 1·15 metres.

- **a** What is the mean height?
- **b** What is the approximate maximum height of a sapling?
- **c** What is the approximate minimum height of a sapling?
- **d** What is the minimum height of the tallest 2·5% of saplings?

4 The fully charged battery life of a new brand of laptop computer is in normal distribution with a mean life of 2 hours. It is found that only 2·5% of the computers have a battery life of longer than 2 hours and 30 minutes.

- **a** What is the standard deviation?
- **b** What is the approximate maximum battery life?
- **c** What is the approximate minimum battery life?
- **d** If a computer was selected at random, between what two times would the battery life almost certainly lie?

5 A machine produces metal rods, the length of which forms a normal distribution with a mean length of 85 cm. It is found that out of every 500 metal rods that are produced, 170 are between 85 cm and 88 cm long.

- **a** What is the standard deviation of the length of the rods?
- **b** How many rods have a length greater than 88 cm?
- **c** If a rod is chosen at random, between what two lengths will it most probably lie?
- **d** What proportion of the rods have a length under 82 cm?

15:04 | Statistics with Two Variables

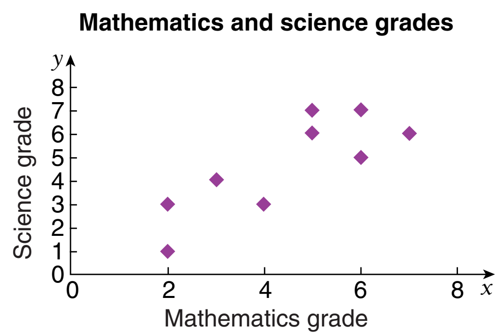
Often data is collected which contains two types of data that may, or may not be related. This is known as **bivariate data**.

worked examples

1 The table below shows the mathematics and science grades for a group of Grade 10 students.

Mathematics	2	7	6	5	4	5	5	3	2	6
Science	1	6	7	6	3	7	6	4	3	5

When graphed against each other, a series of points is obtained as shown. This type of graph is called a **scatter graph** or a **scatter plot**.



It can be seen that there seems to be a relationship between the mathematics and science grades. As the mathematics grades get higher, so do the science grades.

We call such a relationship a *correlation* between variables.

If a student does well in mathematics, it seems he/she also does well in science.

A relationship such as this is called a *positive correlation*.

2 The scatter graph shows the percentage of car accidents for drivers of a given age in a particular town.

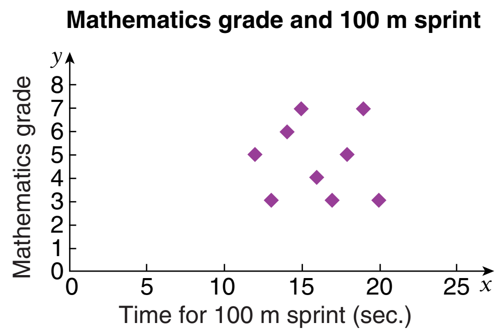
It seems that here there is also a relationship between the variables. However, as one variable increases, the other decreases.

A relationship such as this is called a *negative correlation*.

- A point for discussion: Do you think this trend would continue? Why or why not?

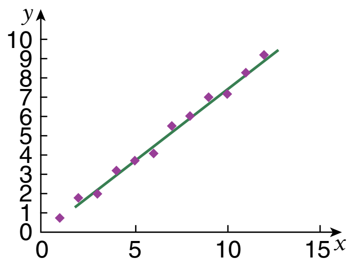
3 This scatter graph compares students' times over a 100 m sprint and their mathematics grade. As perhaps you would expect, there is no relationship between these variables.

When there is no relationship and points are randomly scattered, there is *no correlation*.

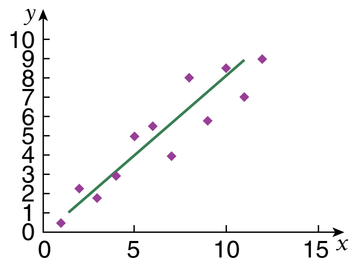


As well as describing whether the correlation is negative or positive, we can describe how strong the correlation is.

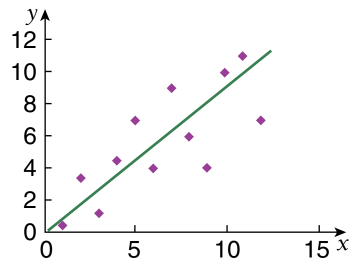
For example:



Here there is a ***strong*** *positive correlation.*



Here there is a ***moderate*** *positive correlation.*



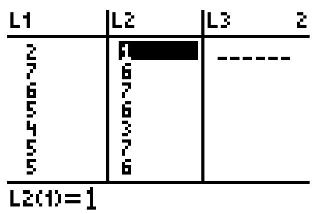
Here there is a ***weak*** *positive correlation.*

Not only are these correlations positive but they are also *linear* — the points seem to form a straight line.

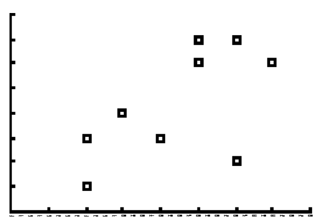
Using a calculator to describe correlation and to find the line of best fit

Consider the data in worked example 1.

This can be entered into the calculator using the STAT function.



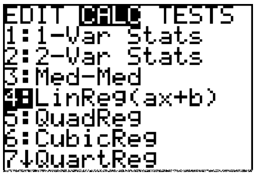
We can then graph these values as a scatter graph.
To do this use the STAT PLOT function.
Turn on only one Stat Plot, choose the scatter graph and make sure you enter the correct lists for the X and Y axes.

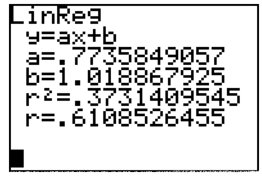


If the data seems to be linear, your calculator will also find the line that best fits the data. This is called the **line of best fit**.

continued →→→

To find the equstion of the line we perform a *linear regression* on the data.

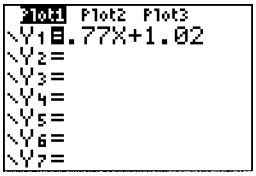




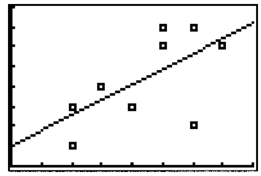
Note: you must have the *diagnostic* turned on. To do this got to *catalog/diagnostic/on*.

This tells us that the gradient of the line is 0·774 and the y-intercept is 0·611.

You can now graph the line



Enter the equation



Graph

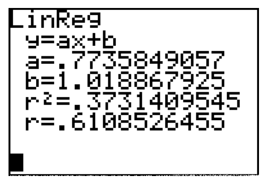
As well as using the graph to describe the correlation as strong, moderate and weak, we can use the *correlation coefficient* to help describe the strength of the correlation.

The correlation coefficient ranges from −1 (a perfect negative correlation) to 1 (a perfect positive correlation).

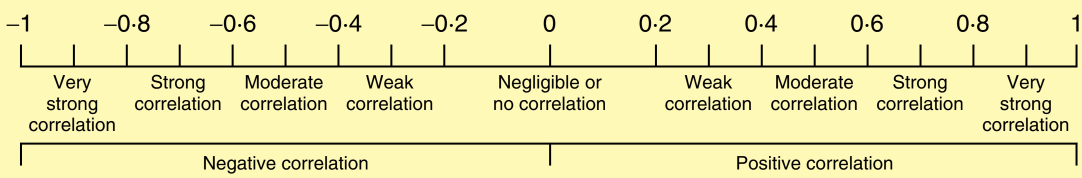
The TI gives a *correlation coefficient* when the linear regression is done.

It is given by the r value.

In this case $r = 0·61$.



To interpret the correlation coefficient, the following guideline can be used:



So, in the example above, it could be said that there is a moderate to strong positive correlation between mathematics grades and science grades.

Exercise 15:04

1 From the following sets of data, determine the correlation coefficient and describe the type and strength of the correlation using the information above.

a The table shows the number of days of rain in the months of a particular year and the number of student absences in the same months in a large international school.

	Jan	*Feb*	*Mar*	*Apr*	*May*	*June*	*July*	*Aug*	*Sept*	*Oct*	*Nov*	*Dec*
Days of rain in the month	5	4	1	0	8	17	15	20	18	11	6	3
Student absences from school	25	22	7	5	46	89	85	100	95	60	30	22

b The table shows the history and mathematics marks scored by 10 students in recent tests.

History mark	50	65	82	98	43	20	68	72	75	69
Mathematics mark	88	90	64	70	60	45	90	65	78	95

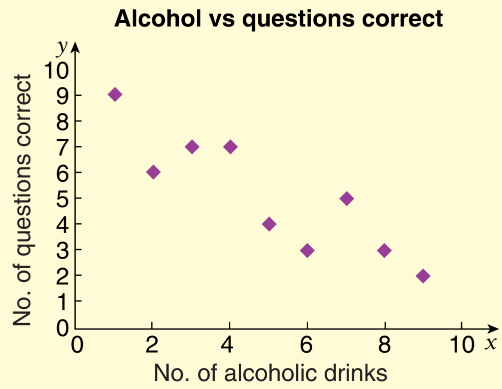
c The table shows the number of hours students spent playing computer games and the average grade they scored in their semester report.

Hours on computer per week	*Average grade*
0	6
2	4
14	3
17	3
10	4
5	6
3	4
18	3
8	4
11	4

d The table shows the percentage capacity at a football ground when games are played in different temperatures.

Temperature (°C)	5	−5	8	12	−1	10	−8	9	7	8
Capacity (%)	75	80	85	60	78	85	80	100	88	90

e The graph shows the results of an experiment in which 10 subjects were asked to drink a prescribed number of alcoholic drinks and complete a multiple choice IQ test with 10 questions.



2 For each of those examples in question 1, find the equation of the line of best fit. Use 3 significant figures.

3 The table shows the yield in tonnes of produce on farms when sprayed with different amounts of insecticide concentrate.

Insecticide (mL)	5	10	15	20	25	30
Yield (tonnes)	220	380	400	320	360	480

a Find the correlation coefficient.
b Describe the correlation between the variables.
c Find the equation of the line of best fit (use 3 significant figures).
d Use the line of best fit to predict the yield if 35 mL of insecticide was used.
e Use the line of best fit to estimate the yield if no insecticide was used.

4 On the same farm, samples of pests were taken and numbers recorded below:

Insecticide (mL)	5	10	15	20	25	30
No of pests per plant	62	52	28	15	10	9

a Find the correlation coefficient.
b Describe the correlation between the variables.
c Find the equation of the line of best fit (use 3 significant figures).
d Use the line of best fit to predict the number of pests if 35 mL of insecticide was used.
e Explain your answer in **d**.

5 The table shows the amount of fuel left in the tank of a car during a trip, compared to the time travelled. The car started the trip with a full tank of fuel.

Length of trip (minutes)	30	90	180	210	270	315
Fuel in tank (litres)	55	32	30	15	14	5

a Find the correlation coefficient.
b Describe the correlation in words.
c Find the equation of the line of best fit (use 3 significant figures).
d Use the line of best fit to approximate how much fuel the tank holds when full.
e Approximate how long the car will last on a tank of fuel if the same pattern of driving continues.
f Give possible reasons why the rate of fuel consumption changes throughout the trip.

Mathematical Terms 15

bivariate data

- data collected that has two variables.

class interval

- The size of the groups into which the data is organised.
  eg 1–5 (5 scores); 11–20 (10 scores).

class centre

- The middle outcome of a class.
  eg The class 1–5 has a class centre of 3.

correlation

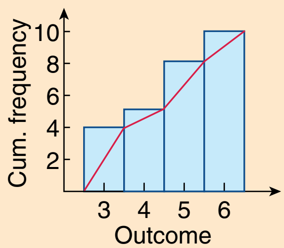
- a connection between sets of data. This can be negative or positive, linear or non linear.

cumulative frequency

- The number of scores less than or equal to a particular outcome.
  eg For the data 3, 6, 5, 3, 5, 5, 4, 3, 3, 6 the cumulative frequency of 5 is 8 (there are 8 scores of 5 or less).

cumulative frequency histogram (and polygon)

- These show the outcomes and their cumulative frequencies.



frequency

- The number of times an outcome occurs in the data.
  eg For the data 3, 6, 5, 3, 5, 5, 4, 3, 3, 6 the outcome 5 has a frequency of 3.

frequency distribution table

- A table that shows all the possible outcomes and their frequencies. (It usually is extended by adding other columns such as the cumulative frequency.)

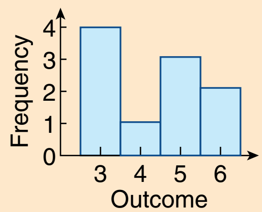
eg

Outcome	*Frequency*	*Cumulative frequency*
3	4	4
4	1	5
5	3	8
6	2	10

frequency histogram

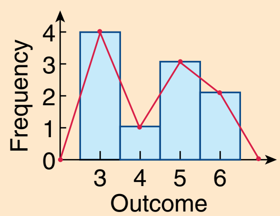
- A type of column graph showing the outcomes and their frequencies.

eg



frequency polygon

- A line graph formed by joining the midpoints of the top of each column. To complete the polygon the outcomes immediately above and below those present are used. The heights of these columns is zero.



grouped data
- The organisation of data into groups or classes.

inter-quartile range
- 3rd quartile – 1st quartile

line of best fit
- the line that best fits the data when graphed

mean
- The number obtained by 'evening out' all the scores until they are equal.
  eg If the scores 3, 6, 5, 3, 5, 5, 4, 3, 3, 6 were 'evened out' the number obtained would be 4·3.
- To obtain the mean use the formula:

$$\text{Mean} = \frac{\text{sum of scores}}{\text{total number of scores}}$$

median
- The middle score for an odd number of scores or the mean of the middle two scores for an even number of scores.

mode (modal class)
- The outcome or class that contains the most scores.

median class
- In grouped data the class that contains the median.

normal distribution
- when the data forms a bell shaped curve in which:
  - Approximately 68% of the scores lie within one standard deviation of the mean.
  - Approximately 95% of the scores lie within two standard deviations of the mean.
  - Approximately 99.7% of the scores lie within three standard deviations of the mean.

ogive
- This is another name for the cumulative frequency polygon.

outcome
- A possible value of the data.

range
- The difference between the highest and lowest scores.

range
- highest score – lowest score

standard deviation (σ_n)
- average distance of the scores from the mean.

statistics
- The collection, organisation and interpretation of numerical data.

Technology Applications Drag and Drops

Mathematical terms 15

Diagnostic Test 15: | Statistics

- Each section of the test has similar items that test a certain type of example.
- Failure in more than one item will identify an area of weakness.
- Each weakness should be treated by going back to the section listed.

diagnostic test 15

	Section
1 The table shows the number of hours it takes overseas students at Vincent's school to fly back to their home countries.	15:01A

Time (hours)	*Class centre*	*Frequency*	*Cumulative frequency*
$1 \leqslant x < 3$	2	13	
$3 \leqslant x < 5$	4	23	
$5 \leqslant x < 7$		30	
$7 \leqslant x < 9$		22	
$9 \leqslant x < 11$		12	

a Complete the table.
b Construct a frequency histogram and polygon.
c Construct a cumulative frequency histogram and polygon.
d Use your cumulative frequency polygon to estimate
  i the median
  ii the first quartile Q1
  iii the third quartile Q3
  iv the inter-quartile range.
e Construct a box and whisker plot of this data.

2 At another international school, the following data is collected. — Section 15:02

Time (hours)	*Class centre*	*Frequency*
$1 \leqslant x < 3$	2	18
$3 \leqslant x < 5$	4	20
$5 \leqslant x < 7$		25
$7 \leqslant x < 9$		22
$9 \leqslant x < 11$		15

a For this data and the data in question 1, calculate at estimate for
  i the mean
  ii the mode
  iii the standard deviation
b Use the mean and standard deviation to compare the times taken for students at the schools to travel to their home countries.

	Section
3 The amount spent by students per week in the school cafeteria forms a normal distribution with a mean of \$28 and a standard deviation of \$8. **a** If a student is chosen at random, what would be the probability that he/she spends between \$20 and \$36 per week? **b** What percentage of students spend **i** between \$44 and \$52? **ii** between \$28 and \$44? **iii** more than \$52? **c** If there are 850 students in the school, how many spend between \$20 and \$28 in the school cafeteria?	15:03
4 A machine is used to pack sweets into a packet. It is known that the number of sweets per packet forms a normal distribution with a mean of 50. It is also known that 2.5% of all packets produced contain less than 46 sweets. **a** What is the standard deviation of this distribution? **b** The number of sweets in the middle 68% of packets would lie approximately between what two numbers? **c** If a packet was chosen at random, what would be the probability of it containing between 52 and 54 sweets? **d** Complete this statement: 'The number of sweets in a packet chosen at random would almost certainly lie between …'	15:03
5 A test was done to find how many metres it would take a car to stop when travelling at different speeds. The same driver was used in each test. The results are shown in the table below.	15:04

Speed (km/h)	*Stopping distance (m)*
30	10
50	20
60	38
80	50
100	100
120	120

a Determine the correlation between the variables and describe the correlation in words.

b Find the equation of the line of best fit for this data.

c Use the line of best fit to approximate the distance it would take a car travelling at 200 km/h.

Chapter 15 | Revision Assignment

assignment 15A

1 The data below gives the average monthly minimum daily temperatures of two Australian cities. The months are in order: January to December.

Adelaide: 15·5, 15·7, 14·3, 11·6, 9·4, 7·4, 6·8, 7·5, 8·6, 10·4, 12·3, 14·3

Alice Springs: 21·2, 20·6, 17·4, 12·5, 8·2, 5·1, 4, 5·9, 9·7, 14·8, 17·9, 20·2

a Calculate the mean and standard deviation for each city.
b Construct grouped frequency distribution tables with the classes 0–<5, etc. for both cities' distributions.
c Draw a cumulative frequency polygon of the each distribution to estimate:
  i the median
  ii Q1
  iii Q3
  iv the interquartile range
d Construct a box and whisker plot of each cities' distributions.
e Compare the measures of centre for each distribution.
f Discuss the spread of each distribution.
g In which city would you rather live? Why?

2 Bags of cement are labelled 25 kg. The bags are filled by machine and the actual weights are normally distributed with mean 25·5 kg. It is known that 16% of bags have less than 25 kg.

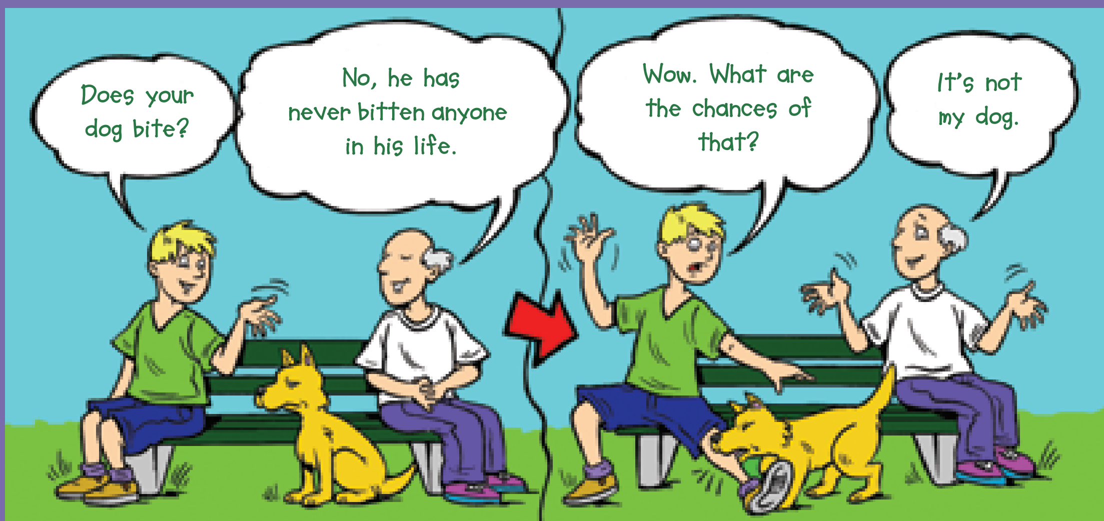
a What is the standard deviation?
b What is the approximate maximum weight of a bag?
c What is the approximate minimum weight of a bag?
d What proportion of the bags weigh between 24 and 26 kg?

3 It is decided to take a random sample of 10 students to see if there is any linear relationship between height and shoe size. The results are given in the table below.

Height (cm)	*Shoe size*
175	8
160	9
180	8
155	7
178	10
159	8
166	9
185	11
189	10
173	9

a Find the correlation coefficient.
b Describe the correlation between height and shoe size.
c Find the equation of the line of best fit.
d Predict the shoe size of a student who is 162 cm in height.
e Predict the height of someone with a shoe size of 13.
f Why is your answer for **e** less reliable than your answer for **d**?

16 Probability



Chapter Contents

Learning Outcomes

Students will be able to:

- Calculate the probability of mutually exclusive events.
- Calculate the probability of combined events.
- Use counting techniques to determine the probability of repeated events.
- Make inferences about the distribution of data given the mean and standard deviation.
- Calculate conditional probability.

Areas of Interaction

Approaches to Learning (Knowledge Acquisition, Reflection, Organisation, Technology), Human Ingenuity — the adaptation of statistics to another theoretical area of mathematics, applications of probability

16:01 | Probability Review

Basic probability was covered in Book 4 and included the following:

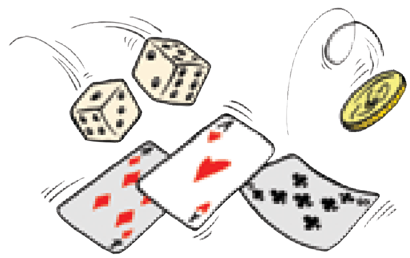
- The sample space of an event (S) is a list of all the total possible outcomes.
- The *theoretical* probability of an event $P(E) = \dfrac{\text{The number of ways that event can occur } n(E)}{\text{The number of elements in the sample space } n(S)}$

$$\text{So } P(E) = \frac{n(E)}{n(S)}$$

- The total probability of all possible events is 1.
- If $P(E)$ is the probability of an event occurring then $P(E')$ is the probability of it not occurring. Where E and E' are known as complementary events.
- $P(E) = 1 - P(E')$
- To calculate the sample space we can use *tree diagrams*, or *dot (grid) diagrams*. These are called *counting techniques*.
- *Venn diagrams* can often be useful when working out probability problems.

worked examples

1 Work out the sample space in each of the following situations. Hence calculate $n(S)$
 a Rolling a die
 b Flipping a coin
 c Flipping 2 coins
 d Rolling 2 dice



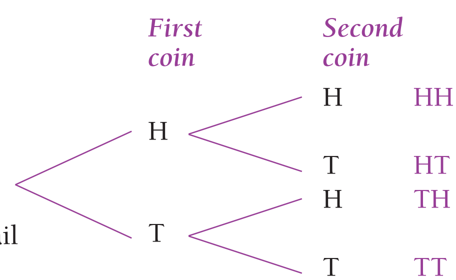
2 Calculate the probability of the following:
 a Rolling a prime number on a die
 b Rolling 2 dice and getting a double
 c Flipping a coin twice and getting at least 1 head

3 In an international school in China, 30 students were asked whether their mothers spoke English or Chinese. Four students said their mother did not speak either language, 24 said their mother spoke English and 20 said their mother spoke Chinese.

If a student's mother is selected at random, what is the probability that she speaks:
 a Both Chinese and English?
 Note: This can be written $P(C \cap E)$ to represent the **intersection** of groups C and E.
 b Chinese but not English?
 Note: This can be written $P(C \cap E')$ to represent the **intersection** of group C and all elements **not in** group E — the intersection of C and E'.
 c Either Chinese or English?
 Note: This can be written $P(C \cup E)$ to represent the **union** of C and E — all the elements in group C or group E or both.

Solutions

1 a $S = \{1, 2, 3, 4, 5, 6\}$, $n(S) = 6$
 b $S = \{\text{Head, Tail}\}$, $n(S) = 2$
 c Using a tree diagram
 $S = \{\text{HH, HT, TH, TT}\}$, $n(S) = 4$
 Note: The result of a head then a tail is different from a tail then a head.

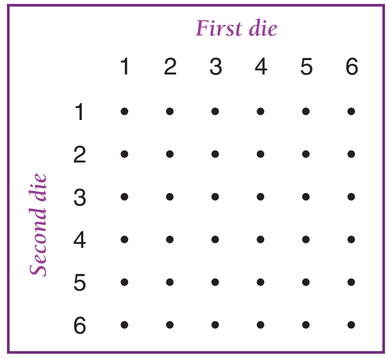


continued →→→

d Using a grid or dot diagram

S = {1&1, 1&2, 1&3, 1&4, 1&5, 1&6,
2&1, 2&2, 2&3, 2&4, 2&5, 2&6,
3&1, 3&2, 3&3, 3&4, 3&5, 3&6,
4&1, 4&2, 4&3, 4&4, 4&5, 4&6,
5&1, 5&2, 5&3, 5&4, 5&5, 5&6,
6&1, 6&2, 6&3, 6&4, 6&5, 6&6}, $n(S) = 36$

Note: The result of a 3 then a 2 is different from a 2 then a 3.



2 **a** $P(\text{prime number}) = \frac{3}{6} = \frac{1}{2}$. There are 3 prime numbers (2, 3 and 5) out of a sample space of 6.

b $P(\text{double}) = \frac{6}{36} = \frac{1}{6}$. There are 6 possible doubles out of a sample space of 36.

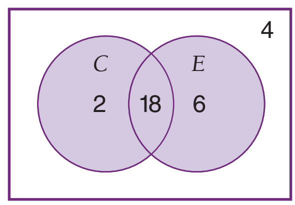
c **Method 1:** $P(\text{at least 1 tail}) = \frac{3}{4}$. There are 3 possible outcomes where there is at least 1 tail (HT, TH, TT) out of a sample space of 4.

d **Method 2:** If you don't flip at least one tail then you must have flipped 2 heads.
So if E = at least 1 tail, then E' = 2 heads
We know that $P(E) = 1 - P(E')$

Since $P(E') = P(\text{2 heads}) = \frac{1}{4}$

$$\text{Then } P(E) = 1 - \frac{1}{4} = \frac{3}{4}$$

3 A Venn diagram is very useful here.
Since 4 of the students' mothers do not speak English or Chinese, this number lies outside the circles.
When we add the numbers of those that speak English and Chinese we get 24 + 20 = 44
However, there are only 26 students left.
We have counted those that speak both languages twice.
This means that the extra students (those whose mothers speak both languages) = 44 − 26
= 18



So this number goes where the circles intersect.
There are 24 that speak English, leaving 6 to go in the remaining part of the English circle.
There are 20 that speak Chinese, leaving 2 to go in the remaining part of the Chinese circle.
Now we can answer the questions:

a $P(\text{both languages}) = P(C \cap E) = \frac{18}{30} = \frac{3}{5}$

b $P(\text{Chinese but not English}) = P(C \cap E') = \frac{2}{20} = \frac{1}{15}$

c $P(\text{Either Chinese or English}) = P(C \cup E) = \frac{26}{30} = \frac{13}{15}$. (Which incidentally $= 1 - \frac{4}{15}$)

Exercise 16:01

1 A bag contains 12 blue discs, 8 red discs and 6 green discs.

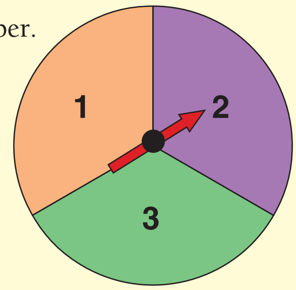
i If a disc is drawn at random from the bag, what is the probability:

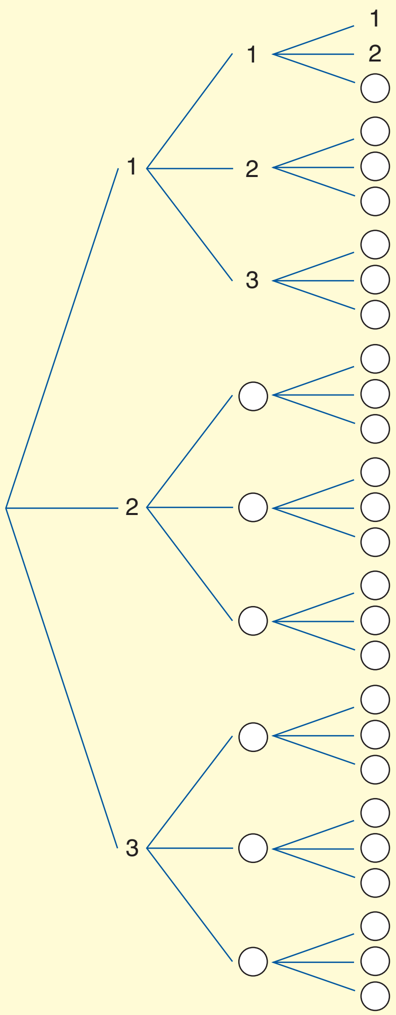
a it is green? **b** it is red? **c** it is not blue?

ii If a disc is drawn at random and not replaced then another disc is drawn at random, what is the probability that the second disc is blue, given that the first disc was also blue?

2 The spinner below is made of a circle divided into three equal sectors.
It is spun 3 times and the results written down to form a 3-digit number.

i Complete the tree diagram below to show the sample space.





ii Find the probability that the 3-digit number that results:

a Has all three digits the same.

b Has all three digits different.

c Is an even number

d Is an odd number

e Has only two digits the same.

3 A die is rolled and a coin is flipped.

a Use a grid diagram to work out the sample space and $n(S)$.

b Find the probability of a head and a multiple of 3 being the result.

4 Three new students are going to join Andy and Steven's homeroom. They try to guess the gender of the new students. Andy guesses 2 boys and 1 girl while Steven says they will all be girls.

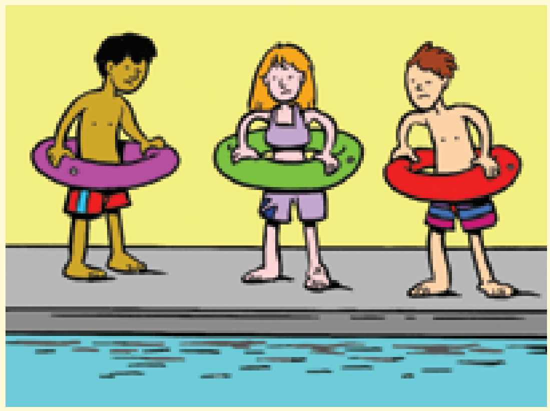
a Draw a tree diagram to show all the possibilities (sample space).
b What is the probability that Andy is correct?
c What is the probability that Steven is correct?
d What is the probability that both Andy and Steven are wrong?
e What is the probability that at least one of the new students is a boy?

5 A bag contains red, white and blue balls. The probability of drawing a red ball is $\frac{1}{4}$ and drawing a white ball is $\frac{3}{5}$.

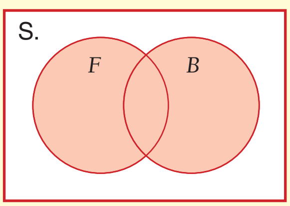
a What is the probability of drawing a blue ball?
b If there are 36 white balls
  i how many red balls are there?
  ii how many blue balls are there?
c If 3 of each colour ball is added to the bag, what is the probability of drawing each at random?

6 In a group of 25 people, it is known that 5 cannot swim.

a What is the probability that a student chosen at random cannot swim?
b If the first person chosen is not a swimmer and taken from the group (not replaced), what is the probability a second person chosen is also a non-swimmer?



7 A group of 35 people were asked whether they had played football (F) or basketball (B) and the results were put into the Venn diagram below.



The results were: $n(F \cap B) = 10$, $n(B) = 20$, $n(F \cup B) = 28$

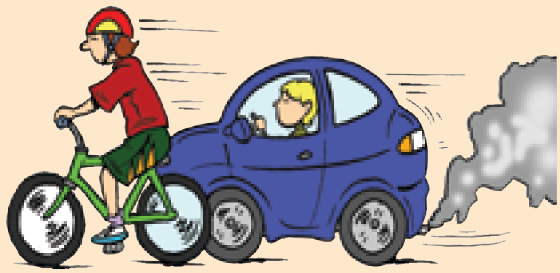
a Complete the Venn diagram.
b If a person is chosen from the group at random, find each of the following, explaining in words what is meant.
i $P(F)$ **ii** $P(F \cap B')$ **iii** $P(F \cup B)$ **iv** $P[(F \cup B)']$ **v** $P(B')$

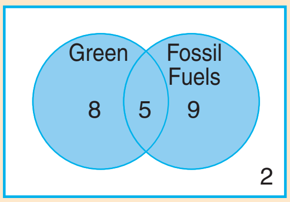
Investigation 16:01 | This AND that OR something else

Please use the Assessment Grid on page 475 to help you understand what is required for this Investigation.

Consider the following situation:

As part of an environmental project, Soo Ah surveyed her class to find out who came to school using 'green power' (eg walking or riding a bicycle), who came to school using power from fossil fuels (eg car, train, bus, tram, trolley bus etc) and who used a combination of both (students who travelled at least 500 m by green power then used fossil fuels). The results are shown in the Venn diagram below.





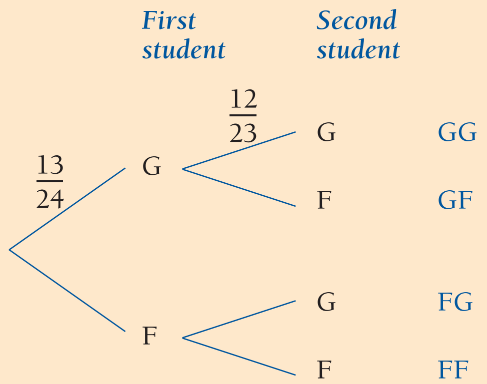
1 Explain how there could be two students who are outside the circles (ie who don't use any of the above to get to school).

2 Complete the following table from the information in the diagram for a student chosen from the above group at random.

	$P(G)$	$P(F)$	$P(G \cap F)$	$P(G \cup F)$	$P(G) + P(F)$
Explain what is meant in words.	The probability that he/she uses green power.				
Find the probability.	$\frac{13}{24}$				

3 Complete the following statement: '$P(G \cup F) = P(G) + P(F) -$ ____________________.' Explain this statement in words.

4 The tree diagram below shows the possible outcomes if two students are chosen at random from the class.



Explain where the fractions $\frac{13}{24}$ and $\frac{12}{23}$ come from and complete the tree diagram.

Consider this statement:

'If two students are to be chosen at random, each student can be paired with all the other students, so each student can be paired with 23 others.'

Try to illustrate this in a grid diagram and then explain why there are 24×23 groups of two students in the class altogether.

How many groups of two students use green power to get to school?

So the probability that two students chosen at random both use green power = $\frac{13}{24} \times \frac{12}{23}$

Use your tree diagram to calculate that if two students are chosen at random:

Both use fossil fuel to get to school.

One uses fossil fuel and one uses green fuel.

5 If the probability of event A occurs is $P(A)$ the probability event B occurs $P(B)$ and the probability that both occur is $P(A \cap B)$, write some rules to calculate the following:

a The probability that A or B occurs = $P(A \cup B)$

b The probability that A and B occurs = $P(A \cap B)$

Assessment Grid for Investigation 16:01 | This AND that OR something else

The following is a sample assessment grid for this investigation. You should carefully read the criteria *before* beginning the investigation so that you know what is required.

<table>
<tr><th colspan="4">Assessment Criteria (B, C, D) for this investigation</th><th>Achieved ✓</th></tr>
<tr><td rowspan="9">Criterion B
Investigating Patterns</td><td>a</td><td>None of the standards below has been reached.</td><td>0</td><td></td></tr>
<tr><td rowspan="2">b</td><td rowspan="2">The student needed some help to complete the tables and answer the questions.</td><td>1</td><td></td></tr>
<tr><td>2</td><td></td></tr>
<tr><td rowspan="2">c</td><td rowspan="2">The student completed independently and recognised the pattern in the table.</td><td>3</td><td></td></tr>
<tr><td>4</td><td></td></tr>
<tr><td rowspan="2">d</td><td rowspan="2">The student completed the tables, recognised patterns and completed question 4 independently.</td><td>5</td><td></td></tr>
<tr><td>6</td><td></td></tr>
<tr><td rowspan="2">e</td><td rowspan="2">All of the above has been completed. In addition the patterns are described in full and the rules for question 5 are correct and explained in full.</td><td>7</td><td></td></tr>
<tr><td>8</td><td></td></tr>
<tr><td rowspan="7">Criterion C
Communication in Mathematics</td><td>a</td><td>None of the standards below has been reached.</td><td>0</td><td></td></tr>
<tr><td rowspan="2">b</td><td rowspan="2">There has been a basic use of mathematical language. Lines of reasoning are hard to follow. Diagrams have been used.</td><td>1</td><td></td></tr>
<tr><td>2</td><td></td></tr>
<tr><td rowspan="2">c</td><td rowspan="2">The use of mathematical language is sufficient. Movement between the problem, diagrams and rules has been done with some success. Lines of reasoning are clear but not always logical.</td><td>3</td><td></td></tr>
<tr><td>4</td><td></td></tr>
<tr><td rowspan="2">d</td><td rowspan="2">There is a good use of mathematical language and an effective movement between the table, diagrams and conclusion. Lines of reasoning are logical and complete.</td><td>5</td><td></td></tr>
<tr><td>6</td><td></td></tr>
<tr><td rowspan="7">Criterion D
Reflection in Mathematics</td><td>a</td><td>None of the standards below has been reached.</td><td>0</td><td></td></tr>
<tr><td rowspan="2">b</td><td rowspan="2">The student has attempted to connect the table with diagrams and to explain the results in context.</td><td>1</td><td></td></tr>
<tr><td>2</td><td></td></tr>
<tr><td rowspan="2">c</td><td rowspan="2">The connection between the table and diagrams has been made correctly and the explanation is correct but brief.</td><td>3</td><td></td></tr>
<tr><td>4</td><td></td></tr>
<tr><td rowspan="2">d</td><td rowspan="2">A full explanation of the connections is given and answers to questions 3, 4 and 5 have been given in full.</td><td>5</td><td></td></tr>
<tr><td>6</td><td></td></tr>
</table>

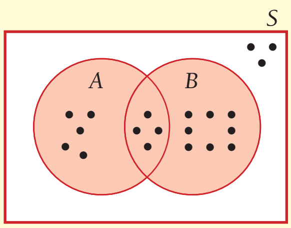
16:02A | Simultaneous Events

Let us now consider situations in which two events can occur at the same time.

We have seen that if there are two possible events, event A and event B, then two of the possible outcomes are:

A and B both occur. **This can be written** $A \cap B$
Either A occurs or B occurs. **This can be written** $A \cup B$

Now remember that $P(E) = \dfrac{n(E)}{n(S)}$ and consider the example shown in the Venn diagram below:



Each dot represents an outcome so that $n(S) = 20$, $n(A) = 9$ and $n(B) = 12$.

Now consider $A \cap B$: $n(A \cap B) = 4$ because this is the number of elements that are in both A and B.

$$\text{So } P(A \cap B) = \frac{n(A \cap B)}{n(S)}$$
$$= \frac{4}{20} = \frac{1}{5}$$

So the probability that **both** A **and** B occur is $\frac{1}{5}$.

Now consider $A \cup B$: $n(A \cup B) = 17$ because this is the number of elements that are in A or B or both.

$$\text{So } P(A \cup B) = \frac{n(A \cup B)}{n(S)}$$
$$= \frac{17}{20}$$

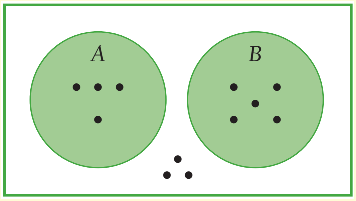
So the probability that **either** A **or** B occur is $\frac{17}{20}$.

However, we don't want to have to draw a diagram every time we solve a problem so we use the rule that $n(A \cup B) = n(A) + n(B) - n(A \cap B)$ since if we add $n(A) + n(B)$ we have counted $n(A \cap B)$ twice.

This means that $P(A \cup B) = P(A) + P(B) - n(A \cap B)$

If two events have no outcomes in common then they are **mutually exclusive events**.

This means that for event A and event B $P(A \cap B) = 0$. This is shown in the Venn diagram below.



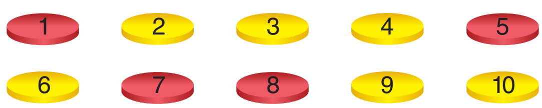
In this diagram, A and B do not overlap so they have no intersection and $P(A \cap B) = 0$.

In these cases then, $P(A \cup B) = P(A) + P(B)$

So if A and B are mutually exclusive events
$P(A \cup B) = P(A) + P(B)$

worked example

1 The diagram shows 10 coloured discs with numbers on them.



If a disc is chosen at random, what is the probability that

a its colour is yellow?
b its colour is red?
c the number is divisible by 4?
d its colour is red and the number is divisible by 4?
e its colour is red or the number is divisible by 4?

Solution

1 **a** $P(Y) = \frac{6}{10} = \frac{3}{5}$

b $P(R) = 1 - P(Y)$
$= 1 - \frac{3}{5}$
$= \frac{2}{5}$

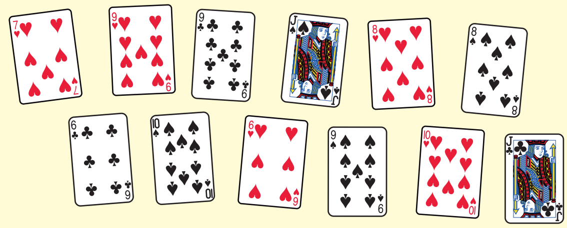
c $P\text{ (divisible by 4)} = \frac{2}{10} = \frac{1}{5}$

d $P(R \cap \text{divisible by 4}) = \frac{1}{10}$

e $P(R \cup \text{divisible by 4}) = P(R) + P\text{ (divisible by 4)} - P(R \cap \text{divisible by 4})$
$= \frac{2}{5} + \frac{1}{5} - \frac{1}{10}$
$= \frac{5}{10}$
$= \frac{1}{2}$

Exercise 16:02A

1 The playing cards shown are placed face down on a table and one is chosen at random. Find the probability that the card chosen is



- **a** a red card
- **b** a black card
- **c** a nine
- **d** a spade
- **e** a nine and black
- **f** a nine or black

2 In a group of 16 students, 8 have been to America, 5 have been to Australia, and 3 have been to both America and Australia. Find the probability that a student chosen at random from this group

- **a** has been to America
- **b** has been to America but not Australia
- **c** has been to America and Australia
- **d** has been to America or Australia
- **e** has been to neither America nor Australia.

3 Events G and H are mutually exclusive.
If $P(G) = \frac{3}{8}$ and $P(G \cup H) = \frac{1}{2}$ find

- **a** $P(G \cap H)$
- **b** $P(H)$

4 In Essi's class, the probability that a student, chosen at random, studies geography is $\frac{1}{3}$; studies history is $\frac{3}{4}$; and studies both history and geography is $\frac{1}{6}$.

- **i** Find the probability that a student chosen at random
  - **a** studies either history or geography or both
  - **b** studies either history or geography but not both
  - **c** studies only history
  - **d** studies neither history nor geography.
- **ii** If Essi's class has 36 students, how many study geography but not history?

5 For events X and Y it is known that $P(X) = \frac{5}{8}$ and $P(Y) = \frac{1}{2}$. If the probability that either event occurs is $\frac{3}{8}$, find the probability that both occur.

6 As part of their school project, Arielle, Ellen and Susan had to count how many people in their apartment building had blonde hair, blue eyes or both.

Arielle noted that $\frac{1}{3}$ of the people had both blonde hair and blue eyes, while Ellen noted that $\frac{1}{2}$ the people had blonde hair, but didn't write down the colour of their eyes. Susan noted that $\frac{3}{4}$ of the people had either blonde hair or blue eyes.

Use the rule $P(A \cup B) = P(A) + P(B) - P(A \cap B)$ to find out the probability of a person chosen at random from the apartment building having blue eyes.

16:02B | Successive Events

Now let us consider events that happen one after the other.

Suppose we have a group of 9 students represented by the letters A, B, C, D, E, F, G, H, I

How many groups of 2 students can be made? To answer this question we use a grid diagram.

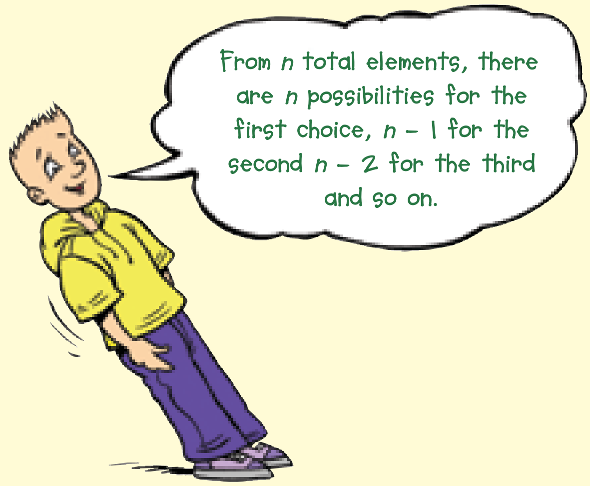
Second student (columns); *First student* (rows)

	A	*B*	*C*	*D*	*E*	*F*	*G*	*H*	*I*
A	~~AA~~	AB	AC	AD	AE	AF	AG	AH	AI
B	BA	~~BB~~	BC	BD	BE	BF	BG	BH	BI
C	CA	CB	~~CC~~	CD	CE	CF	CG	CH	CI
D	DA	DB	DC	~~DD~~	DE	DF	DG	DH	DI
E	EA	EB	EC	ED	~~EE~~	EF	EG	EH	EI
F	FA	FB	FC	FD	FE	~~FF~~	FG	FH	FI
G	GA	GB	GC	GD	GE	GF	~~GG~~	GH	GI
H	HA	HB	HC	HD	HE	HF	HG	~~HH~~	HI
I	IA	IB	IC	ID	IE	IF	IG	IH	~~II~~

When we exclude all the outcomes where the same student is chosen twice, we get a sample space with 72 elements.

So that $n(S) = 72$.

This could have been arrived at by considering the fact that there are 9 possibilities for the first choice and for each of these 9 there are 8 possibilities for the second choice $= 9 \times 8 = 72$ possibilities altogether.



Now suppose the students in blue are boys. How many possible groups of 2 boys can be made? If we make another grid diagram and again ignore the outcomes where the same boy is chosen twice, we get a total of 6.

	B	*C*	*F*
B	~~BB~~	BC	BF
C	CB	~~CC~~	CF
F	FB	FC	~~FF~~

This could have been arrived at by considering the fact that there are 3 possibilities for the first choice and for each of these 3 there are 2 possibilities for the second choice = 3 × 2 = 6 possibilities altogether.

So the probability of choosing 2 boys at random from this group of 9 students $= \dfrac{3 \times 2}{9 \times 8}$

$$= \frac{3}{9} \times \frac{2}{8}$$

$$= \frac{6}{72} = \frac{1}{12}$$

Let's consider the situation again, but in the form of a problem:
From a group of 9 students of which 3 are boys, two are chosen at random. What is the probability that both are boys?

We know the answer is $\frac{3}{9} \times \frac{2}{8}$, but what if we try to solve it a different way?

For event *A*: $P(\text{the first student chosen is a boy}) = \frac{3}{9}$

For event *B*: $P\begin{pmatrix}\text{The second student chosen is a boy} \\ \text{Given that the first chosen was a boy}\end{pmatrix} = \frac{2}{8}$ since there are only 2 boys left out of 8 remaining students

We write this as $P(B|A) = \frac{5}{8}$

The investigation above shows us that if we multiply these we get the correct answer.
So, if event A is choosing a boy first and event B is choosing a boy second, we are trying to find

$$P(A \cap B) = P(A) \times P(B|A)$$

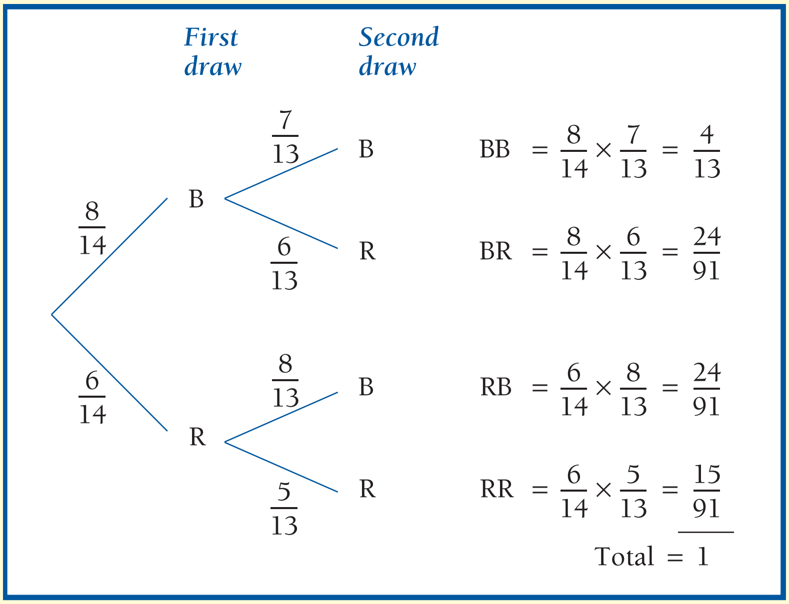
$$= \frac{3}{9} \times \frac{2}{8}$$

$$= \frac{6}{72} = \frac{1}{12}$$

We can apply this knowledge to tree diagrams.

Consider this problem: A bag contains 8 blue marbles and 6 red marbles. One marble is drawn from the bag, replaced and a second is drawn.

This problem can be represented in the tree diagram shown below.



By writing the probabilities of the events on the branches of the tree, it is easy to calculate the probabilities of successive events. It is always a good idea to check that the total of the probabilities is 1.

From this diagram it can be seen that the probability of:

drawing two blue marbles $= P(BB) = \frac{8}{14} \times \frac{7}{13} = \frac{4}{13}$

drawing two red marbles $= P(RR) = \frac{6}{14} \times \frac{5}{13} = \frac{15}{91}$

Drawing marbles of a different colour can happen two ways so we need the total of these

$= P(BR) + P(RB)$

$= \frac{8}{14} \times \frac{6}{13} + \frac{6}{14} \times \frac{8}{13}$

$= \frac{48}{91}$

worked examples

1 A sack of money contains on \$5 and \$10 notes.
There are seven \$5 notes and eight \$10 notes.
Two notes are taken at random *without replacement*.

i Represent the information in a tree diagram.

ii Find the probability of drawing a total of

a \$15

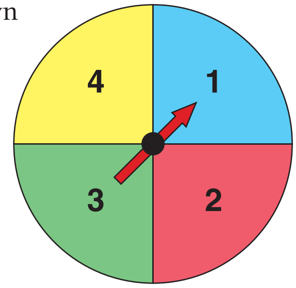
b \$20

c \$10

continued →→→

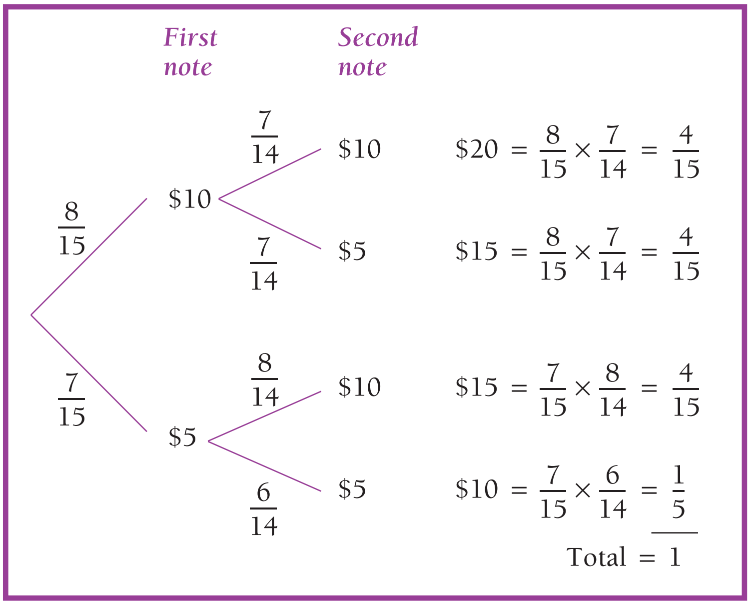
2 The spinner shown is spun twice and the results written down to form a 2-digit number.
Find the probability that the number formed

a is divisible by 6
b is even
c is prime.



Solutions

1 **i**



ii **a** $P(\$15) = P(\$10 \text{ and } \$5) \text{ or } P(\$5 \text{ and } \$10)$

$$= \left(\frac{8}{15} \times \frac{7}{14}\right) + \left(\frac{7}{15} \times \frac{8}{14}\right)$$

$$= \frac{8}{15}$$

b $P(\$20) = P(\$10 \text{ and } \$10)$

$$= \frac{8}{15} \times \frac{7}{14}$$

$$= \frac{4}{15}$$

c $P(\$20) = P(\$5 \text{ and } \$5)$

$$= \frac{7}{15} \times \frac{6}{14}$$

$$= \frac{1}{5}$$

2 Using a grid diagram to work out the possibilities, we find there are 16 possible outcomes since in this problem, the same number can occur twice.
The probability for every digit is the same and is $\frac{1}{4}$.

Second digit across the top, *First digit* down the side:

	1	2	3	4
1	11	12	13	14
2	21	22	23	24
3	31	32	33	34
4	41	42	43	44

a Numbers in the sample space divisible by 6 are 12, 24, 42.

$\therefore P(\text{a number divisible by 6}) = P(1 \text{ and } 2) + P(2 \text{ and } 4) + P(4 \text{ and } 2)$

$$= \left(\frac{1}{4} \times \frac{1}{4}\right) + \left(\frac{1}{4} \times \frac{1}{4}\right) + \left(\frac{1}{4} \times \frac{1}{4}\right)$$

$$= \frac{3}{16}$$

b To be even, the second digit must be a 4 or a 2.

$\therefore P(\text{even}) = P(2 \text{ for the second digit}) + P(4 \text{ for the second digit})$

$$= \frac{1}{4} + \frac{1}{4}$$

$$= \frac{1}{8}$$

c The only prime numbers in the sample space are 11, 13, 31, 43

$\therefore P(\text{prime}) = P(1 \text{ and } 1) + P(1 \text{ and } 3) + P(3 \text{ and } 1) + P(4 \text{ and } 3)$

$$= 4\left(\frac{1}{4} \times \frac{1}{4}\right)$$

$$= \frac{1}{4}$$

It should be obvious now that every 2-digit number has a probability of $\left(\frac{1}{4} \times \frac{1}{4}\right)$ of occurring.

Exercise 16:02B

1 The letters of the word international are written on cards as shown below:

I	N	T	E	R	N	A	T	I	O	N	A	L

The cards are shuffled and two are drawn at random. The first is **not** replaced before drawing the second.

Find the probability that:

a The first card drawn is a vowel (A, E, I, O or U).
b The second card drawn is a vowel.
c Both cards drawn are vowels.
d Both cards have a T on them.
e Both cards have the same letter on them.

2 Three blue, four green and five red balls are put into a bag. One ball is chosen at random and replaced before a second is chosen at random.

- **a** Draw a tree diagram of this information showing the probabilities.
- **b** Find the probability that:
  - **i** Both balls are red.
  - **ii** Both balls are the same colour.
  - **iii** Both balls are a different colour.
  - **iv** Neither of the balls is green.
  - **v** At least one of the balls is green.

3 A group of students has 3 boys and 6 girls. Two are chosen at random to represent the school in a spelling competition. Find the probability that

- **a** Both are girls.
- **b** Both are boys.
- **c** One is a boy and the other a girl.
- **d** At least one is a girl.

4 A bowl contains 10 eggs of which 6 are hard boiled. Two are chosen at random without replacement.
Find the probability that

- **a** Neither of them is hard boiled.
- **b** At least one of them is hard boiled.

5 Toni has forgotten the last 2 digits of her friend's phone number. She has to guess what they are. Find the probability that she guesses the correct numbers on the first guess.

6 At the school dance, Yu Xiao has gone to get two cans of drink, one for himself and one for his friend Jennie. Yu Xiao's favourite drink is lemonade and Jennie's is cola. The drink bucket contains 15 cans of lemonade, 20 cans of cola, 12 cans of orange drink and 18 cans of ginger ale. If Yu Xiao chooses two cans at random, what is the probability of him getting

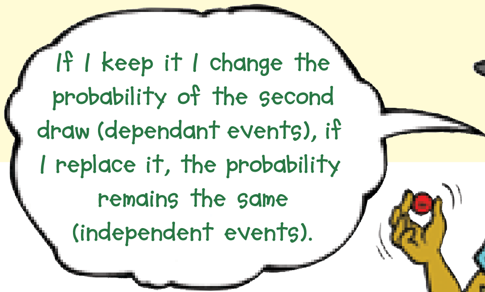
- **a** one of each of their favourite drinks?
- **b** neither of their favourite drinks?
- **c** at least one of their favourite drinks?

7 Roger and Maria both represent their school in tennis. The probability that Roger wins his first game is $\frac{3}{5}$ while the probability that at least one of them wins their first game is $\frac{13}{15}$.

- **a** Find the probability that neither of them wins his/her first game.
- **b** Find the probability that Maria wins her first game.

16:03 | Independent Events and Conditional Probability

If the result of event A has no influence on the result of event B, then A and B are said to be **independent events**. If events are not independent then they are **dependant events**.



worked examples

State whether the events given in each question of the previous exercise describe independent or dependant events.

Solutions

1. Since the cards are not replaced, this describes dependant events.
2. Since the balls are replaced, this describes independent events.
3. Since two are chosen, one cannot be replaced before the second is chosen so this describes dependent events.
4. Since there is no replacement, this describes dependent events.
5. Since there is no restriction on repeating numbers, this describes independent events.
6. Since the first drink is not replaced before getting the second, this describes dependent events.
7. Since Roger and Maria play in separate matches against different opponents (Roger plays in a boy's match while Maria plays in a girl's match), this describes independent events.

We have already seen that $P(A \cap B) = P(A) \times P(B|A)$ where $P(B|A)$ is the probability that B occurs given that A has already occurred OR the probability of B given A.

We call this **conditional probability**.

$$\text{Since} \quad P(A \cap B) = P(A) \times P(B|A)$$

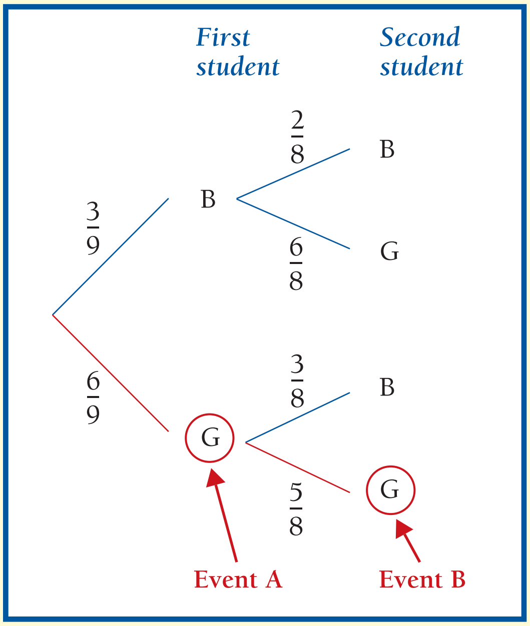
$$\text{Then the probability of } B \text{ given } A \quad P(B|A) = \frac{P(A \cap B)}{P(A)}$$

Consider question **3** in the previous exercise.

A group of students has 3 boys and 6 girls. Two are chosen at random to represent their school in a spelling competition.

Suppose we wish to find the probability that the second student chosen is a girl, given that the first student chosen was a girl. In this case event A is the first student and event B is the second.

This is written $P(B|A)$ and is read 'the probability of B, given A'.



By looking at the tree diagram for this problem, we can see that if the first student chosen is a girl, then we must travel along the red branch and can see then that the probability that the second student chosen is a girl is $\frac{5}{8}$.

$\therefore P(B|A) = \frac{5}{8}$

Also, by using the rule above $\quad P(B|A) = \frac{P(A \cap B)}{P(A)}$

$$\therefore P(B|A) = \frac{\frac{6}{9} \times \frac{5}{8}}{\frac{6}{9}}$$

$$= \frac{5}{8}$$

worked examples

1 It was found that out of 80 students in Grade 10, 60 passed the mathematics test while 50 passed both the mathematics and science tests.
If a student is chosen at random, what is the probability he/she passed the science test, given that he/she also passed the mathematics test?

2 At Seaside International School it is known that 55% of students own a surfboard and that 30% own a surfboard and a skateboard.
Find the probability of a student chosen at random owning a skateboard, given that he/she owns a surfboard.

Solutions

1 $P(\text{passed science} \mid \text{passed mathematics}) = \dfrac{P(\text{passed both Science and Maths})}{P(\text{passed Maths})}$

$$= \frac{\frac{50}{80}}{\frac{60}{80}} = \frac{50}{80} \times \frac{80}{60}$$

$$= \frac{5}{6}$$

2 $P(\text{owns a skateboard} \mid \text{own a surfboard}) = \dfrac{P(\text{owns both a skateboard and a surfboard}}{P(\text{owns a surfboard})}$

$$= \frac{0{\cdot}3}{0{\cdot}55}$$

$$= \frac{6}{11}$$

Exercise 16:03

1 A bag contains red and yellow balls. The probability of drawing a red ball and then a yellow ball is 0.36. The probability of drawing a red ball first is 0.48. Find the probability of randomly choosing a yellow ball on the second draw, given that a red ball has already been drawn.

2 The probability that it is Monday and the school bus is late is 5%.
  a What is the probability of a school day chosen at random being a Monday?
  b Find the probability of the school bus running late, given that it is a Monday.

3 At Elliot's school the probability that a student studies music and drama is $\frac{2}{15}$. The probability a student studies drama is $\frac{1}{3}$. Find the probability that a student studies music, given that the student studies drama.

4 In our apartment building 75% of the apartments have 3 bedrooms and balconies. Fifty per cent of the apartments have balconies. Find the probability of choosing a 3-bedroom apartment at random, given that it has a balcony.

5 Lilly noticed that 4 out of every 5 of the customers at her café have milk in their coffee. She also noticed 2 out of every 5 have sugar in their coffee, given that they have milk in their coffee. Find the probability that a customer, chosen at random, has milk and sugar in his/her coffee.

6 At Capital International School, 5% of the students are on the swim team and the basketball team. Given that a student is on the basketball team, the probability that the student is on the swim team is 40%. Find the probability that a student, chosen at random, is on the swim team.

mathematical terms 16

Mathematical Terms 16

complementary events
- If $P(E)$ is the probability of an event occurring then $P(E')$ is the probability of it not occurring where E and E' are known as complementary events.

conditional probability
- The probability of an event occurring, given that another event has already occurred: $= P(B|A) = \frac{P(A \cap B)}{P(A)}$.

dependent events
- Where one event has an effect on the probability of another.

independent events
- Where one event has no influence on another.

intersection of sets
- Where two sets overlap. In probability it is where two events occur simultaneously.

mutually exclusive events
- Events with no intersection. This means that both events cannot possibly occur.

probability of an event
- The chance that event will occur:

$$p(E) = \frac{n(E)}{n(S)}.$$

sample space
- Includes all the possible outcomes.

union of sets
- Includes the elements that belong to each set or both sets. In probability it is where either or both events can occur.

Venn diagram
- A diagram using circles to represent events (sets) with a numbers inside the circles representing how many times the event can possibly occur.

diagnostic test 16

Diagnostic Test 16: | Probability

- Each section of the test has similar items that test a certain type of example.
- Failure in more than one item will identify an area of weakness.
- Each weakness should be treated by going back to the section listed.

	Section
1 Cards numbered 0 to 9 are placed face down on a table and mixed. **a** If a card is chosen at random, what is the probability that the number on the card **i** is divisible by 3? **ii** is prime? **iii** is five? **b** If two cards are chosen at random without replacement, what is the probability that the total on the cards is **i** 15? **ii** 9? **iii** 20?	16:01

Section

2 Of 300 passengers on a flight from Sydney to London via Singapore, 120 are getting off the plane in Singapore, 180 are going to London and 80 are going to London after a stopover in Singapore. 16:01

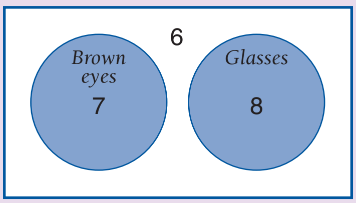
a Represent this information in a Venn diagram.

b How many people are not staying in Singapore or going to London? Explain this.

c What is the probability of a passenger chosen at random only travelling as far as Singapore?

3 Events X and Y are mutually exclusive. The probability of either X or Y happening is $\frac{4}{5}$. If the probability of event X is $\frac{2}{3}$ find the probability of Y. 16:02A

4 Of the 23 students in his homeroom class, Flynn has recorded the number of students with brown eyes, the number who wear glasses and the number who have neither brown eyes nor wear glasses. He recorded his findings in the Venn diagram below. 16:02A



If G = choosing a student at random who wears glasses and B = choosing a student at random with brown eyes:

a What can be said about events G and B?

b Find i $P(B \cap G)$ ii $P(B \cup G)$

5 In Flynn's school, two students are chosen from each homeroom class to compete in a general knowledge quiz. Flynn's homeroom class has 12 boys and 11 girls. Find the probability that: 16:02B

a Two boys are chosen.

b Two girls are chosen.

c At least one boy is chosen.

6 In Flynn's mathematics class, 80% of the students passed both their mathematics and English examinations. If it is known that 95% of the class passed mathematics, what is the probability that a student chosen at random passed English, given that the student had passed mathematics (write your answer as a fraction)? 16:03

7 At a restaurant it is known that on a typical night 40% of the patrons will order red wine and that 35% of the patrons will order steak, given that they have ordered red wine. Find the probability that a patron will order both steak and red wine. 16:03

assignment 16B

Chapter 16 | Revision Assignment

1 A bag contains 2 red, 3 yellow and 5 green sweets. Without looking, Mary takes one sweet out of the bag and eats it. She then takes out a second sweet.
   a If the first sweet is green, what is the probability that the second sweet is also green?
   b If the first sweet is not red, what is the probability that the second sweet is red?

2 Nene and Deka both play netball. The probability that Nene will score a goal on her first attempt is 0·75. The probability that Deka will score a goal on her first attempt is 0·82.
Calculate the probability that:
   a Nene and Deka will both score a goal on their first attempts;
   b neither Nene nor Deka will score a goal on their first attempts.

3 Heinrik rolls two 6-sided dice at the same time. One die has three red sides and three black sides. The other die has the sides numbered from 1 to 6. By means of a tree diagram, table of outcomes or otherwise, answer each of the following questions.
   a How many different possible combinations can he roll?
   b What is the probability that he will roll a red and an even number?
   c What is the probability that he will roll a red or black and a 5?
   d What is the probability that he will roll a number less than 3?

4 The table below shows the number of left and right handed tennis players in a sample of 50 males and females.

	Left handed	*Right handed*	*Total*
Male	3	29	32
Female	2	16	18
Total	5	45	50

If a tennis player was selected at random from the group, find the probability that the player is:
   a male and left handed;
   b right handed;
   c right handed, given that the player selected is female.

5 Amos travels to school either by car or by bicycle. The probability of being late for school is $\frac{1}{10}$ if he travels by car and $\frac{1}{5}$ if he travels by bicycle. On any particular day he is equally likely to travel by car or by bicycle.
   a Draw a probability tree diagram to illustrate this information.
   b Find the probability that:
      i Amos will travel by car and be late.
      ii Amos will be late for school.
   c Given that Amos is late for school, find the probability that he travelled by bicycle.

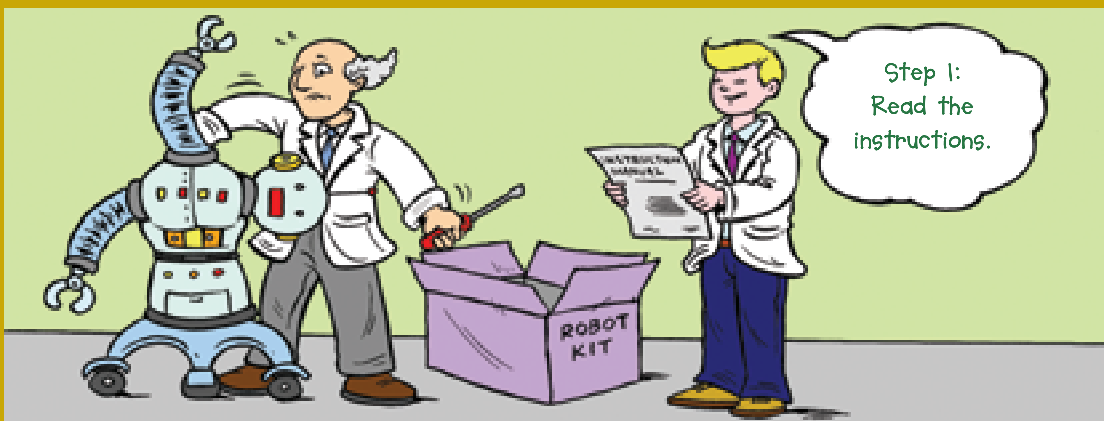
6 Events A and B have probabilities $P(A) = 0{\cdot}4$, $P(B) = 0{\cdot}65$, and $P(A \cup B) = 0{\cdot}85$.
   a Calculate $P(A \cap B)$.
   b State with a reason whether events A and B are independent.
   c State with a reason whether events A and B are mutually exclusive.

7 In a club with 60 members, everyone attends either on Tuesday for Drama (D) or on Thursday for Sports (S) or on both days for Drama and Sports.
One week it is found that 48 members attend for Drama and 44 members attend for Sports and x members attend for both Drama and Sports.
   a i Draw and **label fully** a Venn diagram to illustrate this information.
      ii Find the number of members who attend for both Drama and Sports.
      iii Describe, in words, the set represented by $(D \cap S)'$.
      iv What is the probability that a member selected at random attends for Drama only or Sports only?

The club has 28 female members, 8 of whom attend for both Drama and Sports.
   b What is the probability that a member of the club selected at random
      i is female and attends for Drama only or Sports only?
      ii is male and attends for both Drama and Sports?

17

An Introduction to Algorithms and Number Theory



Chapter Contents

Learning Outcomes

Students will be able to:

- Understand what is meant by an algorithm.
- Invent their own algorithms given a task.
- Understand the purpose of algorithms in mathematics.
- Understand what is meant by absolute value and divisibility.
- Find the greatest common divisor of two integers using prime factors.
- Use the Euclidean algorithm to find the greatest common divisor of two integers.
- Understand and use properties of the greatest common divisor.

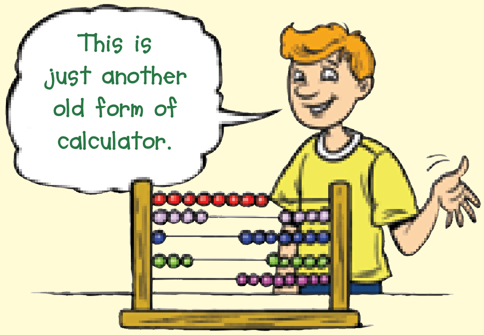
Areas of Interaction

Approaches to Learning (Knowledge Acquisition, Reflection, Organisation, Technology), Human Ingenuity — ordering instructions and the use of algorithms to simplify arithmetic

17:01 | Algorithms

An **algorithm** is a clearly defined set of instructions which, when followed in order, always lead to the solution of a particular problem. Algorithms solve problems in a finite number of steps.

For example:
Multiply 23 and 48.



There are a number of ways to do this (without a calculator). One is to write down 48 twenty-three times and add them up. Another is to set the problem out in a special way:

$$\begin{array}{r} 48 \times \\ 23 \\ \hline 144 \\ 960 \\ \hline 1104 \\ \hline \end{array}$$

This is also the same as multiplying by 3 and then by 20 and adding the two results. However, using addition and this method are different algorithms.

This algorithm can be used when multiplying any two 2-digit numbers. With small modifications it can be used to multiply any two numbers.

Algorithms are not restricted to mathematics. Directing a friend how to get to your place could also be considered an algorithm. In fact, anything that requires a set of instructions is an algorithm.

For example:
The following is a recipe for making a chocolate sponge cake.

Quantity	Ingredients
115 g	self-raising flour
115 g	caster sugar
115 g	margarine
2 small bars	milk chocolate
2	eggs
175 g	icing sugar
115 g	margarine

Method

1. Preheat oven to 180°C and grease two 18 cm cake tins.
2. Melt 1 bar of chocolate over a saucepan of boiling water.
3. Put the butter and caster sugar in a bowl and mix until light and fluffy and white in colour.
4. Add the eggs and a little of the flour and mix.
5. Then add the chocolate and the flour, a little a time, folding in till all is gone.
6. Add the mixture equally between the two tins.
7. Bake for about 20 mins.
8. Leave to cool in tins for 10 mins then turn out onto a cooling rack to fully cool.
9. Melt the rest of the chocolate and mix with the margarine and icing sugar.
10. Put some of the chocolate butter icing on one of the cakes and then sandwich together.
11. With the rest, spread over the top and sides of the cake then leave to set.
    You could decorate with chocolate sprinkles if like.

Source: http://www.chocolate-source.co.uk/chocolate_recipes_cakes_chocolate_cake.htm

A recipe requires a clear set of instructions for people to follow so that the turns out right.

As we have seen, there can be more than one algorithm to solve a problem. However, some algorithms are more efficient than others. This means they solve the problem with the least amount of steps or using the least amount of memory. This is particularly important when creating algorithms for computers to use. We want computers to be fast and efficient.

Exercise 17:01

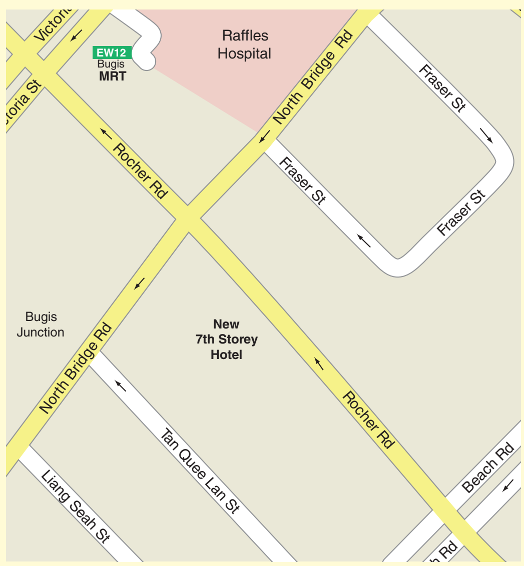
1 Write algorithms for the following and check your answers with others in the class. Decide whose algorithm is the most efficient.

- **a** Multiplying a number by 10.
- **b** Multiplying a number by 15.
- **c** How to get from your classroom to the school cafeteria (or canteen).
- **d** How to get from your classroom to the school's front gate.
- **e** How to find a book on your school library catalogue and borrow it.

2 Here is an algorithm for how Michael, who lives in Singapore at Block 357 Clementi Avenue 2, travels to his friend Theo who is staying at the New 7th Storey Hotel, in another part of Singapore. Michael is using the Singapore MRT train.

Leave apartment 2305 on 23rd floor. Turn left. Catch lift to ground floor.
Turn right. Leave apartment block 357. Walk to Clementi Avenue 2.
Turn left. Walk to Commonwealth Avenue West.
Turn right. Walk straight ahead.
Enter Clementi MRT train station.
Buy a ticket to Bugis. Board east bound train.
Get off train at Bugis. Exit station. Walk to Rocher Rd.
Cross Rocher Rd. Turn left. Cross North Bridge Road.
Walk straight ahead.
Stop at the New 7th Story Hotel. Turn right and enter the hotel.

- **a** Use the maps shown to write a similar algorithm for how Theo could travel to Michael's house.
- **b** Would this be the only algorithm for this journey?
- **c** Do a search on the internet to find another algorithm for Michael's trip.



3 List the steps involved when departing and arriving from your city's international airport.

4 Write an algorithm for how to simplify a fraction — use diagrams if it helps.

5 A simple computer can only determine whether an answer to a question is positive or negative. Write an algorithm for this computer to help it determine whether 3 numbers *a*, *b* and *c* are written in ascending order.

investigation 17:01

Investigation 17:01 | Sorting it out

Please use the Assessment Grid on the following page to help you understand what is required for this Investigation.

We find it very easy to order numbers because our brains can do lots of calculations very quickly and can do many simultaneously. We also recognise numbers from their appearance. Computers, however, need to perform calculations to compare numbers.

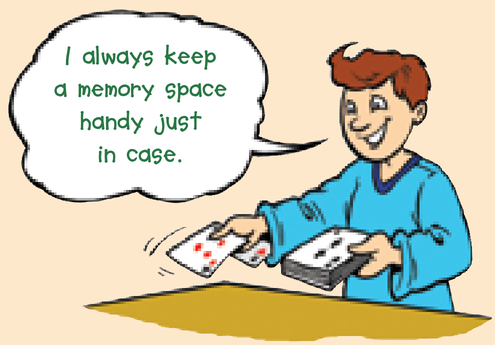
In this exercise you are to write an algorithm to make a computer arrange five numbered cards in the correct order. Below is an example only.

3	1	4	2	5

Each card is located in a memory space in the computer, so there are already five memory spaces used.

You may decide to use extra memory spaces in the design of your algorithm.

When your algorithm is finished, test it to make sure there are no ambiguities — remember, an algorithm *always* results in the correct solution.



Compare your algorithm to help you evaluate its efficiency. Remember, the efficiency of an algorithm is determined by the number of steps taken *and* the amount of memory used to solve the problem.

Assessment Grid for Investigation 17:01 | Sorting it out

The following is a sample assessment grid for this investigation. You should carefully read the criteria *before* beginning the investigation so that you know what is required.

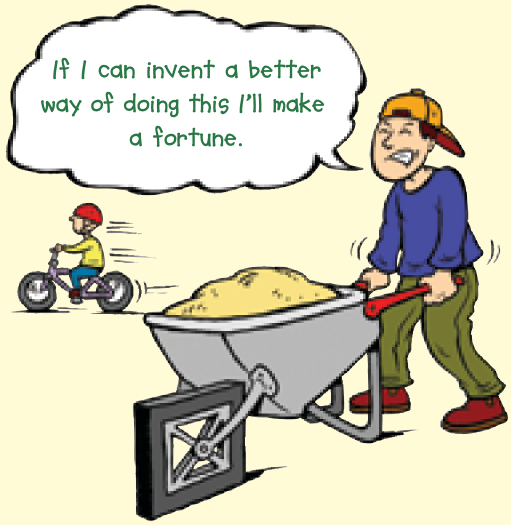
Criterion		Assessment Criteria (B, C, D) for this investigation		Achieved ✓
Criterion B Investigating Patterns	a	None of the standards below has been reached.	0	
	b	The student has recognised the necessity for using a calculation to compare numbers and the continuation of a pattern of calculations.	1	
			2	
	c	The student has recognised the type of calculation to be used and has recognised a pattern of instructions with some assistance.	3	
			4	
	d	The student has independently recognised a pattern for instructions and has written the pattern independently.	5	
			6	
	e	All of the above has been completed with recognition of efficiency as stated in the problem.	7	
			8	
Criterion C Communication in Mathematics	a	None of the standards below has been reached.	0	
	b	There is a basic use of mathematical language. There is some ambiguity in the instructions.	1	
			2	
	c	The use of mathematical language is sufficient and the steps are easy to follow. Some diagrams have been used to illustrate the use of memory spaces.	3	
			4	
	d	There is a good use of mathematical language and the steps given are effective. Diagrams have been used to illustrate the use of memory spaces and examples are worked through.	5	
			6	
Criterion D Reflection in Mathematics	a	None of the standards below has been reached.	0	
	b	The student has attempted to recognise the need for efficiency as outlined in the problem.	1	
			2	
	c	Modifications have been made to the design of the algorithm in an effort to make it more efficient. These modifications are documented.	3	
			4	
	d	All of the above has been completed. A full evaluation of the algorithm is given, after examining those of others in the class.	5	
			6	

17:02 | Absolute Value and the Division Algorithm

Algorithms are very common in mathematics and computer science. By developing algorithms to solve common problems, it saves having to solve the same problem over and over again. It is like having the formula for the circumference of a circle and applying it, rather than starting from scratch and developing it all over again.

It is the same in many occupations: by doing a course at university or college, you will learn how to solve many problems associated with your career choice because others have come across these problems before and have already solved them.

This is sometimes called 'reinventing the wheel': why solve it again when someone has already found a way.



Absolute Value

The **absolute value** of an integer is written $|a|$

so that $|a| = \begin{cases} a \text{ if } a \geqslant 0 \\ -a \text{ if } a < 0 \end{cases}$

worked examples

Find $|a|$ if

a $x = 4$ **b** $x = -4$ **c** $2x + 6 = 4$

Solutions

a $|x| = 4$

b $|x| = -(-4)$
$= 4$

c $2x + 6 = 4$
$2x = -2$
$x = -1$
$\therefore |x| = -(1)$
$\therefore |x| = 1$

The Division Algorithm

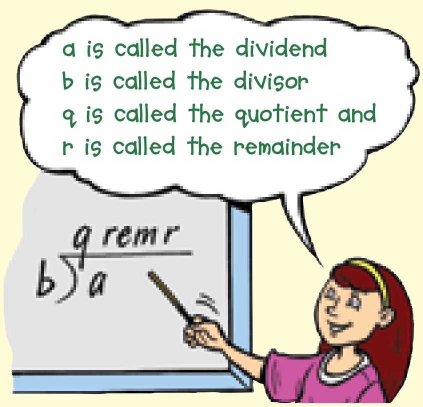
If a and b are two integers where $b \neq 0$, then the integers q and r exist so that

$$a = bq + r \text{ where } 0 \leqslant r < |b|$$

In other words, r is a positive integer smaller than $|b|$

You might recognise this if it is written a different way:

$$b\overline{)a}\ \text{with } q \text{ rem } r \text{ above}$$



worked examples

1 For each of the following, find q and r and write the division algorithm in the form

$a = bq + r$ where $0 \leqslant r < |b|$

a $a = 110, b = 3$

b $a = -354, b = 5$

c $a = 208, b = 4$

2 Show that any integer, n, can be written in the form

$n = 4p + k$ where p is an integer and $k = 0, 1, 2, 3$

Solutions

1 a $110 \div 3 = 36$ remainder 2 $\quad p = 36$ and $r = 2$

$\therefore 110 = 3 \times 36 + 2$

b $-354 \div 5 = -70$ remainder -4

However, *the remainder must be positive* so we must use $p = -71$ and $r = 1$

$\therefore 354 = 5 \times -71 + 1$

c $208 \div 4 = 52$ $\quad p = 52$ and $r = 0$

$\therefore 208 = 4 \times 52$

2 If $n = 4p + k$ then the divisor in this case is 4.

By the division algorithm, if any number n is divided by 4 then we can write

$n = 4p + r$ where r is an integer from 0 up to, but not including, 4.

For any integer n,

$n = 4p$, $n = 4p + 1$, $n = 4p + 2$ or $n = 4p + 3$

$\therefore n = 4p + k$ where $k = 0, 1, 2, 3$

Note: if $k = 4$ then $n = 4p_1 + 4$
$= 4(p_1 + 1)$
$= 4p_2$ which is the same form as $n = 4p$

also if $k = 5$ then $n = 4p_1 + 5$
$= 4p_1 + 4 + 1$
$= 4(p_1 + 1) + 1$
$= 4p_2 + 1$ which is the same form as $n = 4p + 1$

The same can be shown for $k = 6$ and $k = 7$ and so on.

Exercise 17:02

1 For each of the following, find the value of $|x|$

a $x = 6$ **b** $x = -7$ **c** $2x + 7 = 1$ **d** $5 - 3x = 14$ **e** $x^2 = 36$

2 Write in the form of the division algorithm $a = bq + r$ given that:

a $a = 95$ and $b = 6$

b $a = -25$ and $b = 4$

c $a = 50$ an $b = -8$

3 A number, when divided by 6, gives 7 with a remainder of 3. Find the number.

4 Use the division algorithm to solve the following:

a If the quotient is 16, the remainder is 5 and the dividend is 19, find the divisor.

b If the dividend is 157, the divisor is 10 and the remainder is 7, find the quotient.

c If the divisor is 17, the quotient is 14 and the remainder is 10, find the dividend.

5 Show that any integer n, can be written in the form $n = 5q + r$ where r is an integer and $0 \leqslant r < 5$.

6 Prove that if three consecutive integers are chosen, one is divisible by 3 (there is no remainder when it is divided by 3).

17:03 | Divisibility, Greatest Common Divisor and the Euclidean Algorithm

Divisibility

If an integer a is *divisible* by another integer b, then by the division algorithm, $a = kb$, where k is an integer and there is no remainder.

In this case we say that b **divides** a or we can write $b|a$.

For example: $3|6$ since $3 \times 2 = 6$

Greatest Common Divisor

The *greatest common divisor* (gcd), d, of two integers a and b is the largest integer that divides both a and b. This is sometimes called the highest common factor.

In this case we can write $d = \gcd(a, b)$

For example: $\gcd(12, 18) = 6$
since 6 is the largest integer that divides both 12 and 18

worked examples

1. By listing all the factors of 24 and 32, find gcd(24,32)
2. By using prime factors, find gcd(24,36)
3. Use any method to find gcd(1120, 1680)

Solutions

1. Factors of 24 = {1, 2, 3, 4, 6, 8, 12, 24}
   Factors of 32 = {1, 2, 4, 8, 16, 24}
   $\therefore$ gcd(24, 32) = 8

2. $24 = 2 \times 12$
   $= 2 \times 2 \times 6$
   $= 2 \times 2 \times 2 \times 3$ These are the prime factors of 24.

   $36 = 2 \times 18$
   $= 2 \times 2 \times 9$
   $= 2 \times 2 \times 3 \times 3$ These are the prime factors of 36.

   By selecting the common prime factors from each, we obtain $2 \times 2 \times 3 = 12$
   $\therefore$ gcd(24, 36) = 12

3. $1120 = 2 \times 2 \times 2 \times 2 \times 2 \times 5 \times 7$
   $1680 = 2 \times 2 \times 2 \times 2 \times 3 \times 5 \times 7$
   $\therefore \text{gcd}(1120, 1680) = 2 \times 2 \times 2 \times 2 \times 5 \times 7$
   $= 560$

   OR

   factors of 1120 = {1, 2, 4, 7, 8, 10, 14, 16, 70, 80, 20, 28, 32, 35, 40, 56, 112, 140, 160, 280, 560, 1120}
   factors of 1680 = {1, 2, 3, 3, 4, 5, 336, ... 420, 560, 840, 1680}
   We do not need to complete the factors of 1680 as we can already see that 560 is the gcd
   $\therefore$ gcd(1120, 1680) = 560

Euclidean Algorithm

As the last example shows, finding the greatest common divisor using the methods shown can sometimes be very time consuming. The **Euclidean algorithm** was developed to help find the gcd quickly and efficiently.

The Euclidean algorithm applies the division algorithm over and over again until the remainder is zero.

For example, suppose we wish to find gcd(1120, 1680). Using the division algorithm we find:

$1680 = 1(1120) + 560$ Now we divide the quotient by the remainder to get:
$1120 = 2(560)$ Here the remainder is zero.

The greatest common divisor is the *last non-zero remainder*. In this case it is 560.
$\therefore$gcd(1120, 1680) = 560

Often this division process needs to be applied many times to get a remainder of zero.

worked examples

Find the following:

a gcd(168, 240) **b** gcd(210, 336) **c** gcd(639, 852)

Solutions

a $240 = 1(168) + 72$
$168 = 2(72) + 24$
$72 = 3(24)$
$\therefore \text{gcd}(168, 240) = 24$

b $336 = 1(210) + 126$
$210 = 1(126) + 84$
$126 = 1(84) + 42$
$84 = 2(42)$
$\therefore \text{gcd}(210, 336) = 42$

c $852 = 1(639) + 213$
$639 = 3(213)$
$\therefore \text{gcd}(639, 852) = 213$

Exercise 17:03

1 In each of the following, find the greatest common factor of a and b by listing all their factors.

a $a = 24, b = 28$ **b** $a = 36, b = 42$ **c** $a = 108, b = 72$
d $a = 289, b = 408$ **e** $a = 1024, b = 1856$

2 In each of the following, find the greatest common factor of a and b by writing them as a product of their prime factors.

a $a = 120, b = 630$ **b** $a = 525, b = 675$ **c** $a = 490, b = 6125$
d $a = 1050, b = 1350$ **e** $a = 1386, b = 1764$

3 In each of the following, find the greatest common factor of a and b by using the Euclidean algorithm.

a $a = 70, b = 154$ **b** $a = 126, b = 360$ **c** $a = 216, b = 312$
d $a = 1274, b = 1470$ **e** $a = 2850\ b = 6150$

Investigation 17:03 | Common divisors

Please use the Assessment Grid on the following page to help you understand what is required for this Investigation.

In this investigation you are going to explore some of the properties of the greatest common divisor.

1 Complete the following table — remember to give evidence of your working out.

	a	b	$d = \gcd(a, b)$	Does $d\|na$, $n = 2, 3, 4, 5$?	Does $d\|nb$, $n = 2, 3, 4, 5$?	Does $d\|a+b$?	Does $d\|n(a+b)$, $n = 2, 3, 4, 5$?
	24	32	8	Yes	Yes	Yes	Yes
	24	36	12	Yes	Yes	Yes	Yes
	8	12					
	18	42					
	64	84					
	110	132					
Choose your own integers here.							

2 From your results in the table above, form some conjectures about d [$=\gcd(a, b)$], multiples of a [na], multiples of b [nb], $a + b$ and multiples of $a + b$ [$n(a + b)$].

3 By making further investigations (eg choose more numbers of your own, and multiples other than $n = 2, 3, 4$ and 5), show that your conjectures are true.

Assessment Grid for Investigation 17:03 | Common divisors

The following is a sample assessment grid for this investigation. You should carefully read the criteria *before* beginning the investigation so that you know what is required.

Assessment Criteria (B, C, D) for this investigation				Achieved ✓
Criterion B Investigating Patterns	a	None of the standards below has been reached.	0	
	b	The table has been completed but patterns and objective have not been recognised.	1	
			2	
	c	The table has been completed independently; however, the student required some assistance in choosing integers and in recognising patterns.	3	
			4	
	d	The student has independently completed the table and recognised patterns. Conjectures have been made but no further investigation has been carried out.	5	
			6	
	e	The student has independently completed the table and recognised the patterns. Correct conjectures have been made and supported by further investigation.	7	
			8	
Criterion C Communication in Mathematics	a	None of the standards below has been reached.	0	
	b	There is a basic use of mathematical language. Working out is not shown as instructed.	1	
			2	
	c	The use of mathematical language is sufficient and the working out is shown and easy to follow.	3	
			4	
	d	There is a good use of mathematical language and working out is complete. Conjectures are clear and easy to understand.	5	
			6	
Criterion D Reflection in Mathematics	a	None of the standards below has been reached.	0	
	b	The student has made an attempt to explain whether the results make sense.	1	
			2	
	c	An explanation of whether the results make sense is given in a concluding discussion of the investigation.	3	
			4	
	d	A full explanation of whether the results make sense is given and justified in a concluding discussion of the investigation.	5	
			6	

17:04 | Properties of gcd(*a*, *b*)

From investigation 17:03 we can deduce the following properties of the greatest common divisor:

If $d = \gcd(a, b)$ then:

d divides any multiple of a or b	$d\mid na$, $d\mid nb$	where $n \in Z$ {*integers*}
d divides any multiple of $a + b$	$d\mid n(a + b)$	where $n \in Z$ {*integers*}

If we combine these two properties it follows that:

d divides the sum of all multiples of a and b: $d\mid(ma + nb)$, where $m, n \in Z$

Now, if d divides all multiples of a and b then it follows that one of these combinations of a and b equals d.

So we get this special result:

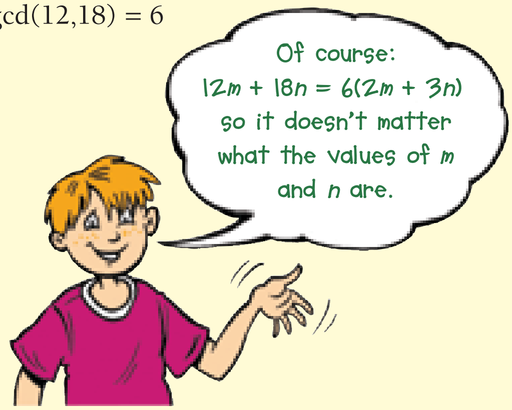
If $d = \gcd(a, b)$ then there exists integers x and y so that $d = ax + by$

For example:
From one of the previous examples it was found that $\gcd(12,18) = 6$

This means that 6 divides all expressions of the form $12m + 18n$, where $m, n \in Z$

This also means that there must be one $m = x$ and one $n = y$ so that $d = 12x + 18y$

In this case, $6 = -1(12) + 1(18)$
so that $x = -1$ and $y = 1$



This becomes a little harder when we consider, for example, gcd(168, 240)

From the previous section we know that $\gcd(168, 240) = 24$, so this means that there must be an x and y so that $24 = 168x + 240y$

To find x and y we go back to the Euclidean algorithm solution:

$240 = 1(168) + 72$ (1)
$168 = 2(72) + 24$ (2)
$72 = 3(24)$ (3)
$\therefore \gcd(168, 240) = 24$

From line (2) we can see that
$24 = 168 - 2(72)$ (4)
From line (1) we can see that $72 = 240 - 1(168)$
By substituting this for 72 in expression (4) we get that

$$\begin{aligned} 24 &= 168 - 2(240 - 1(168)) \\ &= 168 - 2(240) + 2(168) \\ &= 3(168) - 2(240) \end{aligned}$$

$24 = 3(168) - 2(240)$,
$\therefore x = 3$ and $y = -2$

worked examples

For each of following vales of a and b, find:

i gcd(a, b) using the Euclidean algorithm

ii values of x and y such that gcd(a, b) = $ax + by$

a $a = 460$, $b = 1196$

b $a = 748$, $b = 1020$

c $a = 322$, $b = 506$

Solutions

a **i**
$1196 = 2(460) + 276$ (1)
$460 = 1(276) + 184$ (2)
$276 = 1(184) + 92$ (3)
$184 = 2(92)$ (4)
$\therefore$ gcd(460, 1196) = 92

ii From (3): $92 = 276 - 1(184)$ (5)
From (2): $184 = 460 - 1(276)$
Substituting into (5):
$92 = 276 - 1(460 - 1(276))$
$= 276 - 460 + 276$
$= 2(276) - 460$ (6)
From (1): $276 = 1196 - 2(460)$
substituting into (6):
$92 = 2(1196 - 2(460)) - 460$
$= 2(1196) - 4(460) - 460$
$= 2(1196) - 5(460)$
$\therefore$ $x = -5$ and $y = 2$
Note that $a = 460$ and $b = 1196$.

b **i**
$1020 = 1(748) + 272$ (1)
$748 = 2(272) + 204$ (2)
$272 = 1(204) + 68$ (3)
$204 = 3(68)$ (4)
$\therefore$ gcd(748, 1020) = 68

ii From (3): $68 = 272 - 1(204)$ (5)
From (2): $204 = 748 - 2(272)$
Substituting into (5):
$68 = 272 - 1(748 - 2(272))$
$= 272 - 748 + 2(272)$
$= 3(272) - 748$ (6)
From (1): $272 = 1020 - 1(748)$
Substituting into (6):
$92 = 3(1020 - 1(748)) - 748$
$= 3(1020) - 3(748) - 748$
$= 3(1020) - 4(748)$
$\therefore$ $x = -4$ and $y = 3$

c **i**
$506 = 1(322) + 184$ (1)
$322 = 1(184) + 138$ (2)
$184 = 1(138) + 46$ (3)
$138 = 3(46)$ (4)
$\therefore$ gcd(322, 502) = 68

ii From (3): $46 = 184 - 138$ (5)
From (2): $138 = 322 - 184$
Substituting into (5):
$46 = 184 - (322 - 184)$
$= 184 - 322 + 184$
$= 2(184) - 322$ (6)
From (1): $184 = 506 - 322$
Substituting into (6):
$92 = 2(506 - 322) - 322$
$= 2(506) - 2(322) - 322$
$= 2(506) - 3(322)$
$\therefore$ $x = -3$ and $y = 2$

An extra note: If two numbers a and b are prime, and a does not divide b and b does not divide a, then gcd(a, b) = 1

In this case, a and b are known as *relatively prime*.

Some examples of relatively prime pairs of numbers: 11 and 13, 23 and 47, 19 and 37.

Exercise 17:04

1 In each of the following, use the results from Exercise 17:03 question **3** to find x and y so that $\gcd(a, b) = ax + by$:

a $a = 70, b = 154$ **b** $a = 126, b = 360$ **c** $a = 216, b = 312$
d $a = 1274, b = 1470$ **e** $a = 2850\ b = 6150$

2 **a** Show that gcd(2176, 2944) = 128
b Hence find the values of x and y so that $128 = 2176x + 2944y$

3 If $d = \gcd(1232, 4592)$ find the values of x and y such that $d = 1232x + 4592y$

4 **a** Find gcd(47, 111)
b What does this infer about 47 and 111? See if you can find a name for this relationship between numbers.

5 **a** Find the two smallest values of a and b so that $256 = \gcd(a, b)$, where $a, b \neq 256$
b Use the Euclidean algorithm to prove 256 is the $\gcd(a, b)$.

Mathematical Terms 17

mathematical terms 17

absolute value
- absolute value of an integer is written $|a|$ so that $|a| = \begin{cases} a \text{ if } a \geqslant 0 \\ -a \text{ if } a < 0 \end{cases}$

algorithm
- A clearly defined set of finite instructions which, when followed in order, always lead to the solution of a particular problem.

divisibility
- If an integer a is *divisible* by another integer b, then by the division algorithm, $a = kb$, where k is an integer as there is no remainder. In this case we say that *b divides a* or we can write $b|a$.

division algorithm
- If a and b are two integers where $b > 0$, then the integers q and r exist so that $a = bq + r$ where $0 \leqslant r < |b|$

Euclidean algorithm
- Developed to help find the gcd quickly and efficiently.

greatest common divisor
- The greatest integer that divides the given integers.
- Sometimes called the highest common factor.
- If d is the greatest common divisor of a and b we write $d = \gcd(a, b)$

relatively prime numbers
- Two numbers whose greatest common divisor is 1. eg: 13 and 23

diagnostic test 17

Diagnostic Test 17: | An Introduction to Algorithms and Number Theory

- Each section of the test has similar items that test a certain type of example.
- Failure in more than one item will identify an area of weakness.
- Each weakness should be treated by going back to the section listed.

	Section
1 Write an algorithm for withdrawing money from an automatic teller machine (ATM).	17:01
2 Write an algorithm, giving an example, of how to divide a three-digit number by a two-digit number.	17:01
3 Write in the form of the division algorithm $a = bq + r$ given that: **a** $a = 108$ and $b = 6$ **b** $a = -259$ and $b = 8$ **c** $a = 1267$ and $b = -10$	17:02
4 If the quotient is 52, the remainder 12 and the dividend 1312, find the divisor.	17:02
5 Find gcd(126, 1050) by listing all necessary factors.	17:03
6 Find gcd(208, 512) by using prime factors.	17:03
7 **a** Use the Euclidean algorithm to find $d = \gcd(2392, 5336)$ **b** Hence find values of x and y such that $d = 2392x + 5336y$	17:03 17:04

Chapter 17 | Revision Assignment

assignment 17A

1 Write an algorithm explaining how to get from your house to your school.

2 Find the value of $|x|$:
   a $3x + 5 = 2$ b $x^2 = 25$

3 A certain number when divided by 7 gives 5, with a remainder of 2. Find the number.

4 Find the greatest common divisor of 120 and 56 by listing all their factors.

5 By using prime factors find gcd(480, 704).

6 a Find gcd(5700, 9225) by using the Euclidean algorithm.
   b Hence find x and y such that $\gcd(5700, 9225) = 5700x + 9225y$.

Answers

Chapter 1: Basic Skills and Number — Review of Books 1 to 4

Exercise 1:01A

1 **a** 2 **b** 1 **c** 0 **d** 14 **e** –3 **f** 17 **g** 12 **h** 30
i 70 **j** 20 **k** 300 **l** 25 **m** 19 **n** 11 **o** 18

2 **a** 9 **b** 0 **c** 10 **d** 2 **e** $1\frac{1}{5}$ **f** 2 **g** 81 **h** 196 **i** 100

Exercise 1:01B

1 **a** $1\frac{3}{4}$ **b** $8\frac{1}{6}$ **c** $3\frac{3}{4}$ **d** $1\frac{3}{8}$
2 **a** $\frac{11}{2}$ **b** $\frac{22}{7}$ **c** $\frac{35}{4}$ **d** $\frac{200}{3}$
3 **a** $\frac{3}{5}$ **b** $\frac{7}{15}$ **c** $\frac{2}{3}$ **d** $\frac{5}{9}$
4 **a** $\frac{18}{24}$ **b** $\frac{20}{50}$ **c** $\frac{8}{28}$ **d** $\frac{40}{120}$
5 **a** $\frac{8}{15}$ **b** $\frac{1}{4}$ **c** $\frac{37}{40}$ **d** $\frac{9}{35}$
6 **a** $9\frac{1}{10}$ **b** $2\frac{9}{20}$ **c** $10\frac{17}{20}$ **d** $3\frac{19}{40}$
7 **a** $\frac{12}{35}$ **b** $\frac{27}{40}$ **c** $\frac{2}{15}$ **d** $\frac{7}{15}$
8 **a** $4\frac{1}{2}$ **b** $4\frac{1}{2}$ **c** 20 **d** 15
9 **a** $1\frac{7}{20}$ **b** $\frac{5}{8}$ **c** $\frac{2}{15}$ **d** $1\frac{5}{6}$

Exercise 1:01C

1 **a** {0·066, 0·6, 0·606, 0·66} **b** {0·153, 1·053, 1·53} **c** {0·017, 0·7, 0·77, 7} **d** {3·05, 3·4, 3·45, 3·5}
2 **a** 9·301 **b** 3·45 **c** 3·104 **d** 6·32 **e** 1·97 **f** 8·105 **g** 4·888 **h** 159·3
3 **a** 0·036 **b** 0·006 **c** 0·585 **d** 0·0025
4 **a** 31·4 **b** 500 **c** 0·03 **d** 38 000
5 **a** 0·03 **b** 0·265 **c** 3·07 **d** 0·0025
6 **a** $0{\cdot}4\dot{3}$ **b** $0{\cdot}8\dot{2}\dot{7}$ **c** $1{\cdot}\dot{5}$ **d** $0{\cdot}\dot{8}5714\dot{2}$
7 **a** 4·804 **b** 0·016 **c** 0·0009 **d** 0·000 65
8 **a** 21 **b** 10·45 **c** 1500 **d** 2·8
9 **a** $3\frac{17}{1000}$ **b** $\frac{1}{25}$ **c** $\frac{43}{50}$ **d** $16\frac{1}{200}$
10 **a** 0·8 **b** 0·035 **c** 0·625 **d** $0{\cdot}\dot{7}\dot{2}$
11 **a** $\frac{5}{9}$ **b** $\frac{257}{999}$ **c** $\frac{8}{11}$ **d** $\frac{214}{333}$
12 **a** $\frac{5}{6}$ **b** $\frac{151}{165}$ **c** $\frac{98}{225}$ **d** $\frac{1489}{1665}$

Exercise 1:01D

1 **a** $\frac{27}{50}$ **b** $\frac{203}{100}$ or $2\frac{3}{100}$ **c** $\frac{49}{400}$ **d** $\frac{91}{1000}$
2 **a** 55% **b** $44\frac{4}{9}$% **c** 125% **d** $66\frac{2}{3}$%
3 **a** 0·16 **b** 0·086 **c** 0·03 **d** 0·1825
4 **a** 47% **b** 6% **c** 37·5% **d** 130%
5 **a** 144 m **b** 7·56 g **c** \$2.72 **d** \$86 360
6 **a** \$60 **b** 25 kg **c** \$5 **d** 180 min or 3 h
7 **a** 42·5% **b** 45% **c** 18·75% **d** 12% (to nearest whole %)

Exercise 1:01E

1 **a** **i** 3 : 5 **ii** 1 : 10 **iii** 15 : 7 **iv** 3 : 1 **b** 14 : 1 **c** 2 : 3 **d** 15 : 4
e **i** 8 : 5 **ii** 41 : 130 **f** **i** $\frac{3}{5}$: 1 **ii** $\frac{2}{7}$: 1 **iii** $\frac{10}{3}$: 1 **iv** $\frac{25}{4}$: 1
g **i** 1 : $\frac{5}{3}$ **ii** 1 : $\frac{7}{2}$ **iii** 1 : $\frac{3}{10}$ **iv** 1 : $\frac{4}{25}$

2 **a** $x = 50$ **b** 910 million **c** 2·5 people per km^2 **d** 3·6 million

3 **a** Naomi gets 48, Luke gets 36 **b** 40°, 60°, 80° **c** Tokyo, 12 million; Moscow, 10 million
d 36 males, 24 females

Exercise 1:01F

1 **a** 300 km/h **b** 8 m/mL **c** 14·4 t/day **d** 2075 cm^3/kg
2 **a** 6 miles per hour **b** \$46.20 **c** $4\frac{2}{3}$ minutes/book **d** 10 cm^3/s

Exercise 1:01G

1 **a** 2 **b** 2 **c** 3 **d** 3 **e** 4 **f** 3 **g** 4 **h** 3 **i** 3 **j** 4 **k** 1 **l** 1
m 2 **n** 1 **o** 1 **p** 2 **q** 3 **r** 3 **s** 5 **t** 3 **u** 5 **v** 1 **w** 3 **x** 2
y ambiguous, 2 **z** ambiguous, 2

2 **a** 2 **b** 3 **c** 1 **d** 1 **e** 2 **f** 2 **g** 3 **h** 5

Exercise 1:01H

1 **a** 4·6 **b** 0·8 **c** 3·2 **d** 0·1 **e** 15·2 **f** 8·1 **g** 1·0 **h** 121·6
i 0·1 **j** 47·4 **k** 0·4 **l** 2·8

2 **a** 0·54 **b** 2·61 **c** 7·13 **d** 1·17 **e** 12·02 **f** 8·40 **g** 412·68 **h** 0·08
i 0·44 **j** 100·33 **k** 0·02 **l** 0·01

3 **a** **i** 7 **ii** 7·3 **b** **i** 80 **ii** 85 **c** **i** 0·6 **ii** 0·63 **d** **i** 3 **ii** 2·6 **e** **i** 4 **ii** 4·2
f **i** 0·007 **ii** 0·0073 **g** **i** 0·08 **ii** 0·083 **h** **i** 3 **ii** 3·1 **i** **i** 0·009 **ii** 0·0093 **j** **i** 0·01 **ii** 0·0098
k **i** 8 **ii** 7·5 **l** **i** 0·04 **ii** 0·036

4 **a** 2 **b** 14·6 **c** 2·2 **d** 0·9 **e** 4·1 **f** 7·37 **g** 0·724 **h** 6 **i** 31·69 **j** 0·007 **k** 0·8 **l** 0·0072

5 **a** 5·6 **b** 0·2 **c** 0·44 **d** 15·4 **e** 8·33 **f** 413·8 **g** 72·0 **h** 3·067 **i** 10·0 **j** 4·800 **k** 0·08 **l** 0·004

Exercise 1:01I

1 Answers near these are acceptable. **a** 74 **b** 120 **c** 31 **d** 7·2 **e** 110 **f** 60 **g** 18 **h** 5·8
i 7·7 **j** 1·8 **k** 0·4 **l** 52 **m** 15 **n** 21 **o** 310 **p** 1·1 **q** 7·0 **r** 17 **s** 59

Exercise 1:02

1 **a** $3a + 4b$ **b** $12ab$ **c** $k - m$ **d** $m - k$ **e** $\frac{x+y+z}{3}$ **f** $2(m + 5)$ **g** $(a - b)^2$ **h** $\sqrt{5m + 4n}$
i $n + 2$ **j** $m + (m + 1) + (m + 2) = 3m + 3$

2 **a** 21 **b** 2 **c** 16 **d** −15 **e** −43 **f** 45 **g** −2 **h** 5
i $\frac{7}{2}$ or $3\frac{1}{2}$ **j** 3 **k** 1 **l** 3

3 **a** $4a + 4b$ **b** $3ab$ **c** $2x^2 + 2x$ **d** $15xy$ **e** $18a^2b$ **f** $-10m^2n$ **g** $3a$ **h** 2
i $2a$ **j** $\frac{1}{3}$ **k** $\frac{3m}{2n}$ **l** $\frac{3y}{2x}$ **m** 21 **n** $10y$ **o** $5x$

4 **a** $\frac{6a}{5}$ **b** $\frac{2x}{7}$ **c** $\frac{7}{y}$ **d** $\frac{7a}{12}$ **e** $\frac{7m}{15}$ **f** $\frac{1}{6n^2}$ **g** $\frac{ab}{12}$ **h** $\frac{2m^2}{15}$
i $\frac{3ay}{2}$ **j** $\frac{5}{2}$ **k** $2b$ **l** xz

5 **a** $a + 3$ **b** $8m - 10$ **c** $5a + 15$ **d** $8n + 7$ **e** $5a - 19$ **f** $3 - 2x$
g $x^2 + 10x + 21$ **h** $y^2 - 5y + 4$ **i** $k^2 + 2k - 63$ **j** $2p^2 - 7p - 15$ **k** $18x^2 - 9x - 2$ **l** $6m^2 - 17m + 5$
m $m^2 - 49$ **n** $9a^2 - 16$ **o** $100 - 9q^2$ **p** $a^2 + 16a + 64$ **q** $4m^2 - 4m + 1$ **r** $16a^2 + 40a + 25$
s $x^2 - xy - 2y^2$ **t** $a^2 - 4b^2$ **u** $m^2 - 6mn + 9n^2$

6 **a** $5(3a - 2)$ **b** $3m(m - 2)$ **c** $2n(2 + 3m)$ **d** $2m(3n - 2)$
e $5y(2y + 1)$ **f** $2a(3a - 1 + 2b)$ **g** $(x - 7)(x + 7)$ **h** $(10 - a)(10 + a)$
i $(4a - 3b)(4a + 3b)$ **j** $(x + 6)(x + 2)$ **k** $(x - 4)(x + 3)$ **l** $(x - 4)(x - 2)$
m $(a + 3)^2$ **n** $(y - 5)^2$ **o** $(1 - 2m)^2$ **p** $(2x + 1)(x + 3)$
q $(3m - 2)(m + 3)$ **r** $(3a - 4)(2a - 1)$ **s** $(2n + 3)^2$ **t** $(5x - 1)^2$
u $(3 - 4m)^2$ **v** $(a + x)(b - 4)$ **w** $(x - 2)(x + a)$ **x** $(2m - 1)(m + 3n)$

7 **a** $2(x - 3)(x + 3)$ **b** $4(x + 3)(x - 2)$ **c** $3(a - b)(a - 2)$ **d** $2(2n - 1)^2$
e $9(1 - q)(1 + q)$ **f** $m^2(m - 1)(m + 1)$ **g** $(k^2 + 4)(k - 2)(k + 2)$ **h** $(y^2 + 1)(y + 1)$
i $(x + 1)(x - 1)^2$

8 **a** $a + 4$ **b** 5 **c** $\frac{1}{a+2}$ **d** $\frac{m}{m+1}$ **e** $\frac{n-3}{n+3}$ **f** $\frac{x+1}{2x+3}$

9 **a** $6x$ **b** $a + 1$ **c** $\frac{3}{2}$ **d** 5 **e** $\frac{a+3}{a+7}$ **f** $\frac{1}{3(n-1)}$

10 **a** $\frac{2a+7}{(x+3)(x+4)}$ **b** $\frac{2x+14}{(2x-1)(4x+3)}$ **c** $\frac{5x+2}{x(x+2)(x+1)}$ **d** $\frac{2x+5}{(x+3)(x+4)(x+5)}$

11 **a** $\frac{a}{(a-1)(a+1)}$ **b** $\frac{2x-7}{3(x+2)(x-2)}$ **c** $\frac{5x-4}{(x+3)(x-2)(x+1)}$ **d** $\frac{3x-12}{(x-2)(x+1)(x-3)}$
e $\frac{2x^2+x-5}{(x-3)(x+3)(x-2)}$ **f** $\frac{10n+18}{(2n-1)(n+1)(n+3)}$

Exercise 1:03

1 **a** $\frac{1}{3}$ **b** $\frac{1}{10}$ **c** $\frac{11}{30}$ **d** $\frac{7}{10}$ **2** **a** $\frac{1}{6}$ **b** $\frac{1}{3}$ **c** $\frac{1}{2}$ **d** 1

3 **a** $\frac{1}{3}$ **b** $\frac{7}{12}$ **c** $\frac{3}{4}$ **d** 0

4 **a** $\frac{1}{2}$ **b** $\frac{1}{2}$ **c** $\frac{1}{13}$ **d** $\frac{12}{13}$ **e** $\frac{3}{13}$ **f** $\frac{1}{26}$ **g** $\frac{1}{4}$ **h** 0 **i** $\frac{3}{26}$ **j** $\frac{15}{52}$ **k** $\frac{1}{13}$ **l** $\frac{3}{4}$ **m** $\frac{11}{26}$ **n** $\frac{37}{52}$ **o** $\frac{4}{13}$ **p** $\frac{2}{13}$ **q** $\frac{23}{52}$

Exercise 1:04

1 **a** $x = 79$ (corresp. ∠s, ∥ lines)

b $x + 76 = 180$ (co-int. ∠s, ∥ lines)
$x = 104$

c $x + 55 = 130$ (ext. ∠ of Δ)
$x = 75$

d $\angle ACB = b°$ (base ∠s of isos. Δ)
$b + b + 36 = 180$ (∠ sum of Δ)
$\therefore b = 72$

e $x = 105$ (opp. ∠s of a par'm)

f $x = 40$ (∠ sum of a quad.)
$y = 40$ (base ∠s of isos. Δ)

2 **a** **i** 720 **ii** 1440 **b** **i** 108° **ii** 135° **c** 360° **d** **i** 72° **ii** 60°

3 a $\angle BED = \angle ECD + \angle CDE$ (ext. $\angle$ of Δ)
$\angle ABC = \angle ECD$ (alt. $\angle$s, $AB \parallel CD$)
$\therefore \angle BED = \angle ABC + \angle CDE$

b ΔAOB is isosceles
since $AO = BO$ (radii)
$\therefore \angle OAB = \angle OBA$ (base $\angle$s isos. Δ)
$\angle DOA = \angle OAB + \angle OBA$ (ext. $\angle$ of Δ)
$\therefore \angle DOA = 2 \times \angle OBA$
Similarly, $\angle DOC = 2 \times \angle OBC$
$\therefore \angle DOA + \angle DOC = 2 \times (\angle OBA + \angle OBC)$
ie $\angle AOC = 2 \times \angle ABC$

4 a In Δs OCA and OBC
OC is common
$AO = BO$ (radii)
$\angle OCA = \angle OCB = 90°$ (given)
$\therefore \Delta OCA \equiv \Delta OBC$ (RHS)
$\therefore AC = BC$ (corres. sides in cong. Δs)

b In Δs BED and CFD
$BD = CD$ (D is midpt BC)
$\angle BED = \angle CFD = 90°$ (given)
$\angle BDE = \angle CDF$ (vert. opp. $\angle$s)
$\therefore \Delta BED \equiv \Delta CFD$ (AAS)
$\therefore BE = CF$ (corres. sides in cong. Δs)

5 a In Δs ABD and CBD
BD is common
$\angle BDA = \angle BDC = 90°$ ($BD \perp AC$)
$\angle ABD = \angle CBD$ (AC bisects $\angle ABC$)
$\therefore \Delta ABD \equiv \Delta CBD$ (AAS)
$\therefore AB = CB$ (corres. sides in cong. Δs)
$\therefore \Delta ABC$ is isos. (2 equal sides)

b In Δs WXY and YZW
WY is common
$\angle XWY = \angle ZYW$ (alt. $\angle$s, $WX \parallel YZ$)
$\angle XYW = \angle ZWY$ (alt. $\angle$s, $XY \parallel WZ$))
$\therefore \Delta WXY \equiv \Delta YZW$ (AAS)
$\therefore \angle WXY = \angle YZW$ (corres. $\angle$s in cong. Δs)

6 a $YZ = 4{\cdot}55$ cm (to 2 dec. pl.)

b i $AB^2 = x^2 + 4^2$
ii $BC^2 = x^2 + 8^2$
iii ie $AB^2 + BC^2 = 2x^2 + 16 + 64$
$= 2x^2 + 80$
But $AB^2 + BC^2 = 12^2 = 144$
So $2x^2 + 80 = 12^2 = 144$
ie $x = \sqrt{32}$ ($\doteqdot 5{\cdot}6$)

Exercise 1:05

1 a a^3 **b** 2^4 **c** n^5 **d** 10^3
2 a 2^9 **b** a^5 **c** m^5 **d** 10^8 **e** a^8 **f** y **g** b^2 **h** 10^3
i m^{12} **j** a^6 **k** x^8 **l** 10^{10} **m** 3 **n** 2 **o** 6 **p** e^0 or 1
q $30a^2$ **r** $2m^3$ **s** $30a^2$ **t** $16x^8$ or 4^2x^8
3 a $30a^5b^3$ **b** $56a^5b^3$ **c** $24a^4b^7$ **d** $10a^{10}b^3$ **e** $49x^6$ **f** $16m^8$ **g** x^6y^9 **h** $625x^4y^8$
i $6a^2$ **j** $5x^3$ **k** $3a$ **l** 8
4 a $\frac{1}{4}$ **b** $\frac{1}{10}$ **c** $\frac{1}{x}$ **d** $\frac{2}{a}$ **e** $\frac{1}{25}$ **f** $\frac{1}{8}$ **g** $\frac{1}{m^3}$ **h** $\frac{5}{x^2}$
5 a 3^{-1} **b** 8^{-1} **c** a^{-1} **d** $3x^{-1}$ **e** 2^{-4} **f** 10^{-6} **g** y^{-4} **h** $5n^{-3}$
6 a 3 **b** 6 **c** 2 **d** 3 **7 a** $a^{\frac{1}{2}}$ **b** $y^{\frac{1}{3}}$ **c** $5m^{\frac{1}{2}}$ **d** $4x^{\frac{1}{2}}$
8 a x^2 **b** $5m^5$ **c** $12n$ **d** $18y^{\frac{9}{2}}$ **e** $2x$ **f** $3x^2$ **g** $2a$ **h** 2 **i** $6x^2$
9 a $1{\cdot}48 \times 10^8$ **b** $6{\cdot}8 \times 10^4$ **c** $1{\cdot}5 \times 10^{-4}$ **d** $1{\cdot}65 \times 10^{-6}$
10 a 62 000 **b** 1 150 000 **c** 0·0074 **d** 0·000 069 1

Exercise 1:06

1 a rational **b** rational **c** irrational **d** rational **e** irrational **f** irrational **g** rational **h** rational
2 a 2·6 **b** 3·7 **c** 0·3 **d** 0·9
3 a $5\sqrt{2}$ **b** $\sqrt{35}$ **c** $\sqrt{6}$ **d** $6\sqrt{3}$ **e** $\sqrt{10}$ **f** $\sqrt{7}$ **g** $\sqrt{26}$ **h** $\frac{7}{9}$ **i** 7 **j** 6 **k** 75 **l** $8\sqrt{3}$
4 a $5\sqrt{3}$ **b** $6\sqrt{2}$ **c** $6\sqrt{5}$ **d** $-3\sqrt{3}$ **e** $5\sqrt{5}$ **f** $2\sqrt{2}$ **g** $5\sqrt{2}$ **h** $15\sqrt{2}$ **i** $-4\sqrt{6}$
5 a 30 **b** $36\sqrt{35}$ **c** $2\sqrt{2}$ **d** $245\sqrt{2}$ **e** $6\sqrt{2}$ **f** $6 - \sqrt{15}$
6 a $6\sqrt{2} + 7$ **b** $11 - 5\sqrt{5}$ **c** $7 + 3\sqrt{3}$ **d** $5 + \sqrt{10} + \sqrt{15} + \sqrt{6}$ **e** $-1 + 13\sqrt{3}$
f $19\sqrt{6}$ **g** $7 + 4\sqrt{3}$ **h** $14 - 6\sqrt{5}$ **i** $30 + 12\sqrt{6}$ **j** 1 **k** 46 **l** 67
7 a $\frac{\sqrt{3}}{3}$ **b** $\sqrt{5}$ **c** $3\sqrt{2}$ **d** $\frac{\sqrt{2}}{6}$ **e** $\frac{\sqrt{6}}{4}$ **f** $\frac{2\sqrt{5} + 5}{10}$

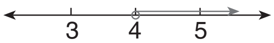
Exercise 1:07

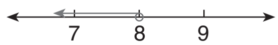
1 a 45·6 m **b** 13·2 cm **c** 39·3 m
2 a 15·12 m^2 **b** 88·25 cm^2 **c** 23·04 m^2 **d** 28·08 cm^2 **e** 17·85 m^2 **f** 11·52 m^2
3 a 63·9 m^2 **b** 35·7 m^2 **c** 23·3 km^2
4 a 370·88 cm^2 **b** 648 m^2 **c** 333·55 m^2 (approx.)
5 a i 77·41 m^2 **ii** 30·41 m^2 **iii** 107·82 m^2 **b i** 1658·76 cm^2 **ii** 760·27 cm^2 **iii** 2419·03 cm^2
c i 38·68 m^2 **ii** 73·49 m^2 **iii** 112·17 m^2

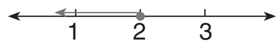
6 a $461 \cdot 04\ \text{cm}^3$ **b** $810\ \text{m}^3$ **c** $247 \cdot 15\ \text{m}^3$ (approx.)
7 a $85 \cdot 15\ \text{m}^3$ (to 2 dec. pl.) **b** $9123 \cdot 19\ \text{cm}^3$ (to 2 dec. pl.) **c** $66 \cdot 14\ \text{m}^3$ (to 2 dec. pl.)

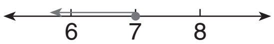
Exercise 1:08

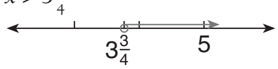
1 a $a = 18$ **b** $m = 5$ **c** $x = 15$ **d** $y = -2$ **e** $p = 2\frac{1}{3}$ **f** $n = 12$ **g** $x = 2$ **h** $m = 2$
i $y = \frac{1}{5}$ **j** $k = \frac{2}{3}$ **k** $x = 2$ **l** $q = 3\frac{1}{2}$
2 a $m = 5$ **b** $x = 4$ **c** $x = 7$ **d** $a = 8$ **e** $m = 4$ **f** $q = -1$ **g** $x = 2$ **h** $z = 1\frac{1}{2}$ **i** $m = -1\frac{1}{3}$
3 a $a = 2$ **b** $x = 7$ **c** $x = 3$ **d** $a = -2$ **e** $x = -3$ **f** $m = 4\frac{1}{3}$ **g** $a = 4$ **h** $p = 8$
i $b = -18$ **j** $y = -1$ **k** $m = 7$ **l** $m = 5$ **m** $y = 2$ **n** $x = 2\frac{1}{2}$
4 a $x = 4$ **b** $a = 9$ **c** $m = 6\frac{2}{3}$ **d** $n = 9$ **e** $x = 6$ **f** $p = -1$
5 a $a = 6$ **b** $x = 15$ **c** $p = 6$ **d** $q = 36$ **e** $k = 24$ **f** $x = 60$ **g** $m = 6$ **h** $n = -1$
i $x = \frac{11}{13}$ **j** $x = \frac{55}{7}$ **k** $m = 33$ **l** $a = \frac{-19}{8}$
6 a i $8n + 11 = 39$ **ii** $2n + 7 = 5$ **iii** $\frac{n+4}{10} = 7$ or $(n + 4) \div 10 = 7$
b i 11 **ii** 8 **iii** width = 34 m, length = 136 m **7 a** $x \leqslant 1$ **b** $x \geqslant 0$ **c** $x > 6$ **d** $x < -2$

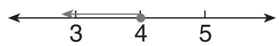
8 a $x > 4$


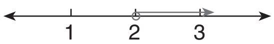
b $a < 8$


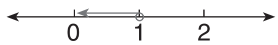
c $y \leqslant 2$


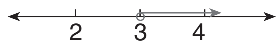
d $m \leqslant 7$


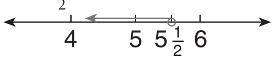
e $x > 3\frac{3}{4}$


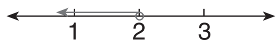
f $m \leqslant 4$


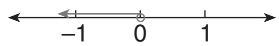
g $x > 2$


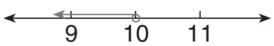
h $n < 1$


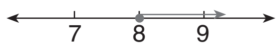
i $x > 3$


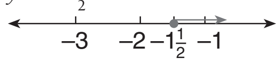
j $x < 5\frac{1}{2}$


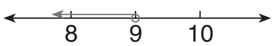
k $a < 2$


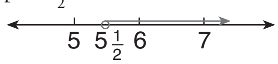
l $m < 0$


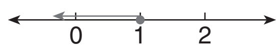
m $x < 10$


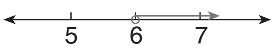
n $x \geqslant 8$


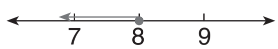
o $y \geqslant -1\frac{1}{2}$


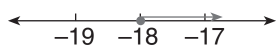
p $p < 9$


q $p > 5\frac{1}{2}$


r $x \leqslant 1$


s $x > 6$


t $a \leqslant 8$


u $x \geqslant -18$


9 a 92 **b** $5 \cdot 81$ **c** $15 \cdot 7$
10 a 4 **b** 10 **c** 25 **d** $3 \cdot 08\dot{3}$ **e** $5 \cdot 2\dot{6}$
11 a $y = \frac{ab - bx}{a}$ (or $y = \frac{b(a - x)}{a}$) **b** $y = \pm\sqrt{\frac{x}{a}}$ **c** $y = \frac{B}{T^2}$ **d** $y = \frac{-1}{a - b}$ (or $y = \frac{1}{b - a}$)

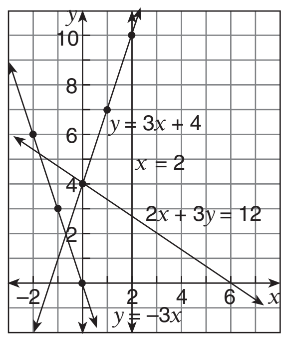
Exercise 1:09

1 a \$96.60 **b** \$238.28 **c** \$350 **d** \$534.90
2 a \$173.47, 30·6% **b** \$8434 **c** \$15 746 **d** \$1336.20
3 a The 15% discount is the best, by \$2.49. **b** Jade's buy **c** Yet tea
4 a i \$7.50 **ii** \$0.68 **iii** \$1.88 **b i** \$82.50 **ii** \$7.48 **iii** \$20.63 **c i** \$20 **ii** \$0.75 **iii** \$1.80
5 a Buying on terms is a way of having an item while spreading the payment over a period of time. Interest is charged.
b He pays \$143.56. Interest is \$56.56. **c** \$220 **d** \$545 700 **e** \$30.60
6 a i $33\frac{1}{3}\%$ **ii** 25% **b i** 25% **ii** $33\frac{1}{3}\%$

Exercise 1:10

1 a 1 **b** 7 **c** 0 **2 a** (5, 8) **b** $(\frac{1}{2}, 1\frac{1}{2})$ **c** $(3\frac{1}{2}, 0)$
3 a $\sqrt{20}$ units **b** $\sqrt{73}$ units **c** 10 units **d** $\sqrt{53}$ units

4



5 **a** gradient = 3, y-intercept = 5 **b** gradient = −1, y-intercept = −2
c gradient = −2, y-intercept = 5

6 **a** $y = 3x + 4$ **b** $y = -\frac{2}{3}x + 4$ **c** $y = -3x$ **d** $x = 2$

7 **a** $y = 5x - 2$ **b** $y = 4$ **c** $y = 2x + 5$ **d** $y = -x + 1$

8 **a** $y = \frac{1}{2}x$ **b** $y = -x + 4$

9 **a** $y = 3x$ and $y = 3x - 1$ **b** yes

c $m_1 = \dfrac{2-4}{4-1} = \dfrac{-2}{3}$ **d** $x = 7$ and $x = -2$

$m_2 = \dfrac{-2-0}{-1-(-4)} = \dfrac{-2}{3}$

$\therefore$ The lines are parallel.

10 **a** yes **b** yes **c** $y = -x + 1$ and $y = x + 7$

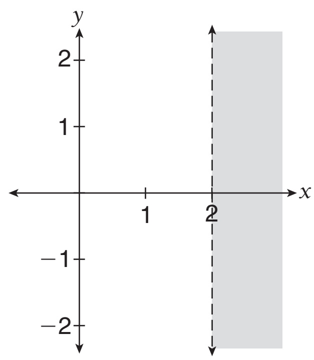
d Gradient of line through (0, −5) and (−3, −4)

is $m_1 = \dfrac{-4-(-5)}{-3-0} = -\dfrac{1}{3}$. $m_2 = 3$

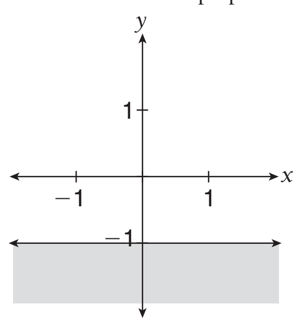
Now $m_1 \times m_2 = -\frac{1}{3} \times 3 = -1$

$\therefore$ The lines are perpendicular.

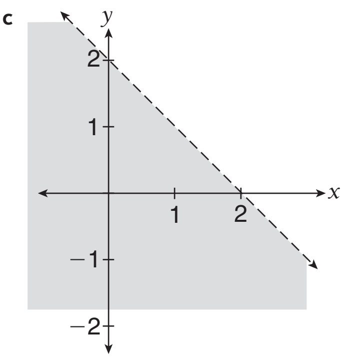
11 **a**



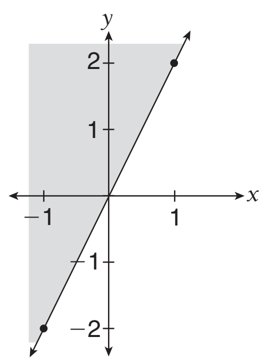
b



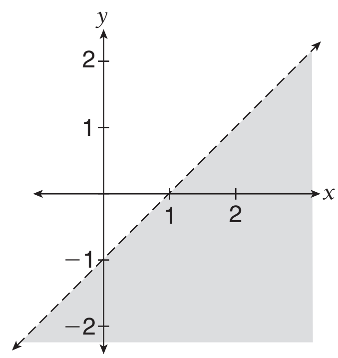
c



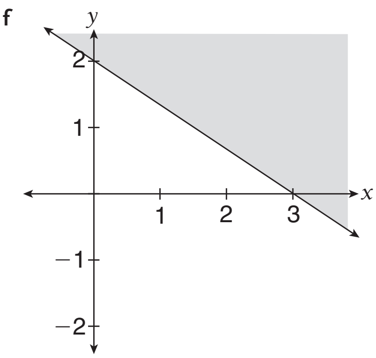
d



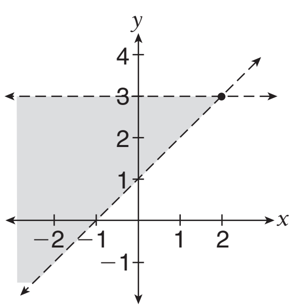
e



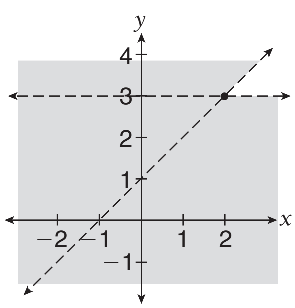
f



12 **a**



b



Exercise 1:11

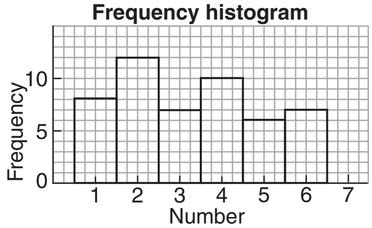
1

Number	*Tally*	*Frequency*
1	𝍸 |||	8
2	𝍸 𝍸 ||	12
3	𝍸 ||	7
4	𝍸 𝍸	10
5	𝍸		6
6	𝍸 ||	7

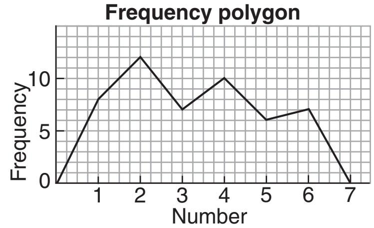
$\sum f = 50$

a 2 **b** 5 **c** 7 **d** 21 **e** 23

2 **a**



b



3 **a** **i** 7 **ii** 5 **iii** $4\frac{1}{9}$ **iv** 4 **b** **i** 5 **ii** 2 **iii** 3·3 **iv** 3

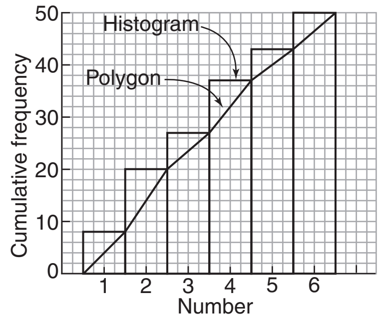
c

Number	*Frequency*	*Cumulative frequency*
1	8	8
2	12	20
3	7	27
4	10	37
5	6	43
6	7	50

$\sum f = 50$

i 37 **ii** 27 **iii** yes

4



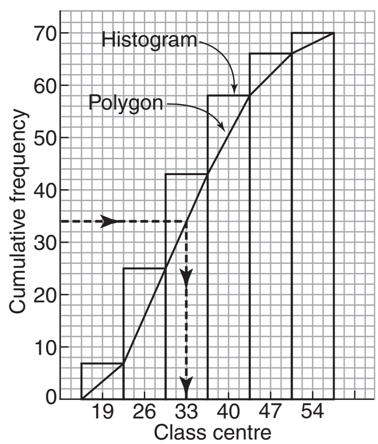
5 **a**

Class	*Class centre*	*Tally*	*Frequency*	*Cumulative frequency*
16–22	19	𝍸 |||	7	7
23–29	26	𝍸 𝍸 𝍸 |||	18	25
30–36	33	𝍸 𝍸 𝍸 |||	18	43
37–43	40	𝍸 𝍸 𝍸	15	58
44–50	47	𝍸 |||	8	66
51–57	54	||||	4	70

$\sum f = 70$

b 34·1 using table (34·6 using the individual numbers)

c



Median class is 30–36

d 23–29 and 30–36 **e** 70

Exercise 1:12

1 **a** $y = 15$ **b** $y = 1\frac{3}{4}$ **2** **a** $x = 5$ **b** $x = 6$

3 **a** yes **b** no

4 **a** $x = 4, y = 2$ **b** $x = -8, y = 2$ **c** $x = -4, y = -2$ **d** $x = 4, y = 2$ **e** $x = 0, y = -6$ **f** $x = 4, y = 2$

5 **a** $x = 2, y = 8$ **b** $x = 7, y = 5$ **c** $x = 3, y = 1$ **d** $a = 1\frac{3}{7}, b = 2\frac{5}{7}$

6 **a** $x = 5, y = 1\frac{2}{3}$ **b** $a = 2\frac{3}{5}, b = -\frac{1}{5}$ **c** $c = 8, d = -2$ **d** $x = -3, y = 5$

7 48 **8** \$20

Exercise 1:13

1 **a** $\sin\theta = \dfrac{\text{opposite side}}{\text{hypotenuse}}$ **b** $\cos\theta = \dfrac{\text{adjacent side}}{\text{hypotenuse}}$ **c** $\tan\theta = \dfrac{\text{opposite side}}{\text{adjacent side}}$
2 **a** $\frac{5}{13}$ **b** $\frac{12}{13}$ **c** $\frac{5}{12}$ **3** **a** 0·385 **b** 0·923 **c** $0{\cdot}41\dot{6}$
4 **a** 0·242 **b** 0·139 **c** 0·997 **d** 0·507 **e** 0·978 **f** 0·995 **g** 0·031 **h** 0·487
i 1 **j** 0·123 **k** 19·855 **l** 0·712
5 **a** 21·97 **b** 2·41 **c** 8·18 **d** 12·85 **e** 2·41 **f** 9·49 **g** 12·01 **h** 10·89
i 34·82 **j** 4·03
6 **a** 31°32′ **b** 67°7′ **c** 41°5′ **d** 49°42′
7 **a** 61 m (to the nearest metre) **b** 76 m (to the nearest metre)
8 **a** 46 626 m **b** N41°E (to the nearest degree)

Exercise 1:14

1 **a** 10 km **b** 20 km **c** 11.30 am **d** Callum **e** 11.30 am **f** 50 km **g** 30 km **h** 90 km
2 **a** 3000 g **b** 4400 g **c** 900 g **d** 500 g **e** 0–2 weeks
3 **a** D **b** C **c** A **d** B

1 Working Mathematically

1 **a**

t	1	3	5
m	3	7	11

, $m = 2t + 1$

b

t	2	4	6
m	5	10	15

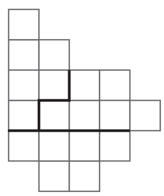
, $m = \dfrac{5t}{2}$

c

t	1	3	6
m	3	9	18

, $m = 3t$

2 Cut along lines as shown



3 **a** 185 **b** 37 **4** 16 min 15 s **5** 32
6 **a** 617 600 **b** 10 290 000 **c** approx. 10 300

Chapter 2: Consumer arithmetic

Exercise 2:01

1 For the answers to this question, see the table on page 94.
2 **a** 0·5% pa for that month **b** 2·65% pa **c** daily **d** monthly **e** 0·007%
3 **a** 4 months **b** 6·90% pa **c** $20 000
4 **a** 5, €3 per transaction **b** **i** 4·25% pa **ii** 2·5% pa **iii** 4·75% pa **c** €5000

Reading Mathematics 2:01 Financial spreadsheets

1 **a** Money earned from typing in 2005 (Y9). **b** Money earned from selling newspapers in 2003 (Y7).
c The category ‘Baby sitting’. **d** The total amount earned washing cars from 2003 to 2006.
e The total income from these sources in 2004 (Y8).
2 **a** B2 + C2 + D2 + E2 **b** C2 + C3 + C4 + C5 + C6 + C7 + C8
3 $7026 **4** The sum of the rows and the sum of the columns are the same as they contain the same numbers.

Prep Quiz 2:02

1 $\frac{1}{12}$ **2** $\frac{5}{12}$ **3** $\frac{11}{12}$ **4** $\frac{1}{365}$ **5** $\frac{7}{365}$ **6** $\frac{128}{365}$ **7** $58.50 **8** $108.80 **9** $15 000 **10** $7.41

Exercise 2:02

1 **a** $446.40 **b** $5850 **c** $7.02 **d** $246.33 **e** $157 500 **f** $247 500 **g** $140 **h** $12 000
i $15 174 **j** $1379
2 **a** $31.36 **b** $2052 **c** ¥1570 **d** €3.23 **e** $139.20 **f** $362.10 **g** €22.77 **h** $210.70
3 **a** ¥6600 **b** $273.60 **c** $6.05 **d** €113.40 **e** $20 160 **f** ¥81 **g** $43.61 **h** $6.72
i $2772.32 **j** $81.61
4 **a** $372 **b** ¥958 **c** €67.08 **d** $1283.33 **e** €1071 **f** $1133.99 **g** $1.41 **h** $4.72
i ¥1 078 000 **j** €12.60
5 **a** $16.32 **b** $161.61 **c** $6.75 **d** €727.10 **e** €67.40 **f** $267.12 **g** $23.75 **h** $1.27
i $126.55 **j** €21.74

Reading Mathematics 2:02 Why not buy a tent?

1 length = 3·5 m, breadth = 2·43 m, floor area = 8.505 m^2 **2** 7·08 m^2
3 deposit = $29.90, number of payments = 156 **4** total cost on terms = $482.30
5 $117.30 **6** the cabin tent (0·036 m^2/$)

Prep Quiz 2:03

1 0·18 **2** 0·05 **3** 0·054 **4** 0·055 **5** 0·0525 **6** 200 **7** 30 **8** 5 **9** 0·1 **10** 3

Exercise 2:03

1 a i $1878 **ii** $6573 **b** $5080 **c** $1108.80 **d** $84 **e** $2850 **f** $225 **g i** $0.24 **ii** $0.71 **iii** $6.83
2 a 644 women **b** 8·4*t* **c** €1045.44 **d** €130.20
3 a £3500 **b** $80 **c** $280 000 **d** ¥100 000
4 a $1210 **b** $4255 **c** $32 310 **d** $15 755.83
5 a $13 000 **b** $276 750 **c** $7550 **d** $30 900
6 a 9% pa **b** €5600 **c** 6 **d** 14·25% pa

Prep Quiz 2:04

1 $11 000 **2** $12 100 **3** $13 310 **4** $14 641 **5** $16 105.10 **6** $101 000
7 $102 010 **8** $103 030.10 **9** $104 060.40 **10** $2420

Exercise 2:04

1 a $2.04 **b** $21.76 **c** $41.82 **d** $70.70
2 a €2420 **b** €7024.64 **c** €11 449 **d** €60 397.48 **e** €33 386.73
3 a A = $501.76, I = $101.76 **b** A = $2741.50; I = $841.50 **c** A = $6356.34, I = $1006.34
d A = $125.97, I = $25.97 **e** A = $3321.27, I = $447.27 **f** A = $730 340, I = $80 340
g A = $122 022.07, I = $36 322.07
4 A = $7320.50, I = $2320.50 **5** $20 776.96
6 a £12.12 **b** £851.39 **c** £160.80 **d** £43.27
7 10% pa compound interest is better by $24.10 **8** 1166 **9** $207 360
10 a $37 791.36 **b** $5.67 **c** $3.59

Prep Quiz 2:05

1 80% **2** 93% **3** 100% **4** 100% **5** $280 **6** $280 **7** $744 **8** $744 **9** yes **10** yes

Exercise 2:05

1 a €11 726 **b** €213.90
2 a $105 318.40 **b** $6906.62 **c** $951.71 **d** $4228.20 **e** 650 **f** 50 165
3 a $30 681.60 **b** $1253.38 **c** $398.29 **d** $1571.80 **e** 190 **f** 6535
4 a $800.15 **b** $1607.87 **5 a** $14 113.50 **b** $16 537.50 **6** £1911.78 **7** 7% pa **8** 3 years

Prep Quiz 2:06

1 0·09 **2** 0·065 **3** 0·1125 **4** 0·75% per month
5 0·666 667% per month **6** 0·541 667% per month **7** 1·15 **8** 0·85 **9** 69 960·25 **10** 20 880·25

Exercise 2:06

1 a $7049.37 **b** $1380.97 **c** $58 361.78 **d** $657.02
2 a $3049.37 **b** $520.97 **c** $41 001.78 **d** $180.52
3 a €967.43 **b** €11 074.22 **c** €38.29
4 a $326.68 **b** $7764.76 **c** $9946.39 **d** $300.52
5 $552 904.08 **6** 11% pa compound interest; 24.54 ringitt **7 a** $5095.57 **b** $1633.52 **c** $616.08
8 a $7 788 322.36 **b** $2 748 960.63 **c** $2 921 255.08 **9** $4893.86 **10** €37 426.70
11 a $1762.34 **b** $1790.85 **c** $1806.11 **d** $1816.70 **e** $1819.61 **f** $1821.94
However small we make the time period, the investment will not exceed $1822.12 in 5 years. Therefore there is a limit to which the investment will grow.
12 a $38.07 **b** $1156.81 **c** $5513.57

Investigation 2:06 Compound interest tables

1 a amount = $45 044, int. = $25 044 **b** amount = $1 477 455, int. = $1 327 455
c amount = $11 953.50, int. = $4453.50
2 a amount = $125 440 **b** amount = $126 970
c For **b**, interest is calculated more often. It accumulates 24 times instead of just twice as in part **a**.

Prep Quiz 2:07

1 $1050 **2** $1080 **3** $420 **4** 1% **5** 0·75% or $\frac{3}{4}$% **6** $0\cdot\dot{6}$% or $\frac{2}{3}$%
7 $900 **8** $19 000 **9** $6930 **10** $11 910

Exercise 2:07

1 a $10 000 **b** $98 000 **c** $9800 **d** $95 800
2 a €29 400 **b** €28 752 **c** €4752 **d** €4800 **e** $48
3 a $40 449.20 **b** $8449.20 **4 a** $249 500 **b** $248 484.95 **5** £79 095.49
6 a $700, $10 700 **b** $10 000 **c i** $10 000 **ii** $10 000 **iii** $10 000
7 a $937.50 **b** $937.50 **8** 5 repayments
9 a $19 175.68 **b** $19 200 **c** quarterly

Exercise 2:08

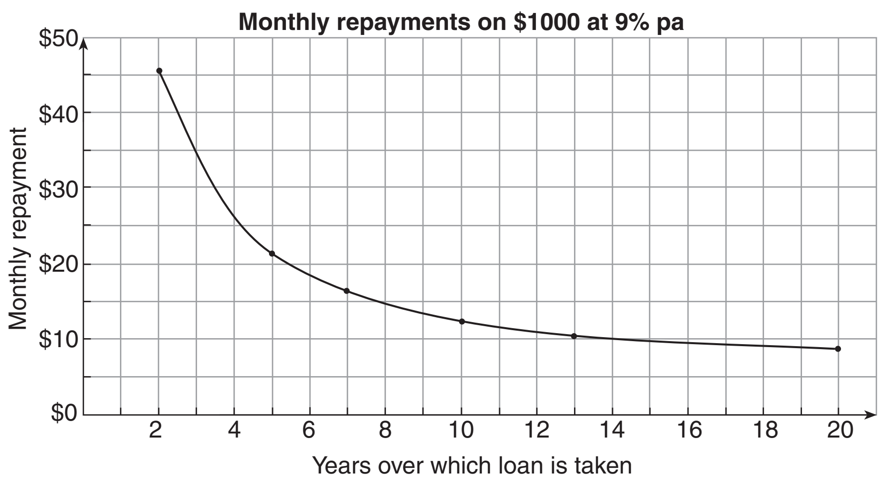
1 a 5% **b** 7% **c** 8% **d** 12% **e** 16% **f** 20%
2 a 5% **b** 7% **c** 8% **d** 12% **e** 16% **f** 20%
3 a 2·15% **b** 3·18% **c** 4·91% **d** 8·21% **e** 8·76% **f** 3·93% **g** 11·79% **h** 5·04%
4 a $1920 **b** $52 200 **c** $954 **d** $81 120 **e** $1135.75 **f** $703.50 **g** $21 763.50 **h** $88 668
5 a 9% flat **b** 3% flat **c** 7% red **d** 11% flat **e** 9% red **f** 12% red
6 a $2377.50 **b** $7377.50 **c** $122.96
7 a 5.80% **b** €81 200 **c** €630
8 a $322.50 **b** $194.64 **c** $289.78 **d** $7343.70 **e** $18.12
9 a $317.90 **b** 20% pa
10 a $9341.40 **b** $2341.40 **c** $468.28 **d** 6.69% **e** 12%
11 a reducible = $129.80, flat = $240; ∴ pay $110.20 more with flat interest
12 €9548 more with flat interest

Challenge 2:08

1 7·22% **2 a** 13·44% **b** 10·47% **c** 14·80% **d** 10·80%

Exercise 2:09

1 a 6·5% **b** $1893 **c** $229 000 **d i** 7·5% **ii** 7% **iii** 6·5%
e $1534 **f i** $6833.33 **ii** $1708.33 **iii** $220 000
2 a $12 000 **b** 25 weeks **c** $376 000 **d** $338 400 **e** $1974 **f** $7200
3 a $1659 **b** $398 160 **c** $184 160 **d** 7% **e** 4·30%
4 a $403 **b** $571.28 **c** $871.38 **d** $920.08
5 a £1085.52 **b** £1216.80 **c** £1455.60 **d** £1720.80 **e** £2006.40 **f** £2316
6 a £85.52 **b** £216.80 **c** £455.60
7 a $74.48 more with flat interest **b** $183.20 more with flat interest **c** $344.40 more with flat interest
8 a $16.12, $24.42, $27.26, $33.92 **b** $30.68, $48.10, $52.64, $65.72
9



a $35 **b** $11 **c** $9

Diagnostic Test 2 Consumer arithmetic

1 **a** €360 **b** €1711.20 **c** €102 000 **2** **a** \$279.50 **b** \$13 950 **c** \$182.25
3 **a** £80 **b** £65.10 **c** £102.38 **4** **a** \$3315 **b** 8% pa **c** \$100
5 **a** \$3986.07 **b** \$12 180.68 **c** \$17 892.57 **6** **a** \$960.40 **b** 512 **c** \$391.55
7 **a** **i** \$48 000 **ii** \$45 880 **iii** \$43 632.80 **b** \$287 635.17
8 **a** **i** 12% **ii** 20% **iii** 7% **b** \$173.03 **9** **a** \$269.70 **b** \$510.75 **c** \$603.75

2A Revision Assignment

1 **a** \$525 **b** \$315.88 **c** \$88 **d** \$89.10 **2** **a** \$85 **b** \$765 **c** \$229.50 **d** \$41.44
3 **a** \$428.40 **b** \$63.40 **4** \$2164.86 **5** C **6** \$7190.40 **7** \$82 978.56
8 **a** \$1114.40 **b** 39·3% **9** 11·70%; \$702
10 **a** approx. $\$2.06 \times 10^{11}$ or \$206 000 000 000 **b** \$135

2B Working Mathematically

1 59 **2** **a** 5 **b** 8 **c** 10 **3** The man is 66, the girl is 6.
4 6 hours, because in 6 hours (half of 12) the hour hand will be pointing the opposite direction to the present, and the minute hand will be in the same place.
5 4
6 **a** **i** 1763 **ii** 9·4 **iii** 740 000
b equal 8th **c** Iceland, Austria

Chapter 3: Quadratic equations

Prep Quiz 3:01

1 $(x+1)(x+3)$ **2** $(x-4)(x-1)$ **3** $x(x+5)$ **4** $3x(2x-1)$ **5** $(x-3)(x+3)$
6 $(2x-5)(2x+5)$ **7** −2 **8** $\frac{1}{3}$ **9** 0 **10** $-1\frac{1}{2}$

Exercise 3:01

1 **a** 0, 5 **b** 0, −7 **c** 0, −1 **d** 0, 2 **e** 0, −5 **f** 0, 7 **g** 2, 1 **h** 7, 3
i 5, 2 **j** −3, −4 **k** −3, −2 **l** −9, −5 **m** 6, −6 **n** −8, 7 **o** −1, 1 **p** $-1, \frac{1}{2}$
q $-\frac{2}{3}, 5$ **r** $0, \frac{1}{3}$ **s** $\frac{1}{4}, -\frac{1}{2}$ **t** $1\frac{1}{3}, \frac{1}{2}$ **u** $\frac{5}{6}, -\frac{3}{4}$ **v** $0, \frac{3}{5}$ **w** $-\frac{1}{9}, -\frac{2}{7}$ **x** $\frac{1}{5}, -\frac{1}{5}$
2 **a** 0, −3 **b** 5, 0 **c** 0, −2 **d** 0, −2 **e** $0, \frac{1}{3}$ **f** 0, −2 **g** 2, −2 **h** 7, −7
i 6, −6 **j** 1, −1 **k** 10, −10 **l** 8, −8 **m** −1, −2 **n** 2, 3 **o** −5, −7 **p** 7, 3
q 8, 2 **r** 8, 3 **s** 4, −5 **t** 5, −7 **u** 9, −5 **v** 7, −8 **w** 7, 1 **x** 1, −10
3 **a** $\frac{1}{2}, -1$ **b** $-\frac{1}{3}, -2$ **c** $-\frac{2}{3}, -5$ **d** $4, 1\frac{1}{2}$ **e** $2\frac{1}{2}, -2$ **f** $7, -1\frac{1}{2}$ **g** $-\frac{1}{4}, -5$ **h** $5, -\frac{1}{4}$
i $5, \frac{1}{4}$ **j** $-\frac{1}{5}, -3$ **k** $1\frac{1}{2}, -8$ **l** $\frac{1}{7}, -7$ **m** $1\frac{1}{2}, -\frac{1}{2}$ **n** $\frac{1}{2}, -\frac{1}{3}$ **o** $-\frac{1}{3}, -\frac{2}{3}$ **p** $-\frac{2}{5}, -\frac{1}{2}$
q $\frac{1}{3}, \frac{1}{4}$ **r** $\frac{4}{5}, \frac{1}{2}$
4 **a** 0, 3 **b** 0, 8 **c** 0, −5 **d** 4, 1 **e** 5, −3 **f** 2, 1 **g** 6, 3 **h** 9, −2
i 8, −4 **j** 1, −2 **k** 1, −3 **l** 5, 2 **m** 3, −6 **n** 4, −7 **o** 3, −5 **p** $\frac{1}{2}, -1$
q $3, -2\frac{1}{2}$ **r** $2, -\frac{3}{4}$ **s** $2\frac{1}{3}, 2$ **t** $3, \frac{2}{5}$ **u** $5, \frac{1}{2}$

Exercise 3:02

1 **a** 3 **b** 4 **c** 1 **d** 2 **e** $\frac{3}{2}$ **f** $\frac{7}{2}$ **g** $\frac{11}{2}$ **h** $\frac{1}{2}$ **i** $\frac{5}{4}$ **j** $\frac{1}{3}$
2 **a** $2 \pm \sqrt{3}$ **b** $-1 \pm \sqrt{2}$ **c** $-5 \pm \sqrt{5}$ **d** $1 \pm \sqrt{10}$ **e** $3 \pm \sqrt{7}$ **f** $-2 \pm \sqrt{11}$
g $-3 \pm 2\sqrt{2}$ **h** $-10 \pm 2\sqrt{3}$ **i** $3 \pm 3\sqrt{2}$ **j** $-\frac{1}{2} \pm \sqrt{5}$ **k** $\frac{2}{3} \pm \sqrt{3}$ **l** $-1\frac{1}{2} \pm 2\sqrt{3}$
m $1 \pm \sqrt{\frac{5}{2}}$ **n** $-3 \pm \frac{3\sqrt{2}}{2}$ **o** $\frac{1}{3} \pm \frac{\sqrt{5}}{3}$
3 **a** 0·41, −2·41 **b** 3·45, −1·45 **c** 5·46, −1·46 **d** 1·12, −7·12 **e** 5·65, 0·35 **f** −0·27, −3·73
g 0·48, −10·48 **h** 1·24, −3·24 **i** 12·08, −0·08 **j** −0·44, −4·56 **k** 0·41, −7·41 **l** 1·30, −2·30
m −0·35, −8·65 **n** 1·19, −4·19 **o** 10·52, 0·48 **p** 2·30, −1·30 **q** 0·56, −3·56 **r** 5·19, −0·19
s 2·22, −0·22 **t** 0·85, −2·35 **u** 3·87, 0·13 **v** 0·72, −1·39 **w** 1·27, −0·47 **x** 0·84, −0·59

Exercise 3:03

1 **a** −2, −3 **b** −2, −4 **c** −1, −9 **d** 5, −2 **e** 5, −3 **f** 2, −6 **g** 7, 2 **h** 6, 2
i 5, 1 **j** $-\frac{1}{3}, -2$ **k** $-\frac{1}{2}, -5$ **l** $-\frac{3}{4}, -2$ **m** $3, -\frac{1}{2}$ **n** $2, -\frac{1}{5}$ **o** $1, \frac{2}{3}$ **p** $-\frac{1}{2}, -\frac{2}{3}$
q $\frac{1}{3}, -1\frac{1}{2}$ **r** $1\frac{1}{2}, \frac{1}{4}$

2 **a** $\frac{-4 \pm \sqrt{8}}{2}$ **b** $\frac{-3 \pm \sqrt{5}}{2}$ **c** $\frac{-5 \pm \sqrt{13}}{2}$ **d** $\frac{-1 \pm \sqrt{5}}{2}$ **e** $\frac{-2 \pm \sqrt{12}}{2}$ **f** $\frac{-4 \pm \sqrt{20}}{2}$
g $\frac{2 \pm \sqrt{8}}{2}$ **h** $\frac{7 \pm \sqrt{41}}{2}$ **i** $\frac{6 \pm \sqrt{24}}{2}$ **j** $\frac{10 \pm \sqrt{136}}{2}$ **k** $\frac{8 \pm \sqrt{52}}{2}$ **l** no solutions
m $\frac{-6 \pm \sqrt{28}}{4}$ **n** $\frac{-3 \pm \sqrt{17}}{4}$ **o** $\frac{7 \pm \sqrt{17}}{4}$ **p** $\frac{-10 \pm \sqrt{76}}{6}$ **q** $\frac{9 \pm \sqrt{57}}{6}$ **r** $\frac{-4 \pm \sqrt{56}}{10}$
s no solutions **t** $\frac{3 \pm \sqrt{21}}{6}$ **u** $\frac{3 \pm \sqrt{41}}{8}$ **v** $\frac{-11 \pm \sqrt{161}}{4}$ **w** $\frac{9 \pm \sqrt{17}}{4}$ **x** $\frac{-2 \pm \sqrt{24}}{10}$

2 **a** 3·73, 0·27 **b** 5·45, 0·55 **c** 0·58, −8·58 **d** −0·11, −8·89 **e** 1·45, −3·45 **f** 0·30, −3·30
g no solutions **h** 7·27, −0·27 **i** no solutions **j** 0·78, −1·28 **k** 2·85, −0·35 **l** −0·74, −2·26
m 3·19, 0·31 **n** 1·24, −0·64 **o** 0·80, −0·63

Investigation 3:03

1 2 **2** 1 **3** 0 **4** 2 **5** 2 **6** 0 **7** 1 **8** 2 **9** 0 **10** 0 **11** 2 **12** 1

Prep Quiz 3:04

1 $5x(x-2)$ **2** $(x-7)(x+2)$ **3** $(x-9)(x+9)$ **4** $(x+2)(x+3)$ **5** 2, −7 **6** $1\frac{1}{2}, -\frac{1}{3}$
7 4, −4 **8** 0, 4 **9** 1, 2 **10** $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

Exercise 3:04

1 **a** −1, −6 **b** 6, 2 **c** 3, −8 **d** 2·62, 0·38 **e** 0·79, −3·79 **f** −0·59, −3·41
g 0, −8 **h** 10, 0 **i** 2, 0 **j** 9, −9 **k** 11, −11 **l** $1\frac{1}{2}, -1\frac{1}{2}$
m −0·29, −1·71 **n** 0·77, −0·43 **o** 2·28, 0·22 **p** −1, −2 **q** −2, −3 **r** 4, −1
s 9, −3 **t** 9, 4 **u** $4, -1\frac{1}{2}$ **v** 5, 5 **w** 9, 4 **x** 1·54, 0·26

2 **a** −3, −4 **b** 5, 3 **c** 4, 2 **d** −2, −3 **e** 0, −7 **f** 3, −3
g $\frac{-2 \pm \sqrt{8}}{2}$ **h** $\frac{-4 \pm \sqrt{8}}{2}$ **i** $\frac{1 \pm \sqrt{5}}{2}$ **j** 1, −6 **k** 9, −2 **l** 8, −4
m 3, −1 **n** 0, −6 **o** $-5 \pm \sqrt{11}$ **p** $\frac{3}{2}, \frac{-5}{2}$ **q** $\frac{3 \pm \sqrt{7}}{5}$ **r** $\frac{7 \pm \sqrt{3}}{6}$
s 5, −3 **t** 7, −4 **u** 1, −2 **v** $\frac{5 \pm \sqrt{13}}{2}$ **w** $\frac{3 \pm \sqrt{21}}{2}$
x $\frac{-4 \pm \sqrt{24}}{4} = \frac{-2 \pm \sqrt{6}}{2}$

Prep Quiz 3:05

1 1, 2, 9, 10 **2** 1, 9 **3** 1, 9 **4** 9, 10 **5** $n+1, n+2$ **6** 12, 14
7 $x+2, x+4$ **8** $x-3$ **9** $a(a+3)$ or $a^2 + 3a$ **10** $4a+6$

Exercise 3:05

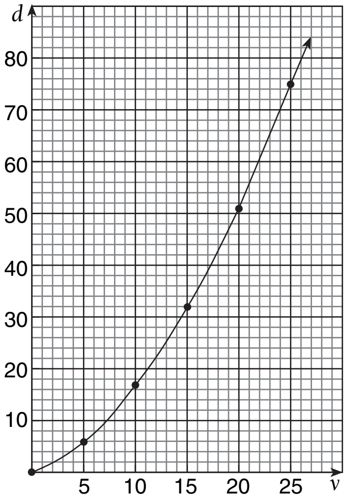
1 **a** 4, 5 **b** 9, 10 **c** 10, 12 **d** 7, 9
2 **a** 9 **b** 11 **c** 8 **d** 0, 5 **e** 7, −6
3 **a** 10 cm, 4 cm **b** 17 m, 15 m **c** 7 cm **d** 17 cm
4 **a** 2 s **b** 12 **c** **i** 2, −9 **ii** $\frac{1}{2}, -6$ **iii** $\frac{-1 \pm \sqrt{85}}{6}$
5 15 **6** 7 **7** 12 or 88

8 a 0·0875

b

v	0	5	10	15	20	25
d	0	6	17	32	51	75

c parabola

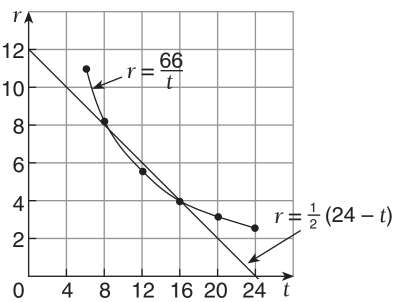


d approx. 17·2 m/s; by formula, 17·2927 m/s

e road and weather conditions, tyre and brake conditions on vehicle, physical conditions of drivers, other possible distractions, etc.

9 a The rise must be no greater than 7·5 inches by formula 1 or $7\cdot\dot{3}$ inches by formula 2.

b



c The points of intersection are:
(8·5, 7·7) and (15·5, 4·3)
By formula the solutions are:

$$\begin{cases} t \doteqdot 8\cdot536 \\ r \doteqdot 7\cdot732 \end{cases} \text{ and } \begin{cases} t \doteqdot 15\cdot464 \\ r \doteqdot 4\cdot268 \end{cases}$$

d

$$\left.\begin{aligned} r &= \frac{1}{2}(60\cdot96 - t) \\ r &= \frac{66 \times (2\cdot54)^2}{t} \end{aligned}\right\} \text{ or } \left\{\begin{aligned} r &= \frac{1}{2}(61 - t) \\ r &= \frac{426}{t} \end{aligned}\right\}$$

where r and t are in centimetres.

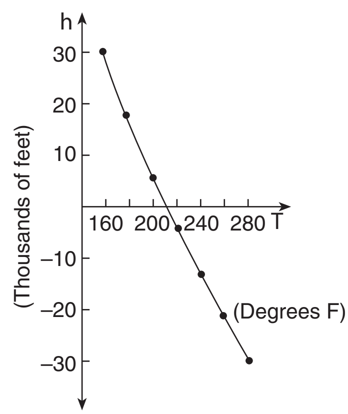
Investigation 3:05

1 $h = (212 - T)(520 + [212 - T])$
$\quad = (212 - T)(732 - T)$

2 i 6384 ft (or 1947 m) **ii** −18 316 ft (or −5586 m), −18 316 ft means 18 316 ft below sea level.

3

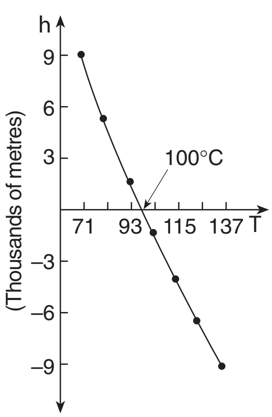
T in °F	160	180	200	220	240	260	280
h in feet	29 744	17 664	6384	−4096	−13 776	−22 656	−30 736



4 i 210°F **ii** 163°F **iii** 291°F (These are given to the nearest °F.)

5 Check each using $T = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

6



7 Discussion

Fun Spot 3:05

If $x = y$, then $x - y = 0$; thus dividing by $x - y$ in the second last line is undefined.

Diagnostic Test 3 Quadratic equations

1 **a** −7, 3 **b** 0, 5 **c** $\frac{1}{2}, -1$ **d** $-\frac{2}{3}, \frac{5}{4}$ **2** **a** 0, −5 **b** −2, −7 **c** −7, 7 **d** $\frac{1}{2}, -3$

3 **a** 9 **b** 4 **c** $\frac{9}{4}$ **d** $\frac{1}{4}$ **4** **a** $-1 \pm \sqrt{3}$ **b** $3 \pm 2\sqrt{2}$ **c** $\frac{3}{2} \pm \frac{\sqrt{29}}{2}$ **d** $\frac{5 \pm 3\sqrt{3}}{2}$

5 **a** $\frac{-1 \pm \sqrt{13}}{2}$ **b** $\frac{5 \pm \sqrt{17}}{2}$ **c** $\frac{-2 \pm \sqrt{2}}{2}$ **d** $\frac{-1 \pm \sqrt{7}}{3}$ **6** **a** 6, −1 **b** −3 **c** $-4 \pm \sqrt{6}$ **d** 4, −2

3A Revision Assignment

1 **a** 5, −6 **b** 0, 7 **c** −4, 2 **d** $\frac{3}{2}, -5$ **e** 4, −6 **f** $-2, \frac{1}{3}$ **g** 7 **h** $0, \frac{3}{2}$

i 10, −10 **j** 7, −2 **k** 7, −4 **l** $\frac{-5 \pm \sqrt{21}}{2}$ **m** $3 \pm \sqrt{2}$ **n** $\frac{1}{2}, -\frac{1}{5}$ **o** $\pm\sqrt{20}$ **p** $0, -\frac{3}{5}$

q −5 **r** $-1 \pm \sqrt{5}$ **s** $1, -\frac{2}{3}$ **t** $\frac{-5 \pm \sqrt{17}}{4}$ **u** 3, −8 **v** −1, −2 **w** $\frac{5 \pm \sqrt{17}}{2}$ **x** $2, -\frac{4}{3}$

2 **a** 4, −8 **b** 8, −5 **c** $5 \pm \sqrt{21}$ **d** $\frac{-3 \pm \sqrt{15}}{2}$

3 **a** 3, 4, 5 **4** **a** 5 **b** 7 **5** 40 m

3B Working Mathematically

1 357 **2** 3 **3** **a** €1 **b** the minimum monthly balance **c** no

4 \$14.40 **5** 8

6 **a** **i** Europe and the former USSR **ii** Americas

b **i** 17% **ii** 8% **c** **i** 300 **ii** 1600

Chapter 4: Number plane graphs and coordinate geometry

Exercise 4:01

1 **a**

x	−3	−2	−1	0	1	2	3
y	9	4	1	0	1	4	9

b

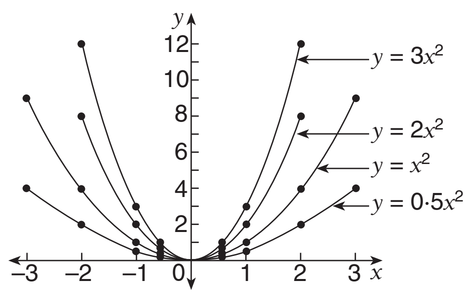
x	−2	−1	−0·5	0	0·5	1	2
y	8	2	0·5	0	0·5	2	8

c

x	−2	−1	−0·5	0	0·5	1	2
y	12	3	0·75	0	0·75	3	12

d

x	−3	−2	−1	0	1	2	3
y	4·5	2	0·5	0	0·5	2	4·5



The value of a changes the curvature of the parabola.

2 **a** B **b** D **c** C **d** A

3 **a**

x	−3	−2	−1	0	1	2	3
y	9	4	1	0	1	4	9

b

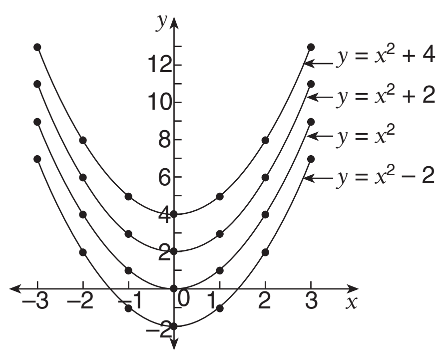
x	−3	−2	−1	0	1	2	3
y	11	6	3	2	3	6	11

c

x	−3	−2	−1	0	1	2	3
y	13	8	5	4	5	8	13

d

x	−3	−2	−1	0	1	2	3
y	7	2	−1	−2	−1	2	7

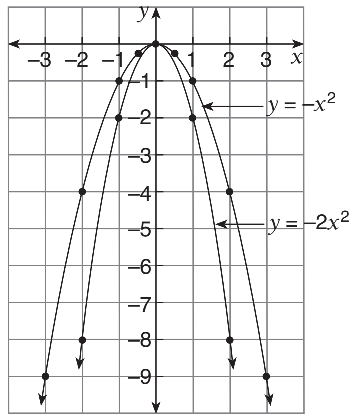


$y = x^2 + 2$ is the same curve as $y = x^2$ but raised two units. The value of c determines where the vertex of the parabola cuts the y-axis.

4 a $y = -x^2$

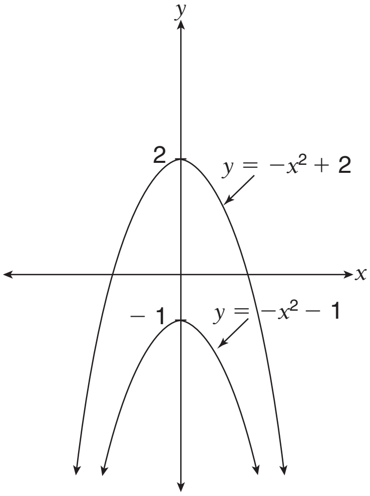
x	-3	-2	-1	$-0{\cdot}5$	0	$0{\cdot}5$	1	2	3
y	-9	-4	-1	$-0{\cdot}25$	0	$-0{\cdot}25$	-1	-4	-9

a, b



c Parabola is 'upside down' or 'concave down'.

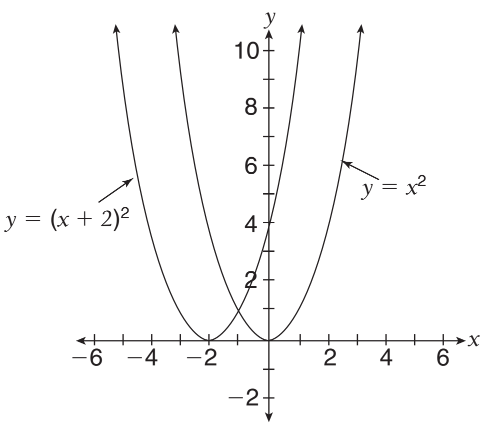
5



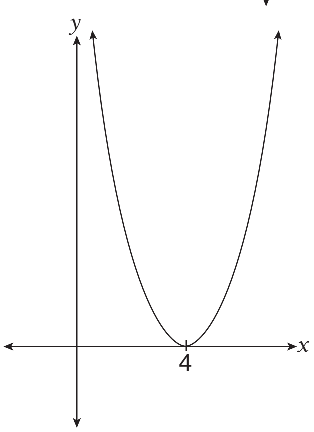
6 a E **b** C **c** B **d** A **e** D

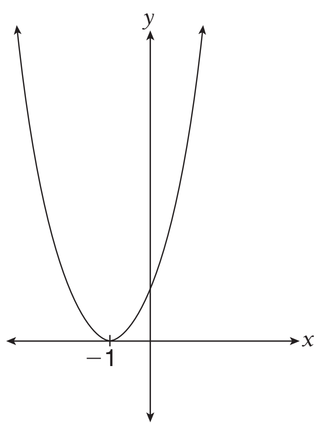
7 a The graphs are congruent. The graph of $y = (x - 2)^2$ can be produced by translating the graph of $y = x^2$ two units to the right.

b The parabola $y = (x + 2)^2$ is congruent to $y = x^2$. It can be produced by translating $y = x^2$ two units to the left.



c



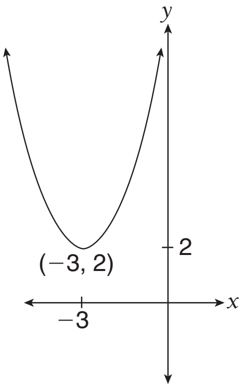


d The graphs of $y = (x - h)^2$ and $y = (x + h)^2$ can be obtained by translating the graph of $y = x^2$, h units to the right and h units to the left respectively. The graphs of all three are congruent.

8 **a** It is the same graph translated up two units.

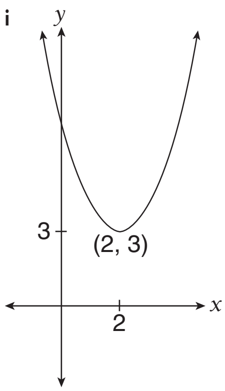
b It could be obtained by translating the graph of $y = (x - 3)^2$ down two units.

c

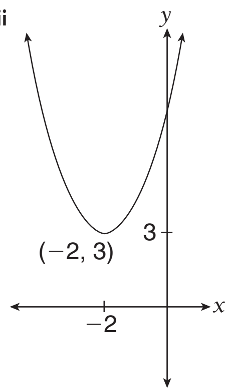


d The parabola $y = x^2$ has its vertex at (0, 0). If the parabola is translated p units sideways and q units vertically so that its vertex is at the point (p, q), then its equation is $y = (x - p)^2 + q$.

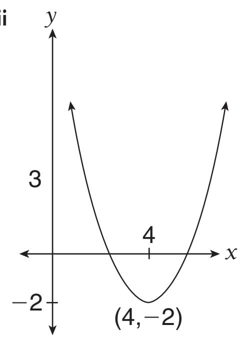
e **i**



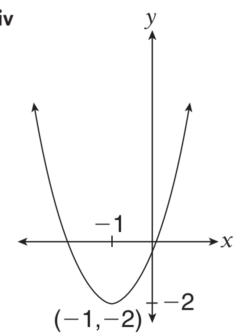
ii



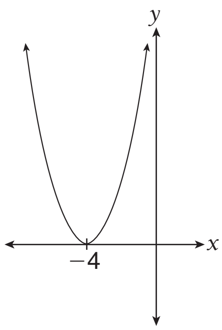
iii



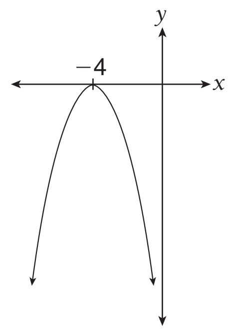
iv



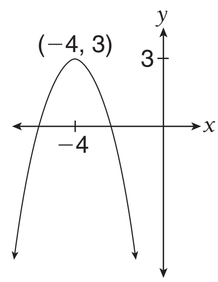
9 **a**



b



c



10 **a** $y = x^2 + 2$ **b** $y = x^2 - 2$ **c** $y = (x - 2)^2$ **d** $y = (x + 2)^2$ **e** $y = -x^2 + 4$
f $y = -x^2 - 2$ **g** $y = -x^2 - 2$ **h** $y = -(x - 2)^2$ **i** $y = (x + 2)^2 + 2$ **j** $y = -(x + 3)^2 - 2$

11 **a** C **b** D **c** B **d** A **e** F **f** E

12 **a** E **b** D **c** A **d** B **e** C **f** F

13 **a** $y = -x^2 + 4$ **b** $y = (x + 3)^2$ **c** $y = -(x - 3)^2 + 3$ **d** $y = (x - 2)^2 - 3$
e $y = (x - 2)^2 + 1$ **f** $y = (x + 2)^2 - 1$

Exercise 4:02

1 **a** **i** 0 **ii** 0, 4 **iii** $x = 2$ **iv** (2, −4)
b **i** 3 **ii** −3, 1 **iii** $x = -1$ **iv** (−1, 4)
c **i** −6 **ii** −1, 3 **iii** $x = 1$ **iv** (1, −8)

2 **a** 5 **b** −8 **c** −6

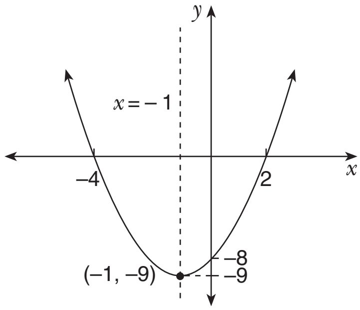
3 **a** −2, 4 **b** $-4, \frac{2}{3}$ **c** $-\frac{7}{4}, 3$

4 **a** $x = 4$, (4, −1) **b** $x = -2$, (−2, −48) **c** $x = -1$, $(-1, -\frac{9}{2})$
d $x = 3$, (3, −2) **e** $x = 1\frac{1}{2}$, $(1\frac{1}{2}, 7\frac{1}{4})$ **f** $x = -1\frac{1}{2}$, $(-1\frac{1}{2}, 6\frac{1}{4})$

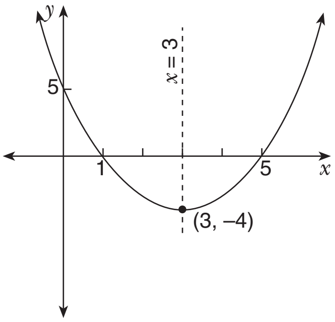
5 **a** −11 **b** 5 **c** −7

6 **a** 2 **b** −2 **c** 11

7 a (0, −8)
b (−4, 0), (2, 0)
c $x = -1$
d (−1, −9)
e



8 a



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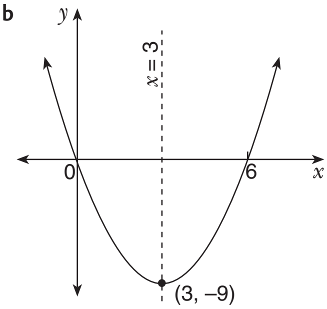
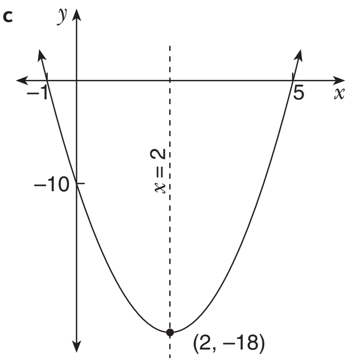
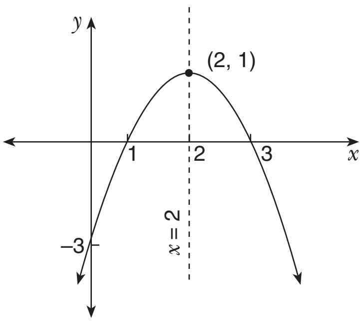


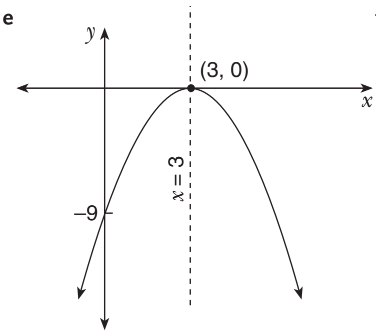
c



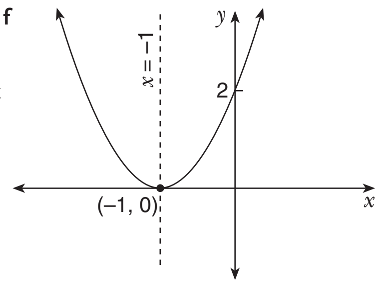
d



e



f



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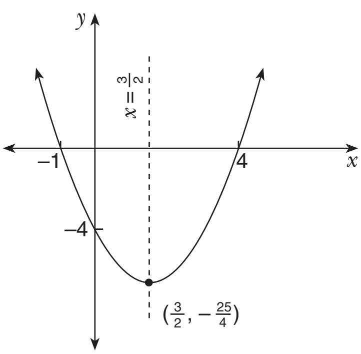

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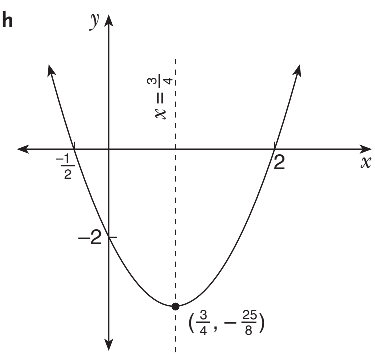
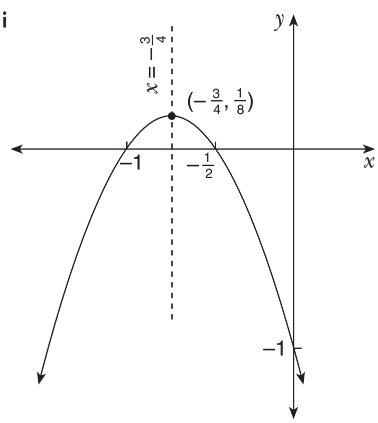
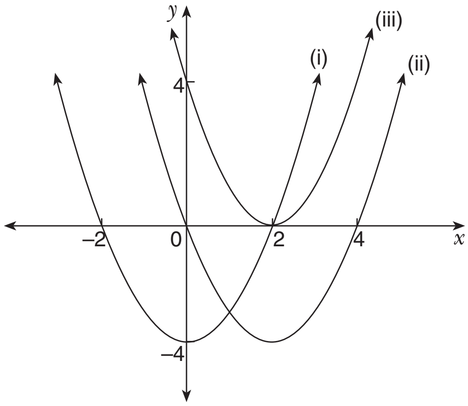


i

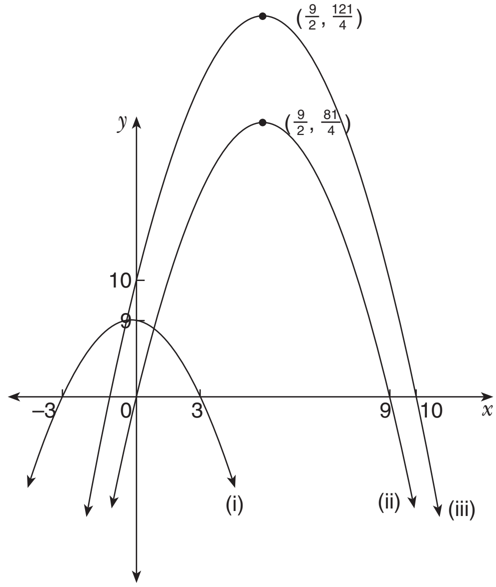


9 a D **b** B **c** F **d** A **e** E **f** C

10 a



b



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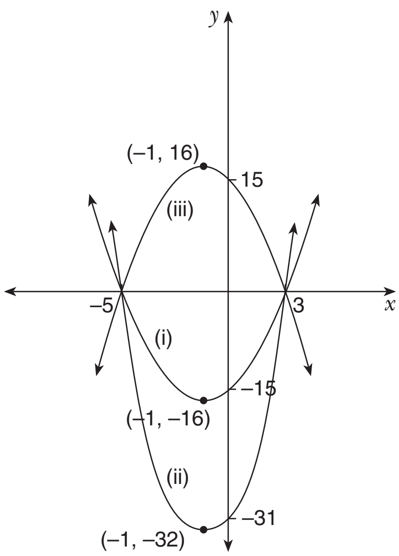

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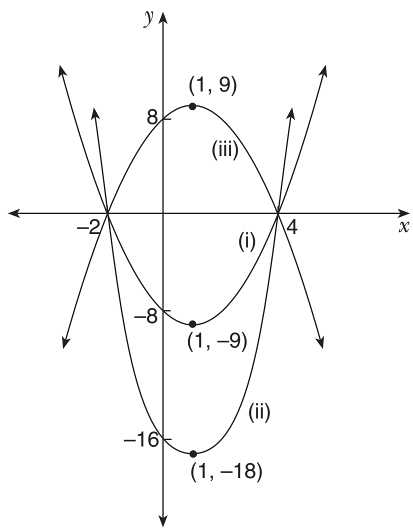
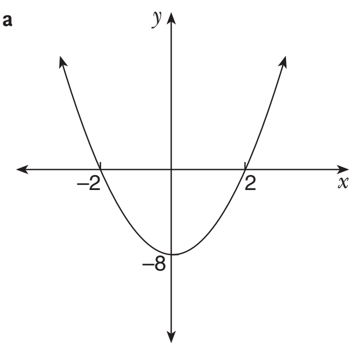


11 a



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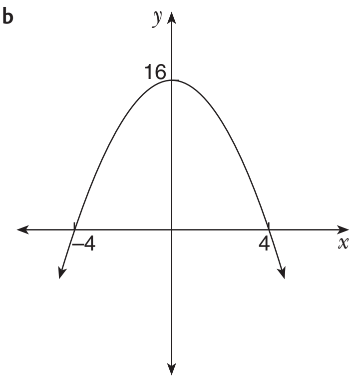

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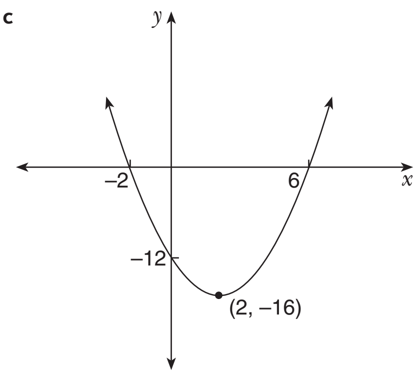
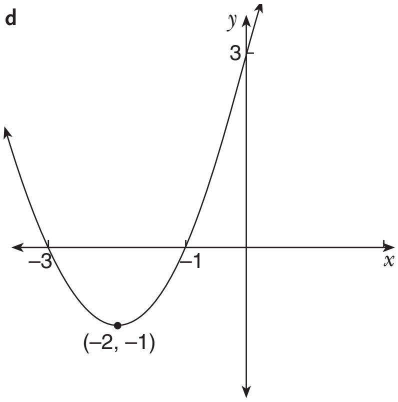
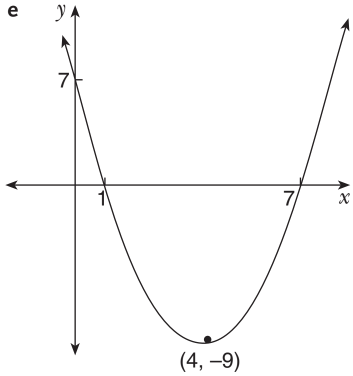


d



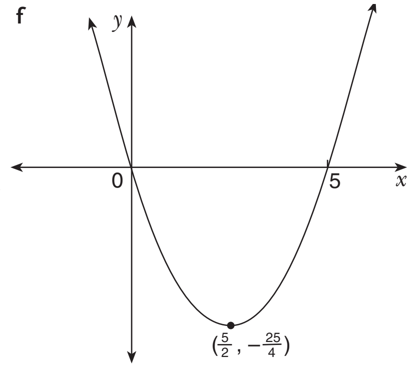
y
3
–3
–1
x
(–2, –1)

e



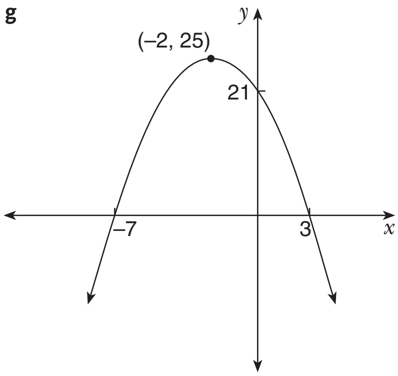
y
7
1
7
x
(4, –9)

f



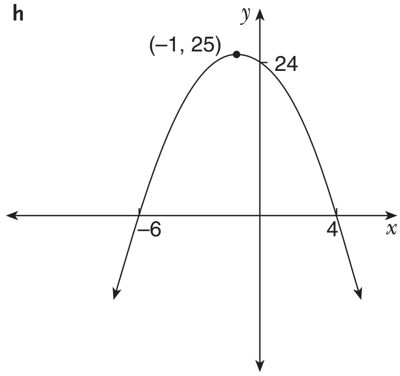
y
0
5
x
($\frac{5}{2}$, $-\frac{25}{4}$)

g



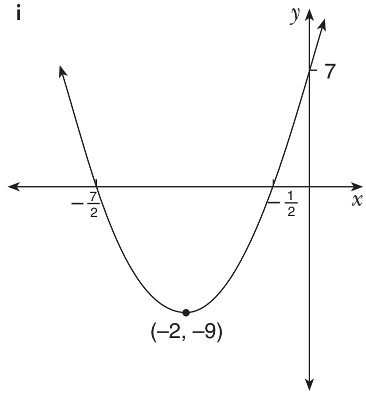
y
(–2, 25)
21
–7
3
x

h



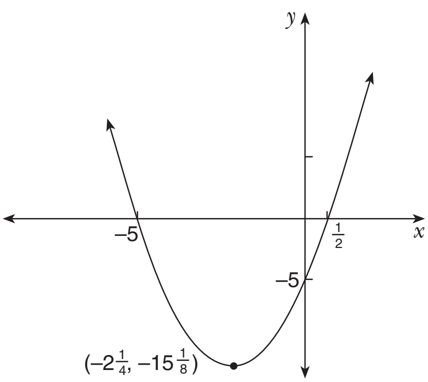
y
(–1, 25)
24
–6
4
x

i



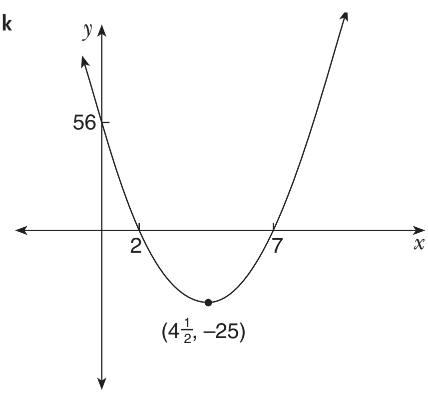
y
7
$-\frac{7}{2}$
$-\frac{1}{2}$
x
(–2, –9)

j



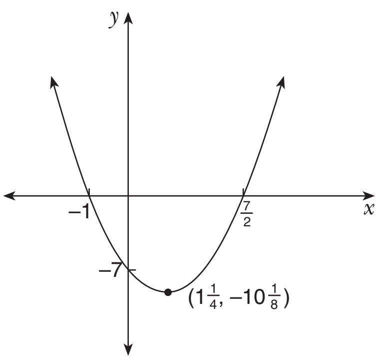
y
–5
$\frac{1}{2}$
x
–5
($-2\frac{1}{4}$, $-15\frac{1}{8}$)

k



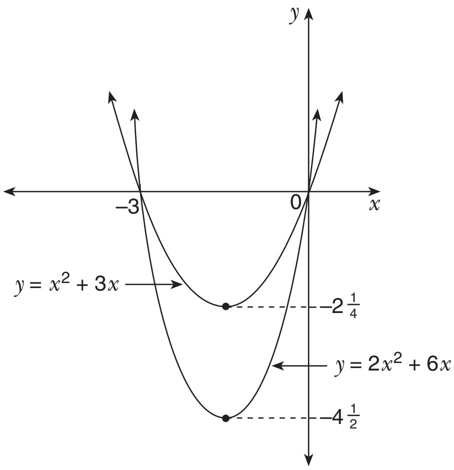
y
56
2
7
x
($4\frac{1}{2}$, –25)

l



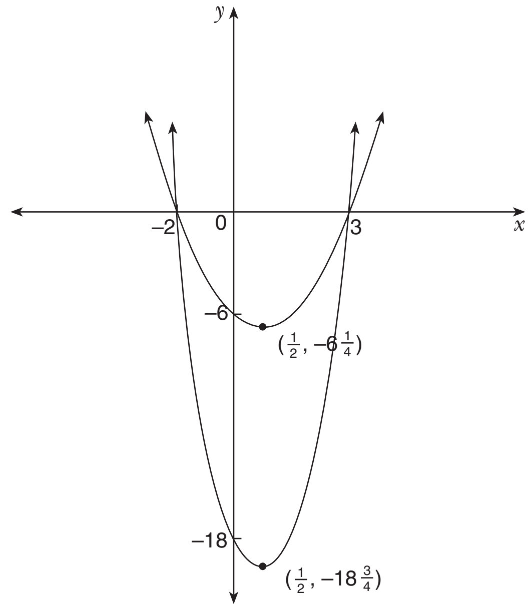
y
–1
$\frac{7}{2}$
x
–7
($1\frac{1}{4}$, $-10\frac{1}{8}$)

12 a



Same x intercepts but $y = 2x^2 + 6x$ has twice the minimum value of $y = x^2 + 3x$.

b



Minimum value of $y = 3(x - 3)(x + 2)$ is three times that of $y = (x - 3)(x + 2)$.

13 a Substituting $(0, -5)$ into $y = ax^2 + bx + c$ gives $c = -5$.

b $\dfrac{-b}{2a} = -1$

$\therefore b = 2a$

$\therefore y = ax^2 + 2ax - 5$

c $y = ax^2 + 2ax - 5$

$4 = a(1)^2 + 2a(1) - 5$

$4 = 3a - 5$

$a = 3$

d $y = 3x^2 + 6x - 5$

14 a $y = \frac{1}{2}x^2 + 2$ **b** $y = -2x^2 + 8x - 8$ **c** $y = x^2 + 6x + 11$

d $y = \frac{1}{2}x^2 - x - 4$ **e** $y = -\frac{1}{2}x^2 + \frac{1}{2}x + 3$ **f** $y = 4x^2 - 16x + 8$

Exercise 4:03

1 a

x	–4	–2	–1	2	4	8
y	–0·5	–1	–2	1	0·5	0·25

b

x	–6	–2	–1	1	2	6
y	–1	–3	–6	6	3	1

c

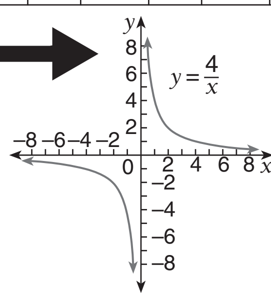
x	–4	–2	–1	1	2	4
y	0·25	0·5	1	–1	–0·5	–0·25

d

x	1	2	3	4	6	8
y	–8	–4	–2·67	–2	–1·33	1

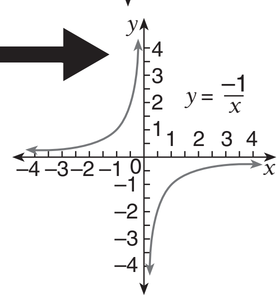
2

x	–8	–4	–2	–1	–0·5	0·5	1	2	4	8
y	–0·5	–1	–2	–4	–8	8	4	2	1	0·5



3

x	–4	–2	–1	–0·5	–0·25	0·25	0·5	1	2	4
y	0·25	0·5	1	2	4	–4	–2	–1	–0·5	–0·25



Negative value of k reflects the graph about the x-axis.

4 a C **b** F **c** D **d** B **e** E **f** A

5 a Yes **b** –18 **c** 20

6 a $(6, -2)$; $y = \dfrac{-12}{x}$ **b** $(4, 5)$; $y = \dfrac{20}{x}$

c $(2, -3)$; $y = \dfrac{-6}{x}$ **d** $(2, 2)$; $y = \dfrac{4}{x}$

Exercise 4:04

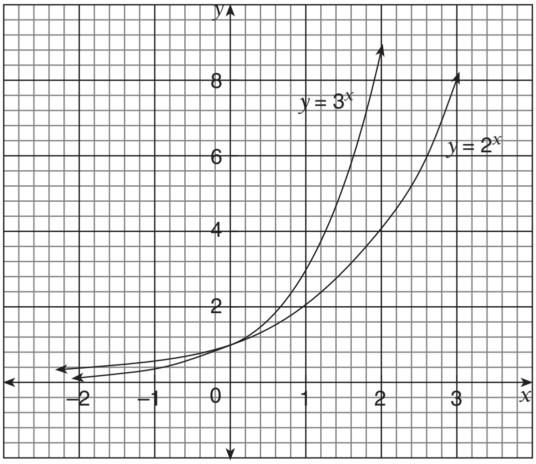
1 a

x	−2	−1·6	−1·2	−0·8	−0·4	0	0·4	0·8	1·2	1·6	2	2·5	3
y	0·25	0·33	0·44	0·57	0·76	1·0	1·3	1·7	2·3	3·0	4·0	5·7	8

b

x	−2	−1·5	−1	−0·5	0	0·5	1	1·5	2
y	0·11	0·19	0·33	0·58	1·0	1·7	3·0	5·2	9·0

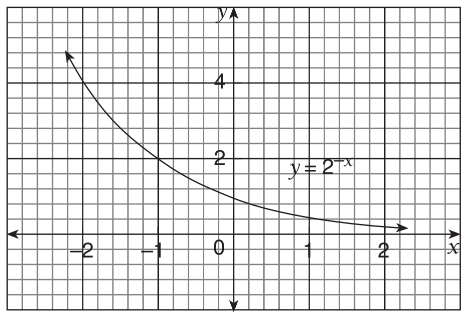
c $y = 3^x$ is 'above' $y = 2^x$ for positive values of x but the curves cross over at the common point of (0, 1).



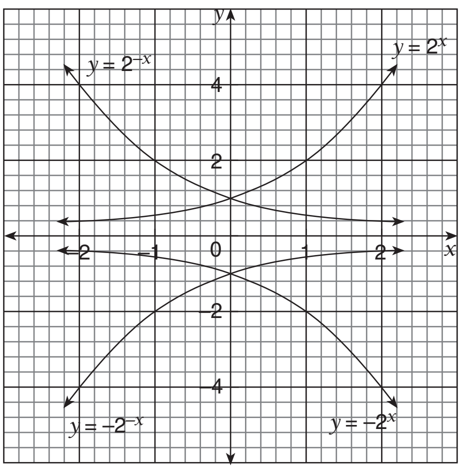
2 a

x	−2	−1·5	−1	−0·5	0	0·5	1	1·5	2
y	4·0	2·8	2·0	1·4	1·0	0·71	0·50	0·35	0·25

b $y = 2^{-x}$ is a reflection of $y = 2^x$ in the y-axis.



3 a b



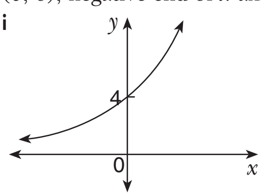
c Reflection in the x-axis.

4 a A is $y = 3 \times 2^x$, B is $y = 2 \times 2^x$
C is $y = 2^x$, D is $y = 0{\cdot}5 \times 2^x$

b It moves the y-intercept to (0, k) and increases the steepness of the curve.

5 a (0, 6), negative end of x-axis. **b** (0, 6), positive end of x-axis.

c i



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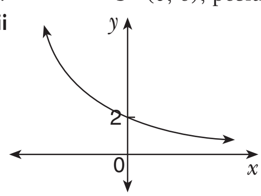
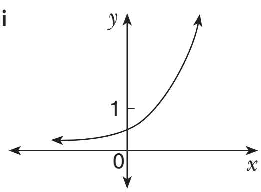
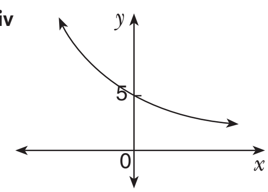


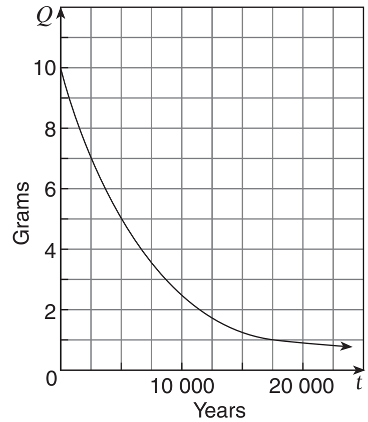
iii



iv

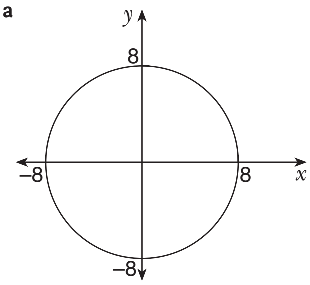


6 **a** $Q = 10$; the quantity present when $t = 0$ is 10 grams.
b 5 g
c **i** 2·5 **ii** 1·25 **iii** 7·07
d

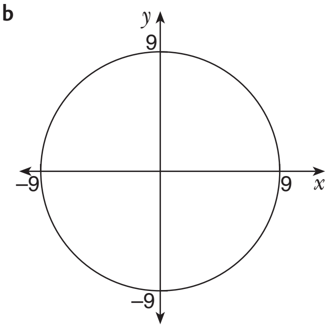


Exercise 4:05

1 **a** $x^2 + y^2 = 9$ **b** $x^2 + y^2 = 25$ **c** $x^2 + y^2 = 1$
2 **a** $x^2 + y^2 = 4$ **b** $x^2 + y^2 = 49$ **c** $x^2 + y^2 = 100$
d $x^2 + y^2 = 3$ **e** $x^2 + y^2 = 6$ **f** $x^2 + y^2 = 8$
g $x^2 + y^2 = \frac{9}{4}$ or $4x^2 + 4y^2 = 9$
h $x^2 + y^2 = \frac{81}{16}$ or $16x^2 + 16y^2 = 81$
i $x^2 + y^2 = 17{\cdot}64$
3 **a** 8 **b** 9 **c** $\sqrt{10}$ **d** $\sqrt{2}$
e 1·5 **f** 2·5 **g** $\frac{3}{2}$ **h** $\frac{4}{3}$
4 **a** **i** (4, 3), (4, −3) **ii** (−3, 4), (−3, −4) **iii** $(2, \sqrt{21})$, $(2, -\sqrt{21})$
b **i** (3, 4), (−3, 4) **ii** (4, −3), (−4, −3) **iii** $(\sqrt{21}, 2)$, $(-\sqrt{21}, 2)$
5 **a**



b



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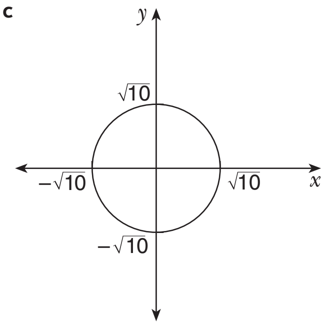
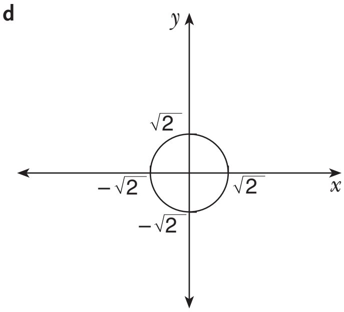


d



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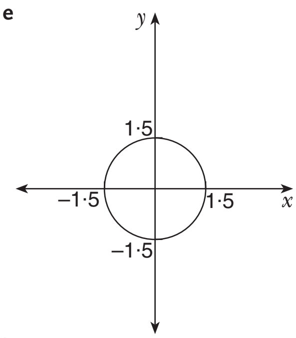

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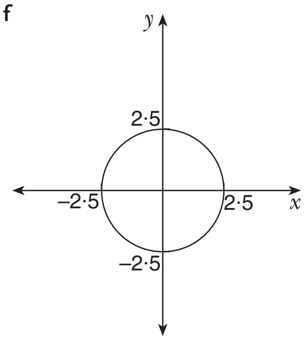

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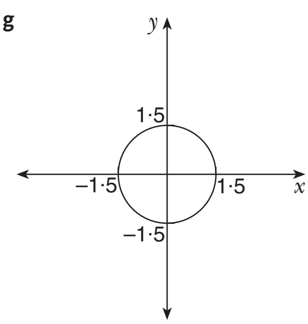

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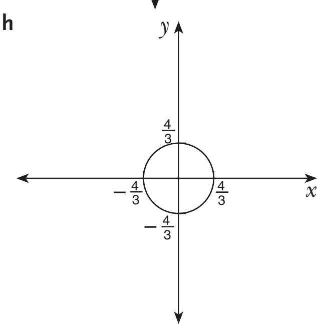


6 **a** $x^2 + y^2 = 1$ **b** $x^2 + y^2 = 9$
c $x^2 = 4 - y^2$ **d** $y^2 = 2 - x^2$
7 **a** Use the distance formula and compare to the radius of the circle.
b **i** on **ii** out **iii** in
iv in **v** out **vi** on
vii in **viii** out
8 **a** $x^2 + y^2 = 25$ **b** $x^2 + y^2 = 13$
c $x^2 + y^2 = 4$

Exercise 4:06

1 **a** **i** B **ii** C **iii** A **b** $y = 3x^3$ **c** $y = \frac{1}{2}x^3$
d By comparing the coefficients of x^3 you can tell which graph is the steepest. The larger the coefficient, the steeper the graph.
2 **a** $y = 3x^3$ **b** $y = x^3$ **c** $y = 3x^3$
3 **a** They are reflections of each other in the x-axis.
b The size of a affects the steepness of the graph while the sign of a indicates whether the graph is increasing or decreasing as you move from left to right.

4 **a** increasing **b** decreasing **c** increasing **d** increasing **e** decreasing **f** decreasing

5 **a**

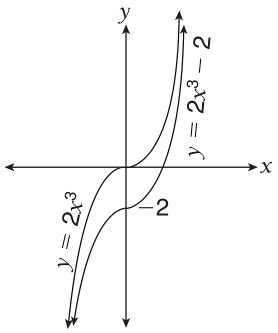
x	−2	−1	0	1	2
x^3	−8	−1	0	1	8
$x^3 + 2$	−6	1	2	3	10
$x^3 - 2$	−10	−3	−2	−1	6

b A is $y = x^3 + 2$
B is $y = x^3 - 2$
c $y = x^3 + 2$ is the curve $y = x^3$ translated up two units.
d $y = x^3 - 2$ is the curve $y = x^3$ translated down two units.

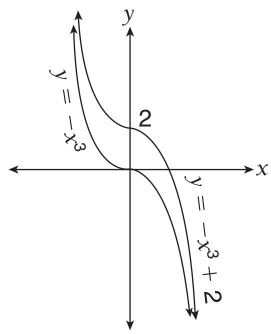
6 **a** Move the graph of $y = \frac{1}{2}x^3$ up one unit. **b** Move the graph of $y = \frac{1}{2}x^3$ down one unit.
c Move the graph of $y = \frac{1}{2}x^3$ up two units. **d** Move the graph of $y = \frac{1}{2}x^3$ down two units.
e Reflect the graph of $y = \frac{1}{2}x^3$ in the x-axis. **f** Reflect the graph in the x-axis and then move it up one unit.

7 **a** **i** $y = x^3 + 10$ **ii** $y = x^3 - 10$ **b** **i** $y = -\frac{x^3}{3} + 6$ **ii** $y = -\frac{x^3}{3} - 5$
c **i** $y = -\frac{1}{10}x^3 + 10$ **ii** $y = -\frac{1}{10}x^3 - 15$

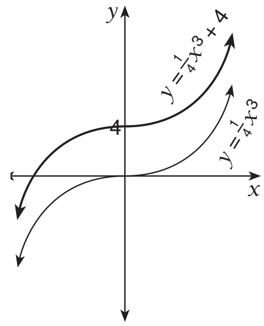
8 **a**



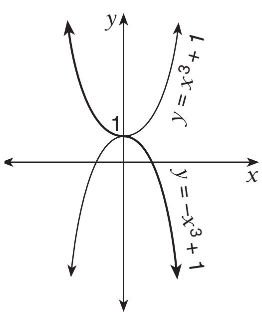
b



c



9 **a**



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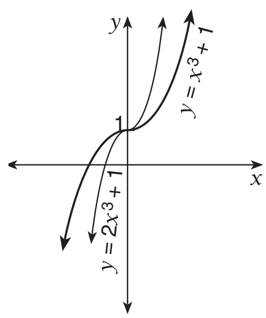


10 **a** B, H **b** D, E **c** C, F, I
d A, F, G **e** G **f** C, E, H

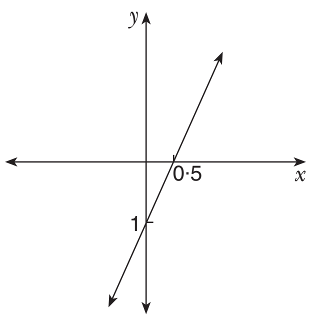
11 **a** $y = \frac{5}{4}x^3 + 10$
b $y = -2x^3 - 4$

12 The size of a affects the steepness of the graph and the sign of a indicates whether the graph is increasing or decreasing as it moves from left to right. The value and sign of d affects how far and in which direction the graph is shifted vertically.

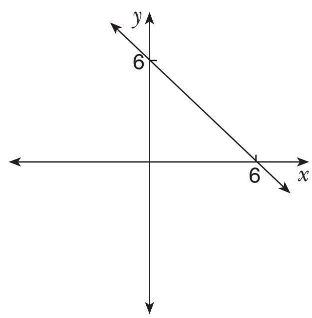
Exercise 4:07

1 **a** J, K **b** A, H **c** B, E **d** F, I **e** D, G **f** C, L

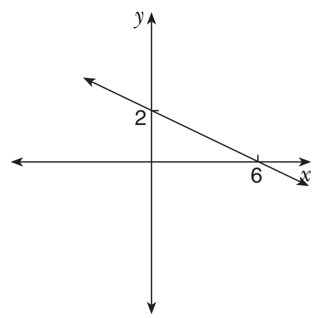
2 **a**



b



c



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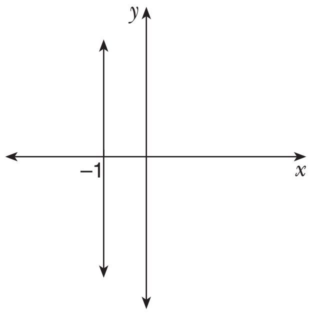
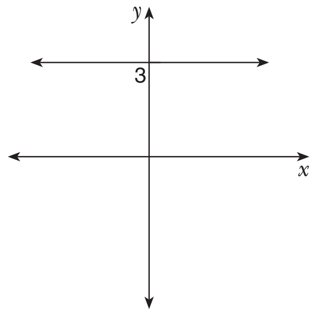
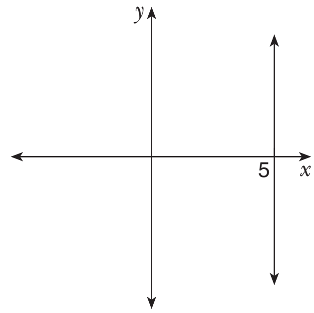


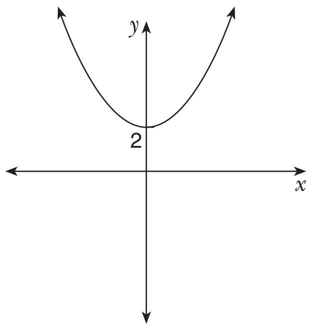
e



f



g



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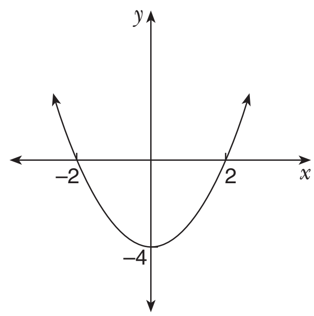

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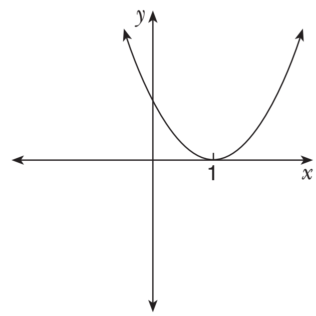

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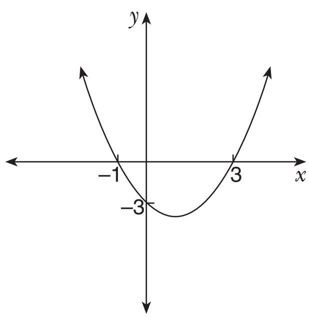

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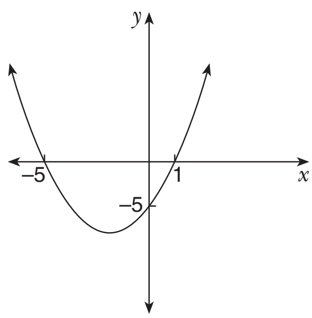

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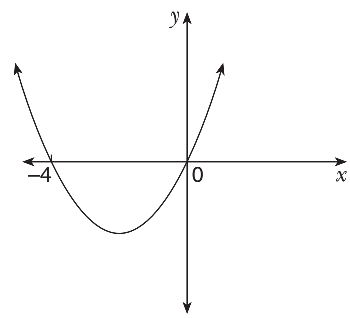

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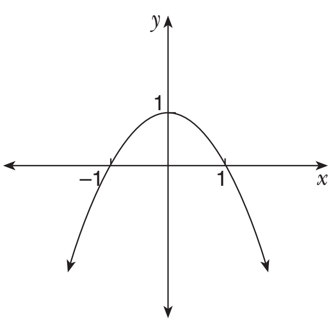
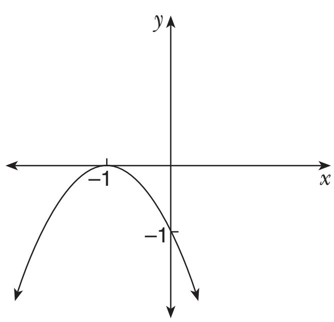
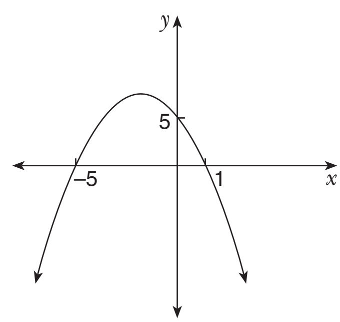


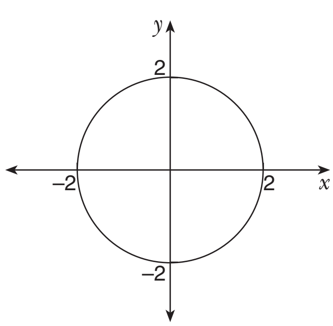
n



o



p



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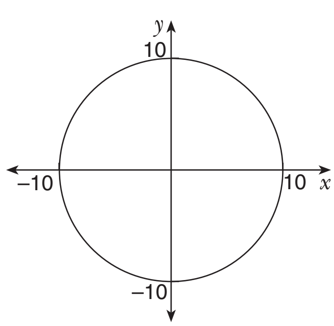
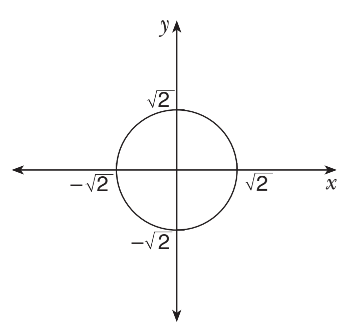


r



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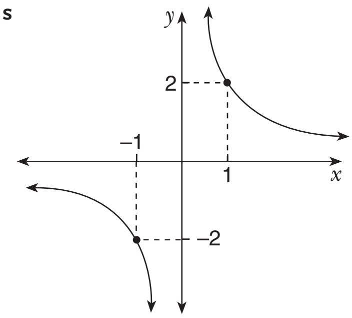
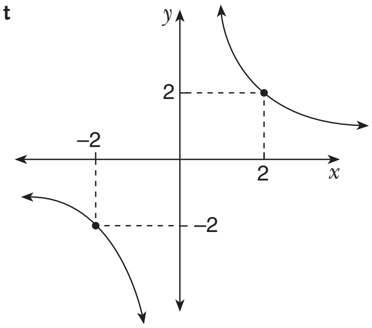
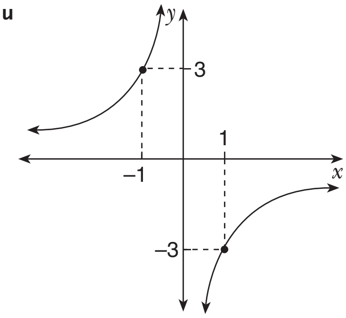


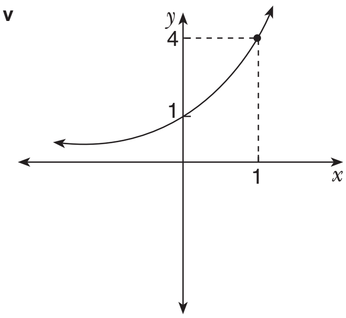
t



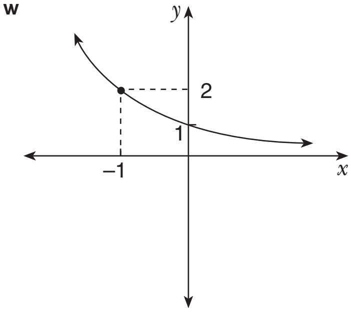
u



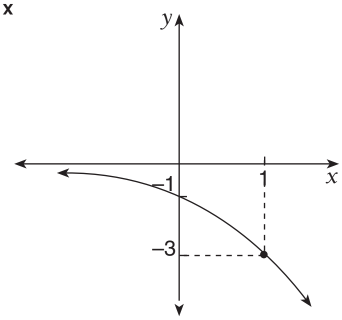
v



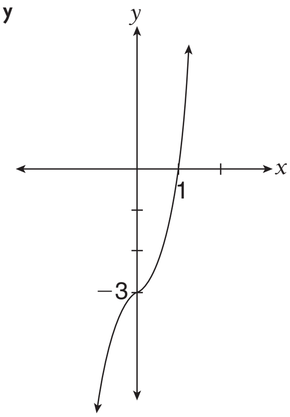
w



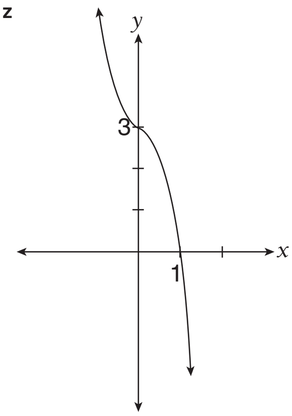
x



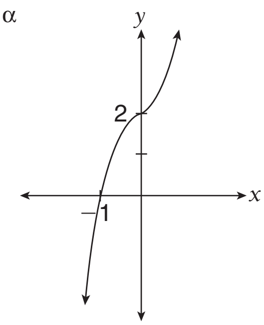
y



z



α



3 **a** D **b** E **c** I **d** A **e** H **f** B **g** C **h** G **i** F

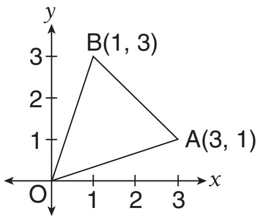
4 **a** $y = x + 1$ **b** $y = -x^2$ **c** $y = 2^x$ **d** $x^2 + y^2 = 36$
e $y = -\frac{1}{3}x + 1$ **f** $xy = -1$ **g** $y = 2x^2$ **h** $y = x(x - 2)$
i $x^2 + y^2 = 4$ **j** $y = -\frac{1}{2}x^3$ **k** $y = 2^{-x}$ **l** $xy = 4$
m $y = (x - 2)^2$ **n** $y = 9 - x^2$ **o** $y = 2x^3 - 6$ **p** $y = (x + 1)(x - 3)$
q $y = 6^x$ **r** $y = (x - 2)(x + 6)$

Prep Quiz 4:08

1 $\sqrt{32}$ units **2** −1 **3** (1, 0) **4** 1 **5** $y = -x + 1$ **6** −3
7 −2 **8** $y = 2x + 8$ **9** 2 **10** $-\frac{1}{2}$

Exercise 4:08

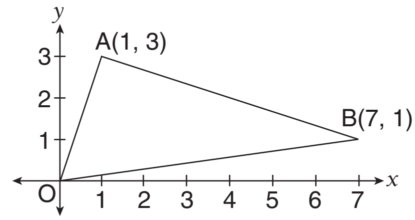
1 a



$$OB = \sqrt{(3-0)^2 + (1-0)^2}$$
$$= \sqrt{10}$$
$$OA = \sqrt{(1-0)^2 + (3-0)^2}$$
$$= \sqrt{10}$$

$\therefore$ ΔOAB is isosceles (2 equal sides).

b



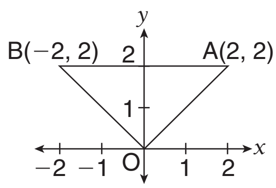
$$\text{Slope of OA} = \frac{3-0}{1-0}$$
$$= 3$$
$$\text{Slope of AB} = \frac{3-1}{1-7}$$
$$= -\tfrac{1}{3}$$

Now, slope of OA $\times$ slope of AB $= -1$

$\therefore$ OA $\perp$ AB

$\therefore$ ΔOAB is right-angled at A.

c



$$OA = \sqrt{(2-0)^2 + (2-0)^2}$$
$$= \sqrt{8}$$
$$OB = \sqrt{(2-0)^2 + (-2-0)^2}$$
$$= \sqrt{8}$$

AB = 4

Now, OA = OB

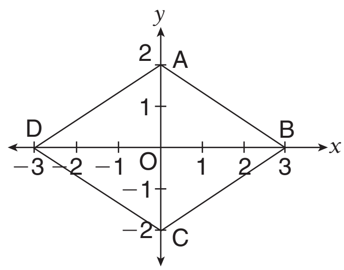
$\therefore$ ΔOAB is isosceles.

Also, $4^2 = (\sqrt{8})^2 + (\sqrt{8})^2$

$\therefore$ $AB^2 = OA^2 + OB^2$

$\therefore$ ΔOAB is right-angled at O (by Pythag. th'm)

2 a



AO = OC = 2 and

OD = OB = 3

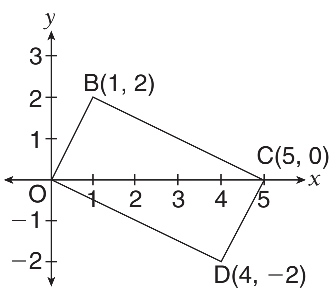
Also, AC $\perp$ BD

$\therefore$ Diagonals bisect each other at right angles.

$\therefore$ ABCD is a rhombus.

Note: This question could be done in other ways.

b



$$OC = 5$$
$$BD = \sqrt{(2--2)^2 + (1-4)^2}$$
$$= \sqrt{4^2 + (-3)^2}$$
$$= 5$$

$\therefore$ OC = BD

$$\text{Midpoint of BD} = \left(\frac{4+1}{2}, \frac{-2+2}{2}\right)$$
$$= (2\tfrac{1}{2}, 0)$$

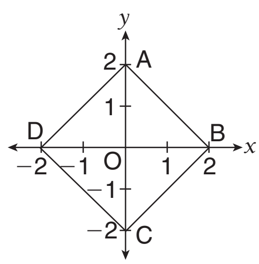
Midpoint of OC $= (2\tfrac{1}{2}, 0)$

$\therefore$ BD and OC bisect each other

$\therefore$ OBCD is a rectangle

(equal diagonals that bisect each other)

c



$$AB = \sqrt{2^2 + 2^2} \quad \text{(Pythag. th'm)}$$
$$= \sqrt{8}$$

Similarly, BC = DC = AD = $\sqrt{8}$

$\therefore$ ABCD is a parallelogram

(both pairs of opp. sides equal)

Now, slope of AB $= -1$

$\therefore$ slope of AD = 1

$\therefore$ AB $\perp$ AD (product of slopes is -1)

$\therefore$ $\angle DAB = 90°$

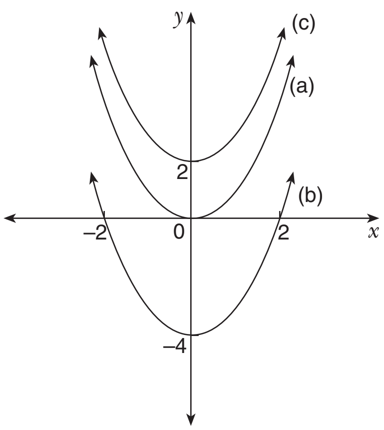
$\therefore$ ABCD is a rectangle

(par'm with one right $\angle$)

3 **a** (3, 2) **b** OB = AC = $\sqrt{13}$; The diagonals of a rectangle are equal in length.
c The midpoint of both diagonals is $(1\frac{1}{2}, 1)$. The diagonals bisect each other.
4 **a** Midpoint of AB is (3, 2); midpoint of AC is (2, –1). **b** 3 **c** 3
d The line joining the midpoints of two sides of a triangle is parallel to the third side.
5 Slope of BD is 1; slope of AC is –1; the diagonals are perpendicular.
6 **a** (3, 2) **b** $\sqrt{13}$ **c** $\sqrt{13}$ **d** E is equidistant from A, B and O.
7 **a** AB = BC = $\sqrt{10}$ (by distance formula). Hence ΔABC is isosceles as it has two equal sides.
b (0, 2) **c** 1 **d** slope of AC = –1; slope of EB = 1; EB ⊥ AC since product of their slopes is –1.
e Find the lengths AC and BE and then evaluate $\frac{1}{2}$AC × BE.
8 **a** Midpoint of OA is $(\frac{1}{2}, 1)$; midpoint of AB is (2, 1). **b** $1\frac{1}{2}$ units. **c** OB is 3 units in length. Hence $\frac{1}{2}\text{OB} = 1\frac{1}{2}$.
9 **a** Midpoints are: AB (–2, 3); BC (3, 2); CD (1, –2); AD (–4, –1). **b** a parallelogram
c Show that the opposite sides are parallel by showing they have the same slope.
10 **a** Perp. bisector of AB is $x + 2y = 3$; perp. bisector of BC is $y = x$; perp. bisector of AC is $x = 1$.
b (1, 1) **c** As (1, 1) satisfies the equation $y = x$, then (1, 1) lies on $y = x$.
∴ Perp. bisector of BC ($y = x$) passes through the point of intersection of the other bisectors.
11 **a** Median from A is $y = -2x$; median from B is $2x - 5y + 4 = 0$; median from C is $x - y + 1 = 0$.
b Point of intersection (of any two medians) is $(-\frac{1}{3}, \frac{2}{3})$.
c Substitute $(-\frac{1}{3}, \frac{2}{3})$ into the equation of the median not used in **b** and show that the coordinates satisfy the equation. This shows that the third median also passes through $(-\frac{1}{3}, \frac{2}{3})$.

Diagnostic Test 4 Number plane graphs and coordinate geometry

1



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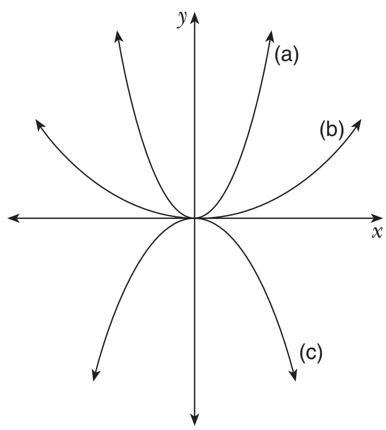

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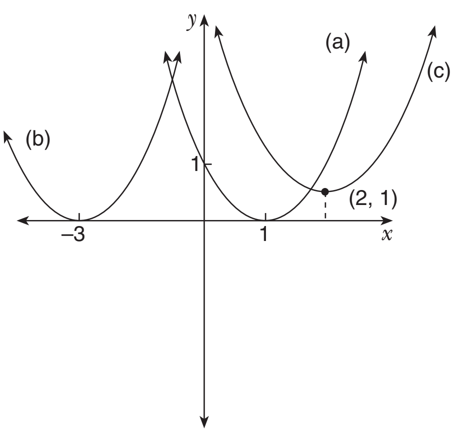
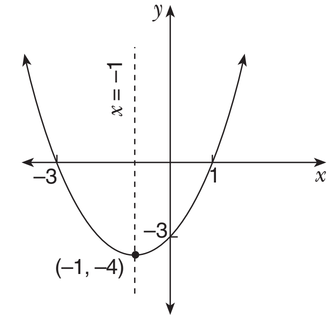
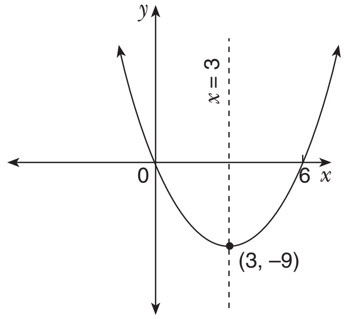


4 **a** (0, –3) **b** (0, 0) **c** (0, 8) **d** (0, –5)
5 **a** (1, 0), (–3, 0) **b** (0, 0), (6, 0) **c** (2, 0), (–4, 0)
d $(\frac{1}{2}, 0), (-\frac{5}{2}, 0)$
6 **a** $x = -1$ **b** $x = 3$ **c** $x = -1$ **d** $x = -1$
7 **a** (–1, –4) **b** (3, –9) **c** (–1, 9) **d** (–1, –9)

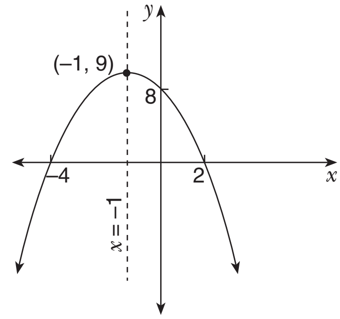
8 **a**



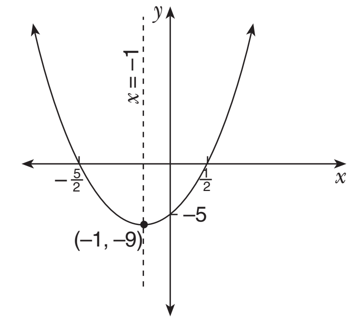
b



c

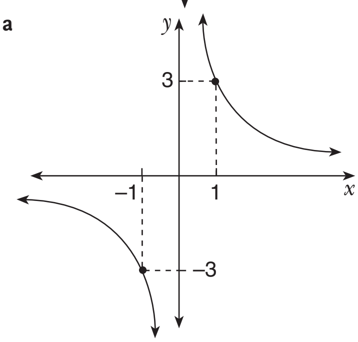


d



9 **a** $y = x^2 - 9$
b $y = (x + 1)(x - 5)$
c $y = 2(x + 2)^2$
d $y = (6 - x)(2 + x)$

10 **a**



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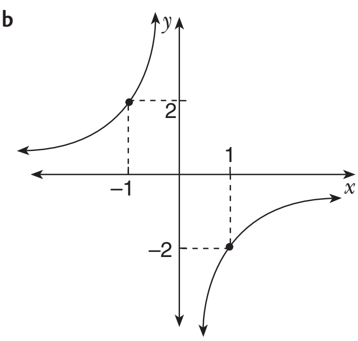
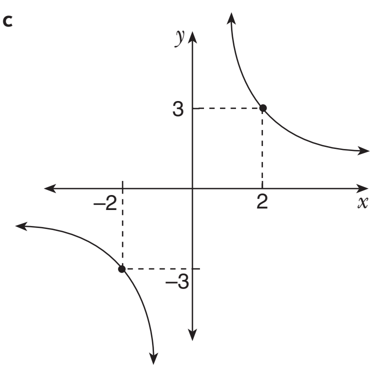
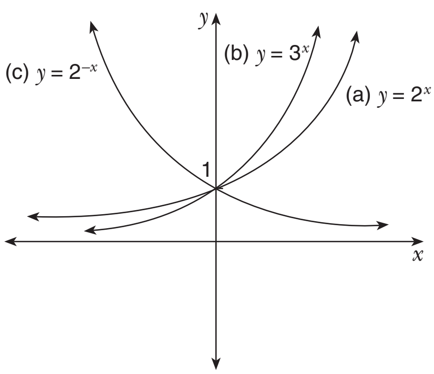


c

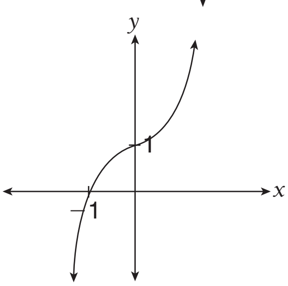


11

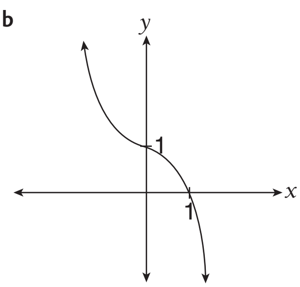


12 **a** $x^2 + y^2 = 4$
b $x^2 + y^2 = 49$
c $\sqrt{3}$

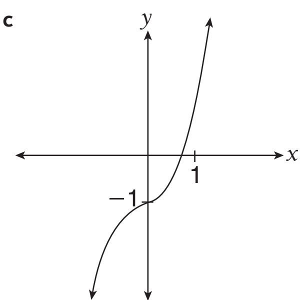
13 **a**



b



c



14 **a** E, H
b B, I
c G, J
d C, L
e A, D
f F, K

15 **a** E **b** B
c C **d** A
e D **f** F

4A Revision Assignment

1 Each of the graphs is congruent to the graph of $y = x^2$. The graphs can be obtained by performing the following translations on $y = x^2$.

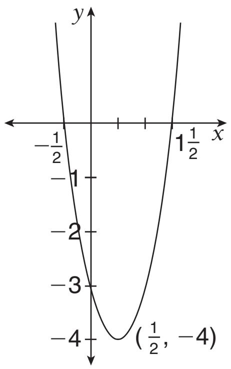
a moving it up five units **b** moving it down five units **c** moving it five units to the right

d moving it five units to the left **e** moving it five units to the right and then five units up

2 A is $y = \frac{1}{2}x^2$, B is $y = 2x^2$, C is $y = -\frac{1}{5}x^2$, D is $y = -4x^2$

3 a i -3 **ii** $1\frac{1}{2}, -\frac{1}{2}$ **iii** $x = \frac{1}{2}$ **iv** $(\frac{1}{2}, -4)$

b



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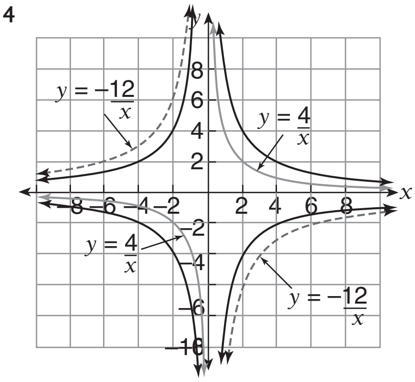
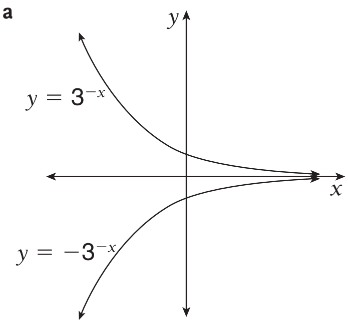


5 a



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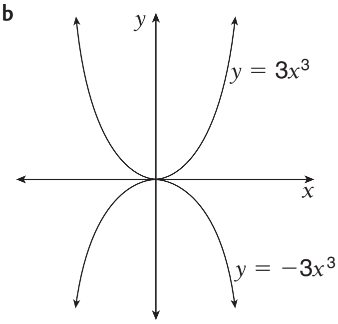
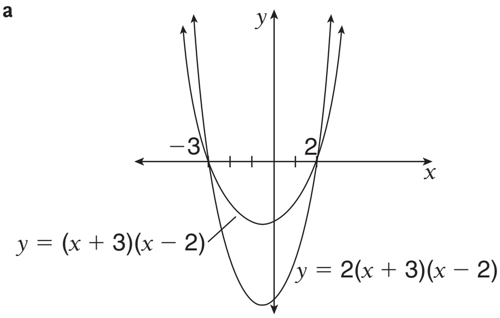


6 a



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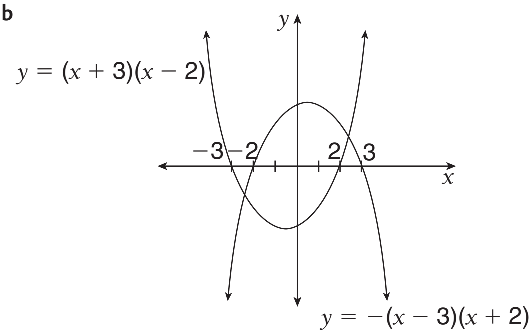


7 A is $y = \frac{1}{2}x^3 - 4$, B is $y = -x^3 + 4$

4B Working Mathematically

1 In a circle, draw two diameters at right angles. Join the end points of these diameters to form a square.
Draw two diameters that bisect the sides of the square. Join the end points of these diameters to form another square.

2 40 kg, 45 kg, 50 kg, 52 kg, 60 kg

3 $1^2 = 1$, $11^2 = 121$, $111^2 = 12\,321$, $1111^2 = 1\,234\,321$, $11\,111\,111^2 = 123\,456\,787\,654\,321$

4 11 seconds **5** 19 January 2004

6 a $0{\cdot}1 = 10^{-1}$, $0{\cdot}01 = 10^{-2}$, $0{\cdot}001 = 10^{-3}$ **b** The higher the pH, the lower the concentration of the acid. (To find the pH write the concentration as a power of 10. The pH is the opposite of the power.)

c 5 **d** $10^{-4{\cdot}5}$ mole/L (or 0·000 031 6 mole/L to three significant figures)

e The acid with a pH of 2 is more concentrated. It is 100 times more concentrated.

Chapter 5: Further algebra

Prep Quiz 5:01

1 $x = -3, 2$ **2** $x = \frac{1}{2}, -7$ **3** $x = 0, -4$ **4** $x = 2, -2$ **5** $x = 1, 2$ **6** $x = \frac{1}{2}, -4$
7 $x = 1, 3$ **8** $x = -\frac{3}{2}, 1$ **9** $x = 2, y = 3$ **10** $x = 4, y = -1$

Exercise 5:01

1 **a** $x = 2, y = 8$ **b** $x = -1, y = 5$ **c** $x = 2, y = 4$ **d** $x = 3, y = 9$ or $x = -2, y = 4$
e $x = 2, y = 4$ or $x = -1, y = 1$ **f** $x = 1, y = 1$ or $x = -3, y = 9$
2 Solutions same as for question 1.
3 **a** $x = -1, y = 1$ or $x = -3, y = 9$ **b** $x = 7, y = 49$ or $x = -7, y = 49$ **c** $x = 0, y = 0$ or $x = 3, y = 9$
d $x = 8, y = 64$ or $x = -7, y = 49$ **e** $x = 3, y = 9$ or $x = 7, y = 49$ **f** $x = -1, y = 1$ or $x = -2, y = 4$
g $x = 9, y = 86$ or $x = -5, y = 30$ **h** $x = 7, y = 56$ or $x = 1, y = 8$ **i** $x = 1, y = -9$ or $x = -10, y = 90$
4 **a** $x = -1, y = 1$ or $x = 2, y = 4$ **b** $x = 1, y = 1$ **c** no solutions
5 **a** $x = 6, y = 36$ or $x = -1, y = 1$ **b** $x = 2, y = 5$ or $x = -1, y = 2$ **c** $x = 6, y = 33$ or $x = -2, y = 1$
d $x = 4, y = 12$ or $x = -1, y = 2$ **e** $x = 7, y = 28$ or $x = -2, y = 10$ **f** $x = 1, y = 4$ or $x = -7, y = 28$
g $x = 5, y = 20$ or $x = -4, y = 2$ **h** $x = 2, y = 0$ or $x = -7, y = 27$ **i** $x = 3, y = 13$ or $x = -\frac{1}{2}, y = -1$
6 **a** (2, 2) and (−1, −1) **b** (4, 6) and (−5, 15) **c** (2, 6)
d (4, 4) and (−7, 15) **e** (−3, 4) **f** no points of intersection

Prep Quiz 5:02

1 $a = 27 - 15$ **2** $m = p - n$ **3** $x = \frac{35}{5}$ **4** $b = \frac{c}{a}$ **5** $x = 3$ **6** $n = -3$
7 $m = 7\frac{1}{2}$ **8** $x = 13$ **9** $x = 81$ **10** $x = \pm 3$

Exercise 5:02

1 **a** $S = \frac{1-k}{2}$ **b** $S = \frac{a-b}{c}$ **c** $S = \frac{n+m}{a}$ **d** $S = \frac{y+z}{x}$ **e** $S = \frac{y-x}{a}$ **f** $S = \frac{g+q}{p}$
g $S = \frac{b}{a}$ **h** $S = \frac{bx}{a}$ **i** $S = \frac{uv}{t}$
2 **a** $x = \frac{3a+b}{3}$ **b** $x = \frac{n-5m}{5}$ **c** $x = \frac{q-ap}{a}$ **d** $x = \frac{8-y}{4}$ **e** $x = \frac{tv-w}{t}$ **f** $x = \frac{m-12}{3}$
g $x = \frac{t-2a}{a}$ **h** $x = \frac{v-u}{u}$ **i** $x = \frac{h-km}{k}$
3 **a** $x = \pm\sqrt{\frac{n}{m}}$ **b** $x = \pm\sqrt{\frac{a}{b}}$ **c** $x = \pm\sqrt{\frac{p}{5}}$ **d** $x = \pm\sqrt{a+b}$ **e** $x = \pm\sqrt{t-u}$ **f** $x = \pm\sqrt{v-w}$
g $x = \pm\sqrt{ay}$ **h** $x = \pm\sqrt{kz}$ **i** $x = \pm\sqrt{\frac{3m}{n}}$
4 **a** $a = \frac{c^2}{b}$ **b** $a = \frac{t^2}{5}$ **c** $a = c^2 + b$ **d** $a = y^2 - 5$ **e** $a = b - r^2$ **f** $a = (m-n)^2$
g $a = (p+q)^2$ **h** $a = \frac{x^2}{y^2}$ **i** $a = \frac{(u-t)^2}{r^2}$
5 **a** $x = \frac{b-a}{2}$ **b** $x = \frac{q}{a-p}$ **c** $x = \frac{a-b}{a-1}$ **d** $x = \frac{c+e}{b+d}$ **e** $x = \frac{15a}{8}$ **f** $x = \frac{4a}{4a-b^2}$
g $x = \frac{2a}{1-a}$ **h** $x = \frac{3a+2}{a-1}$

6

	A	B	C		A	B	C
a	$x = \frac{P}{y}$	$q = p - r$	$R = Q - P$	**b**	$y = \frac{ab}{x}$	$r = \frac{As}{3}$	$m = \frac{2a}{n}$
c	$x = \frac{a-y}{3}$	$a = \frac{m-n}{t}$	$a = \frac{v^2-u^2}{2s}$	**d**	$b = \frac{x-5a}{5}$	$n = \frac{4m}{3}$	$t = \frac{P-Qr}{Q}$
e	$n = \pm\sqrt{m-a}$	$u = \pm\sqrt{v^2-t}$	$x = \pm\sqrt{A-y^2}$	**f**	$c = \frac{a^2}{b}$	$x = \frac{Y^2}{a^2}$	$a = x^2 - y$
g	$Y = aX^2$	$x = \frac{bR^2}{a}$	$a = \frac{2V}{t^2}$	**h**	$a = \frac{6y-3b}{2}$	$m = \frac{15x+5n}{3}$	$A = 2x - 3y$
i	$b = nm^2 - a$	$k = \frac{h}{1-2h}$	$m = na^2T^2$				

7 a $x \geqslant 4$ **b** $x \geqslant -3$ **c** $x \geqslant N$ **d** $x \leqslant 4$ **e** $x \leqslant 10$ **f** $x \leqslant Y$
g $x \geqslant \frac{-1}{3}$ **h** $x \geqslant \frac{2}{5}$ **i** $x \leqslant \frac{Q}{2}$ **j** 0

8 a 3 **b** −10 **c** 1 **d** q **e** $-s$ **f** a
g 2, −2 **h** ±4 **i** 1, 2

9 a $t = 0, H = 10$ **b** $0 = 10 + 9t - t^2 \Rightarrow (10 - t)(t + 1) = 0 \Rightarrow t = 10$

10 a $I = 0 \Rightarrow W = 350$ **b** $r \geqslant 0,\ r = \sqrt{\frac{5}{4\pi}}$ **c i** $B = \frac{A}{L}$ **ii** $L \geqslant 0$ **iii** B will decrease
d i $l \geqslant 0$ **ii** T will increase **iii** $l = g\left(\frac{T}{2\pi}\right)^2 = g\frac{T^2}{4\pi^2}$

Challenge 5:02

a 55, 89, 144 **b** 0·6180 **c i** 610 **ii** 6765 **iii** 832 040

Prep Quiz 5:03

1 −3 **2** −3 **3** 5 **4** 21 **5** $9a^2 - 1$ **6** $x^2 - 2x - 15$ **7** $4y^2 - 12y + 9$ **8** $(10y - 3)(10y + 3)$
9 $(m - 9)(m + 8)$ **10** $2(x - 2)^2$

Exercise 5:03

1 a $6x + 5$ **b** $3x + 8$ **c** $12x - 4$ **d** $15a + 5$ **e** $3a - 1$ **f** $3a^2 - 6a + 8$

2 a $3n$ **b** $7 - n$ **c** $n^2 + 7n + 12$ **d** $n^2 + 6n$ **e** $2n^2 + 11n + 18$ **f** $4n^2 + 4n + 1$

3 a $5 - 2a$ **b** $a^2 + 4$ (unchanged) **c** $2a^2 - 3a - 5$ **d** $a^2 - a^3$ **e** $a^4 - 2a^2 + 1$ (unchanged) **f** $-a + \frac{1}{a}$

4 a $\frac{1}{x} + x$ **b** $\frac{1}{x^2} + 2 + x^2$ **c** $\frac{3}{x} - 5 + 2x$ **5 a** $x = \pm 2, \pm 1$ **b** $x = \pm 4, \pm 2$ **c** $x = \pm 3$

6 a $x = 0, 1$ **b** $x = 2, 3$ **c** $x = -1, 4$ **7 a** $a = 4, 25$ **b** $a = 1, 16$ **c** $a = 4, 9$

8 a $(x^2 - x - 1)(x^2 + x + 1)$ **b** $(x^2 - x + 3)(x^2 + x - 3)$ **c** $(a^3 - a - 2)(a^3 + a + 2)$

9 a $2x + 3$ **b** $20m$ **c** $26a + 13$ **d** $-2n + 19$

10 a $(a + 7)(a + 1)$ **b** $(x - 9)x$ **c** $2(m + 3)(2m - 1)$ **d** $(4y - 11)(4y - 3)$

11 a T **b** F **c** F **d** T **e** F **f** T **g** F **h** T

12 a $p > 2$ **b** $0 < p < \frac{1}{2}$ **c** $p < 0, p > 1$ **d** $p < 2$

13 a $x > 5, y < 5$ **b** y is odd **c** $x + y < 0$ or $y < -x$ **d** $-5 \leqslant x \leqslant 5, -5 \leqslant y \leqslant 5$

14 a $a = 9$ **b** $y = -\frac{3}{2}$ **c** $y = \frac{5}{2}$ **d** $x = 1$

Investigation 5:03

A Make y the subject of the formula. $y = \frac{6x}{x-1}$. Since $x - 1$ is one less than x, $x - 1$ will not divide x (when $x > 2$). Therefore answers that are positive integers can only be found when $x - 1$ divides 6 exactly.
$\therefore x = 2, 3, 4$ or 7. The possible scores are 2.12.24, 3.9.27, 4.8.32 and 7.7.49.

B **1** −1, 0, 1 or 2, 3, 4 **2** The equation would be $n + (n + 2)(n + 4) = n(n + 2 + n + 4)$. This becomes $n^2 - n - 8 = 0$, which has solutions $n = \frac{1 \pm \sqrt{33}}{2}$. Since these answers are not integers, no solution that is an integer exists.
3 No for 3, 4 and 5. Yes for 6. For integers differing by 6 the solution is: −8, −2, 4 or 9, 15, 21.

Diagnostic Test 5 Further algebra

1 a $x = 8, y = 64$ or $x = -1, y = 1$ **b** $x = 5, y = 15$ or $x = -4, y = 6$ **c** $x = 0, y = -5$ or $x = -3, y = -8$

2 a $a = \frac{m - S}{n}$ **b** $a = \pm\sqrt{2P - b^2}$ **c** $a = p\left(\frac{K}{n}\right)^2$ **d** $a = \frac{Lm}{L - 3}$

3 a $x \nless 1$ **b** $x \nless \frac{5}{2}$ **c** $x \neq 4$ **d** $x \neq \pm 3$

4 a $2a - 1$ **b** $x^2 + x - 1$ **c** $-8a^3 + 4a$ **d** $\frac{6}{m} - \frac{3m}{2}$

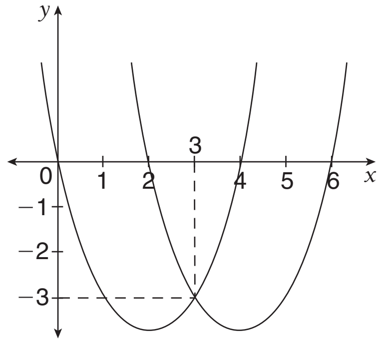
5 a no **b** yes **c** no **d** no

5A Revision Assignment

1 **a** $x = -6, y = 0$ or $x = 1, y = 7$ **b** $x = \frac{1}{2}, y = \frac{9}{2}$ or $x = -3, y = -6$ **c** $x = \frac{1}{2}, y = \frac{5}{4}$ or $x = 3, y = 0$

2 **a** $(3, -3)$

b



3 **a** $a \geqslant 4$ **b** $a \leqslant 4$ **c** $a \geqslant -4$ **d** $a \geqslant 0$

4 **a** $n \neq 3$ **b** $n \neq -3$ **c** $n \neq 0$ **d** $n \neq 3$ or -3

5 **a** $t = 0$ or 6 ∴ ball returns to the ground after 6 seconds

b 90 metres **c** $0 \leqslant t \leqslant 6$, $0 \leqslant h \leqslant 90$

6 **a** $4a^2 + 2a + 1$ **b** $a^2 + 5a + 7$ **c** $a^4 + a^2 + 1$

d $\frac{4}{a^2} + \frac{2}{a} + 1$

7 **a** $x = \pm 2$ or ± 4 **b** $x = \pm 3$ **c** no real solutions

8 **a** $16x$ **b** $14a + 21$ **c** $-24y$

9 **a** $n + 1$ must be even **b** $n + 2$ must be odd

c $2n$ must be even **d** n^2 must be odd

e $n!$ must be even

5B Working Mathematically

1 721 **2** 35·25 m **3** 12 cm **4** 14 h 36 min, assuming a steady rate of climb and a slide back that takes only a second or two **5** 66 **6** 11

Chapter 6: Curve sketching

Prep Quiz 6:01

1 T **2** F **3** T **4** T **5** y becomes smaller **6** y becomes larger **7** 1st **8** 4th **9** 2nd **10** 3rd

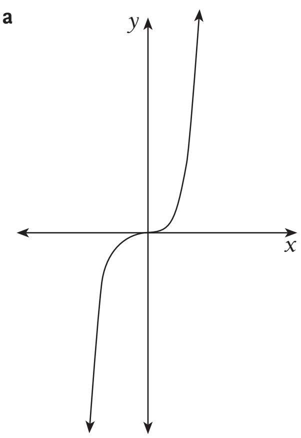
Exercise 6:01

1 **a** B **b** A **c** C **d** D **e** B **f** A **g** C **h** A **i** D

2 **a** **i** $y = \frac{1}{2}x^4 - 4$ **ii** $y = \frac{1}{2}x^4 + 2$ **b** **i** $y = 2x^7 - 5$ **ii** $y = 2x^7 + 3$ **c** **i** $y = -5x^6 - 4$ **ii** $y = -5x^6 - 8$

3 **a** **i** B **ii** A **iii** C **b** **i** C **ii** B **iii** A **c** **i** C **ii** B **iii** A

4 **a**



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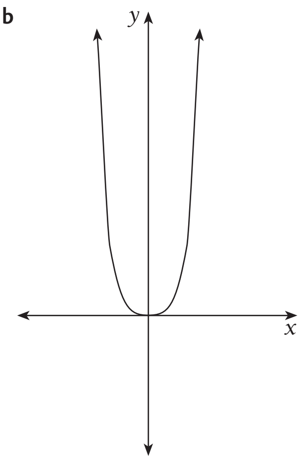

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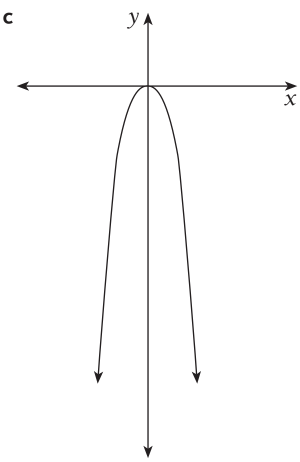

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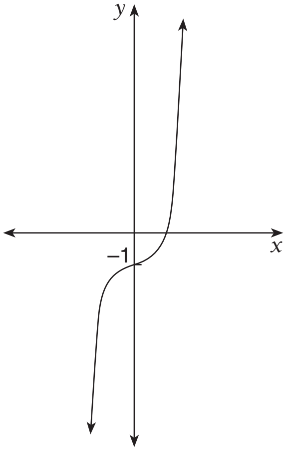

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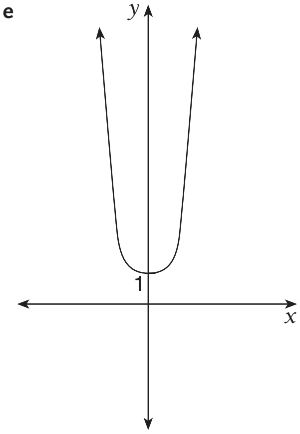

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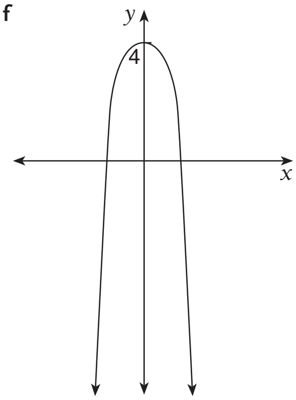

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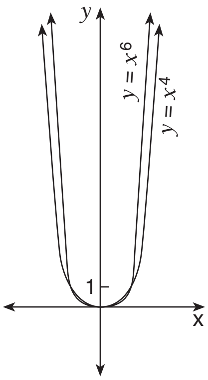

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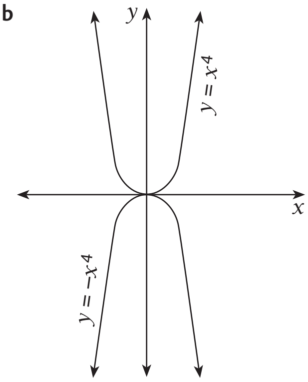

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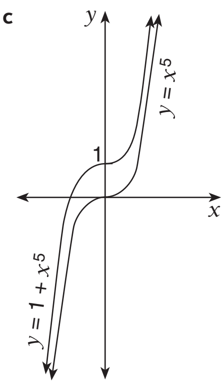

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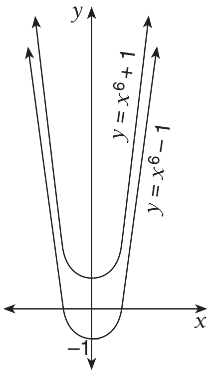

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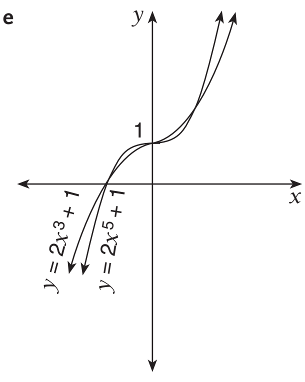

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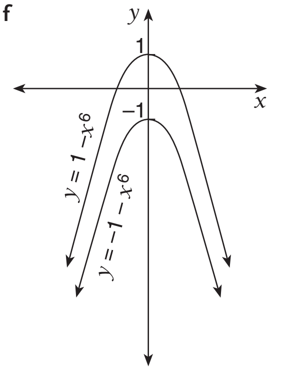
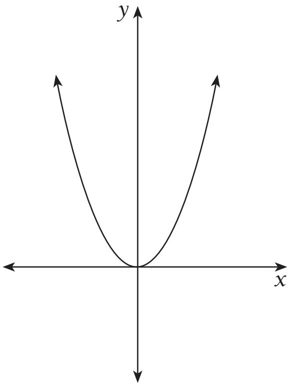


6 a The 4 tells us that the curve is shaped like a parabola, while the –2 tells us that it is upside down. The 1 tells us that the curve has been produced by moving the curve $y = -2x^4$ up one unit.

b The curve is similar in shape to $y = x^3$. It is steeper than this curve and it crosses the y-axis at (0, –3).

c The curves are reflections of each other in the x-axis.

7 a


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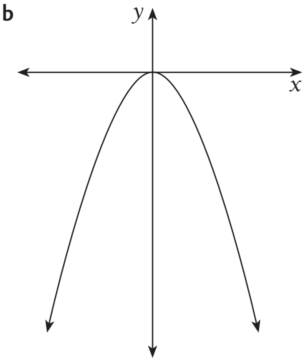

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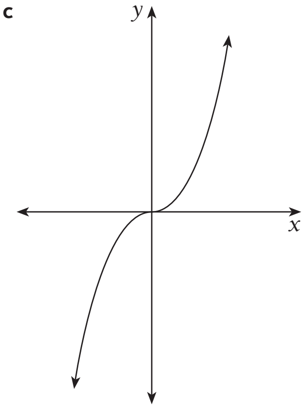

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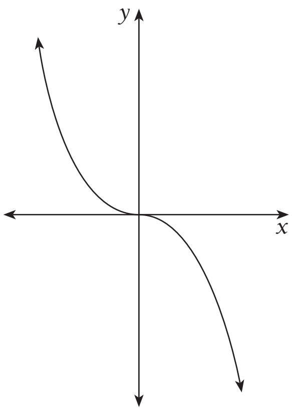

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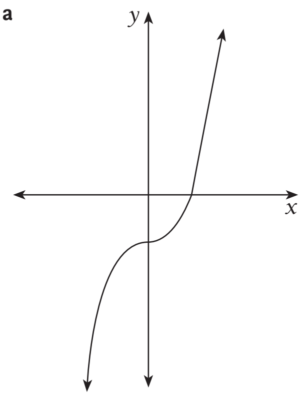

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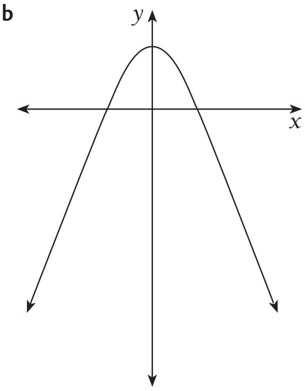


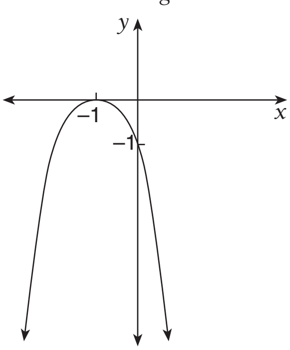
9 a n is odd, a is negative and d is negative. **b** n is even, a is negative and d is positive.
c n is even, a is positive and d is negative.

Exercise 6:02

1 a A is $y = (x - 3)^5$, B is $y = (x + 4)^5$ **b** D

2 a $y = (x + 4)^4$ **b** $y = (x - 4)^4$ **c** $y = (x - 1)^4$ **d** $y = (x + \frac{1}{2})^4$

3 a 1 unit to the right **b** 3 units to the left **c** $\frac{1}{2}$ a unit to the right

4 a


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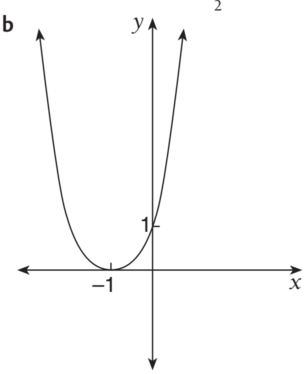

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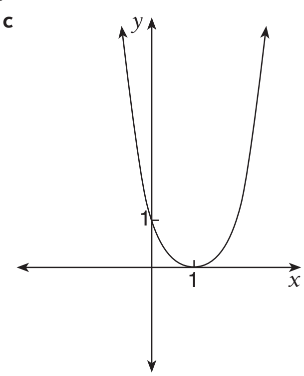
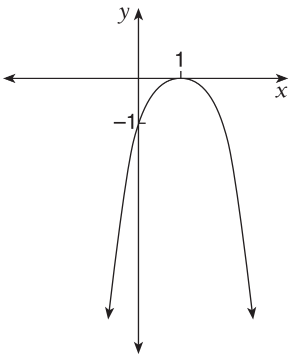
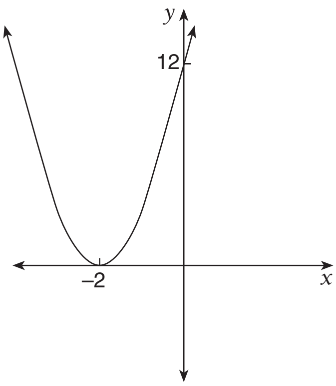


d



5 a



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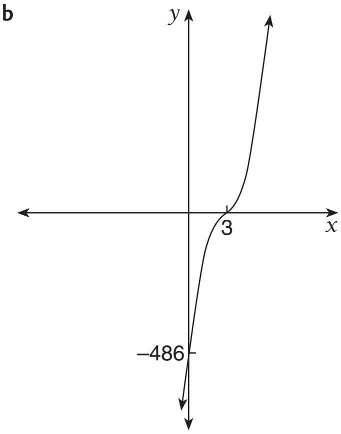

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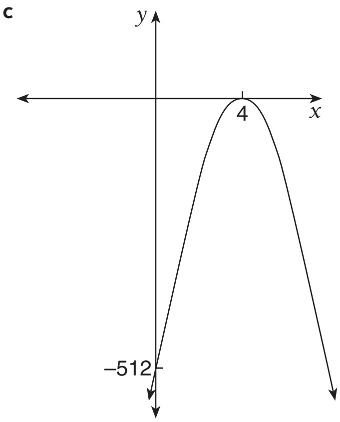

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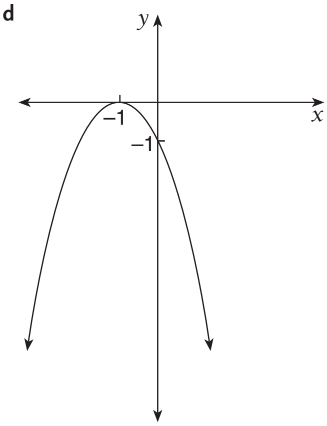
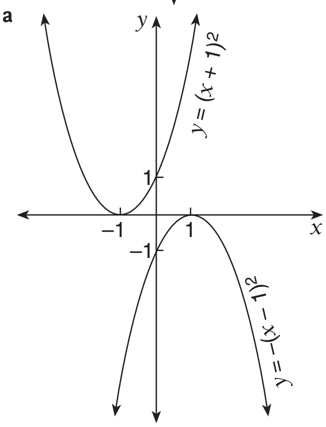


6 a



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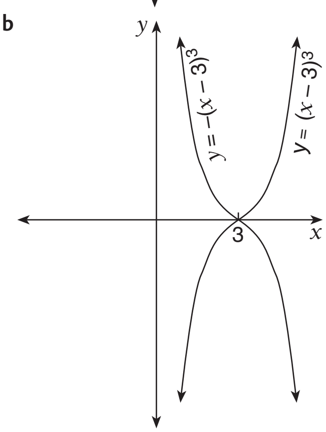

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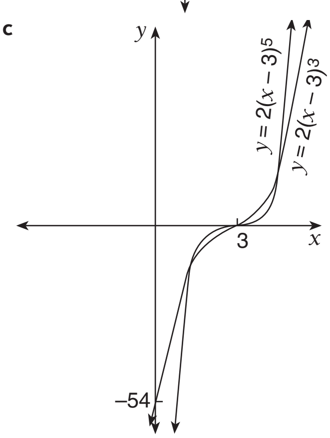

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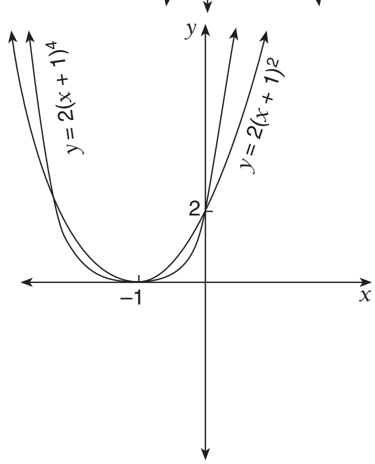

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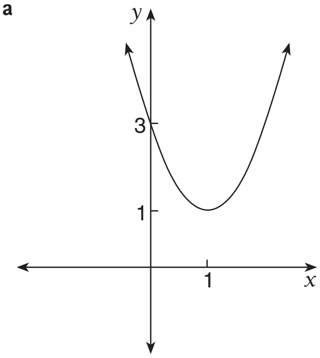

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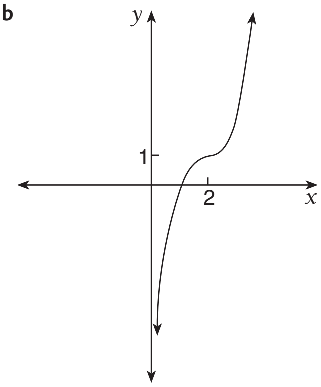

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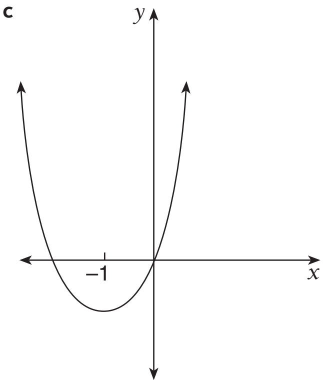
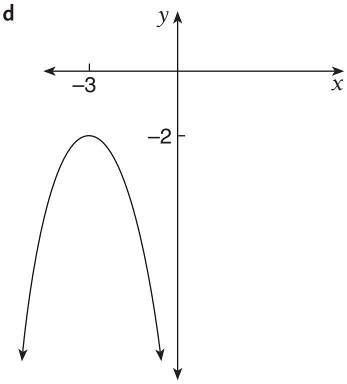


d



Exercise 6:03

1 a

x	−5	−4	−3	−2	−1·5	−1	0
y	−	0	+	0	−	0	+

x	0	1	2	4	5
y	−	0	−	0	+

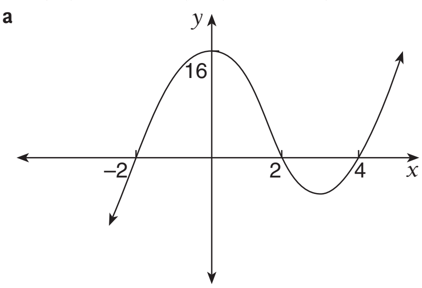
b

x	−3	−2	0	2	4	5	6
y	+	0	−	0	+	0	−

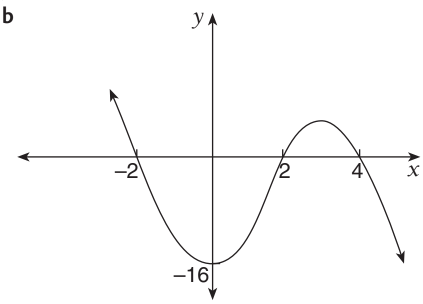
x	−2	−1	0	2	3
y	+	0	−	0	−

2 a 1, 2, 4 **b** 1, −2, 4 **c** 2, −5 **d** 0, −4, 2

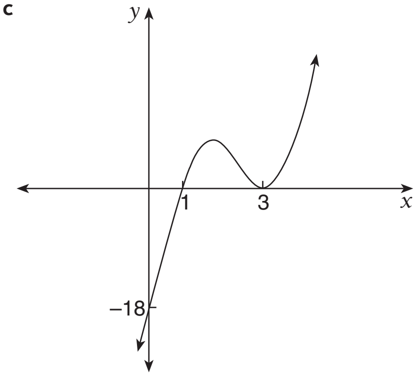
3 a



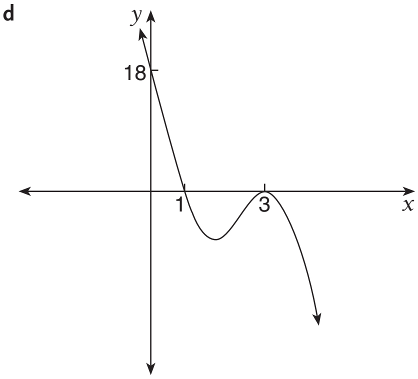
b



c



d

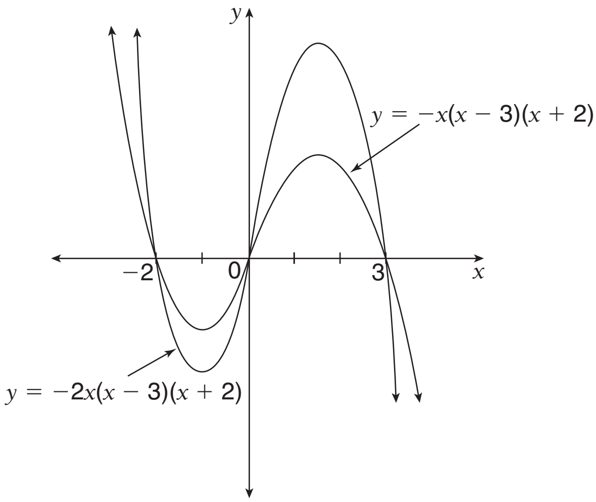


4 **a** 16; −16; −18; 18 **b** Let $x = 0$ to find the y-intercept. Ask yourself whether the graph could have this y-intercept.

5 **a** $y = -(x + 1)(x - 1)(x - 3)$ **b** $y = (x + 2)(x - 1)(x - 3)$ **c** $y = -(x + 4)(x + 2)(x - 2)$
d $y = -(x + 2)(x - 2)^2$ **e** $y = (x + 2)(x - 2)^2$ **f** $y = (x - 1)(x - 3)^2$

6 You would need to know another point that the curve passes through.

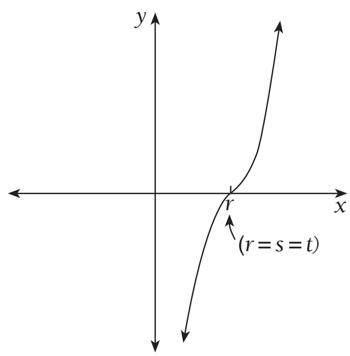
7 **a** **i** B **ii** C **iii** A
b



8 **a** $y = (x - 1)(x + 1)(x - 4)$
b $y = (x + 4)(x + 2)(x - 1)$
c $y = -x(x + 2)(x - 3)$
d $y = x(-x + 2)(x + 3)$

9 **a** $y = 2(x + 1)(x - 1)(x - 4)$
b $y = -2(x + 2)(x - 2)^2$
or $y = -\frac{2}{5}(x - 2)(x + 2)^2$

10



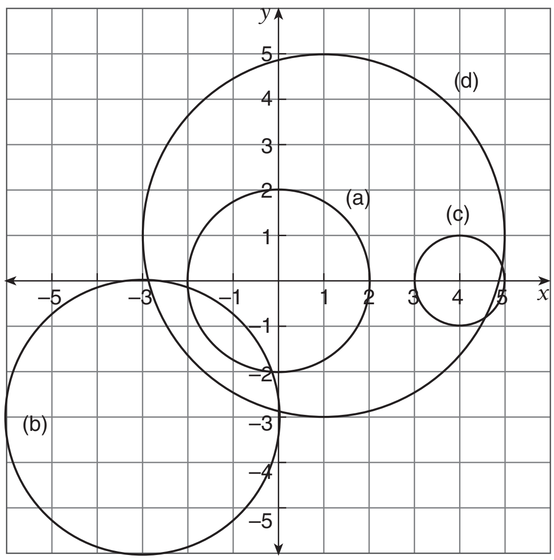
Exercise 6:04

1 **a** $(x - 1)^2 + (y - 1)^2 = 49$ **b** $(x - 5)^2 + y^2 = 4$ **c** $(x + 3)^2 + (y + 5)^2 = 16$ **d** $(x - 2)^2 + (y + 5)^2 = 1$
e $x^2 + (y - 2)^2 = \frac{1}{4}$ **f** $x^2 + y^2 = 9$

2 **a** (2, 3); 8 **b** (−4, 1); 2 **c** (−3, −3); 3 **d** (6, 5); 10 **e** (0, −5); 4 **f** (3, 0); 1
g (0, 0); 9 **h** (0, 0); 7 **i** (0, 0); $\sqrt{11}$ **j** (7, 8); $\sqrt{2}$

3 **a** (5, −4); 3 **b** (−4, 7); 10 **c** (9, 10); 11 **d** $(4\frac{1}{2}, 0)$; $\sqrt{7}$

4



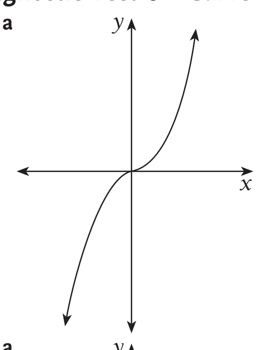
5 a x-intercepts: $\pm\sqrt{5}$
y-intercepts: $-1, 5$
b x-intercepts: $2 \pm \sqrt{15}$
y-intercepts: $1 \pm \sqrt{12}$
6 a $(x-3)^2 + (y-4)^2 = 25$
b $4x^2 + 4y^2 = 8$
7 a $(x-2)^2 + y^2 = 4$
b $(x+2)^2 + y^2 = 4$
c $x^2 + (y-2)^2 = 4$
d $x^2 + (y+2)^2 = 4$
8 $(x-4)^2 + (y+3)^2 = 4$

Exercise 6:05

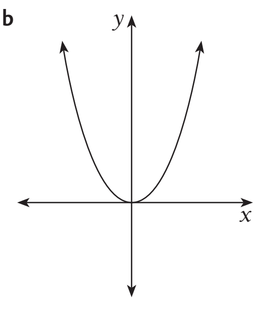
1 a (–1, 1), (2, 4) **b** (–2, 4), (2, 4) **c** (–2, 4), (1, 1) **d** (–8, –1), (1, 8) **e** (–6, –1), (2, 3) **f** (–1, –3), (1, 3)
2 a (–2, 4), (3, 9) **b** (2, 4) **c** (–1, 1), (2, 4) **d** (–3, 9), (1, 1)
3 a (–4, 3), (4, –3) **b** (3, 4) **c** (4·7, –1·8)
4 a A (–0·3, –0·7), B (2·3, 4·7) **b** A (–0·5, 0·2), B (1·2, 1·5)
5 and 6 a (0, 0), (2, 4) **b** (–1, 1), (3, 9) **c** (1, 1), (–1, –1)
d $(1+\sqrt{2}, -1+\sqrt{2}), (1-\sqrt{2}, -1-\sqrt{2})$ **e** $(\sqrt{2}, \sqrt{2}), (-\sqrt{2}, -\sqrt{2})$ **f** (0, –2), (1·6, 1·2)
7 a (2, 3), (–1, 0) **b** $(2+\sqrt{2}, 2-\sqrt{2}), (2-\sqrt{2}, 2+\sqrt{2})$
8 a A $\left(\frac{10-\sqrt{180}}{10}, \frac{10-\sqrt{180}}{5}\right)$; B $\left(\frac{10+\sqrt{180}}{10}, \frac{10+\sqrt{180}}{5}\right)$ **b** A $\left(\frac{3-\sqrt{41}}{8}, \frac{50-6\sqrt{41}}{64}\right)$; B $\left(\frac{3+\sqrt{41}}{8}, \frac{50+6\sqrt{41}}{64}\right)$

Diagnostic Test 6 Curve sketching

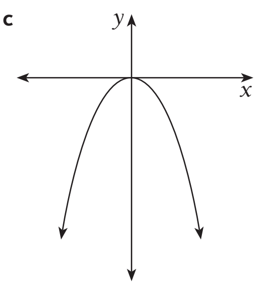
1 a



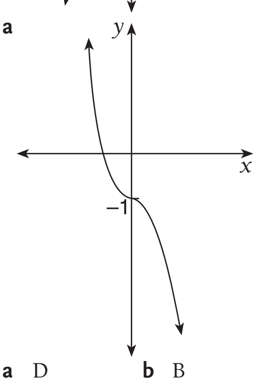
b



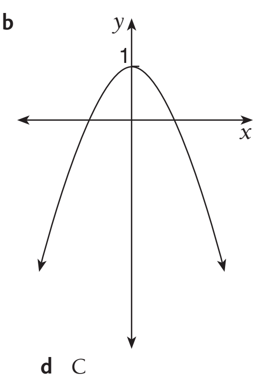
c



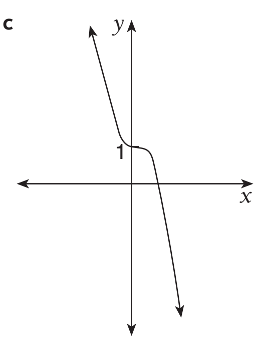
2 a



b

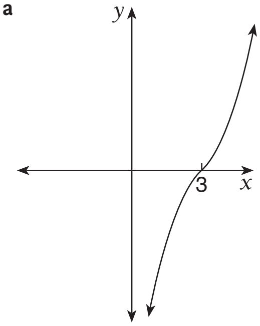


c

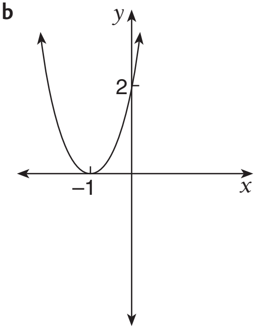


3 a D **b** B **c** A **d** C

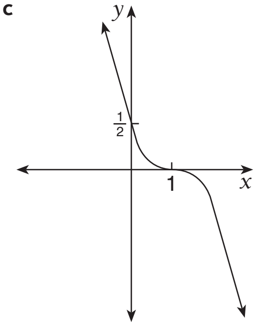
4 a



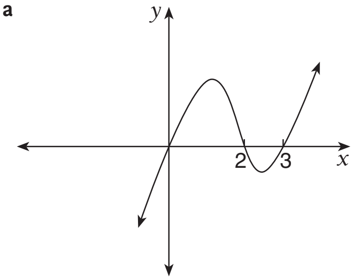
b



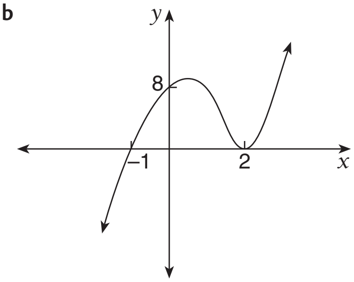
c



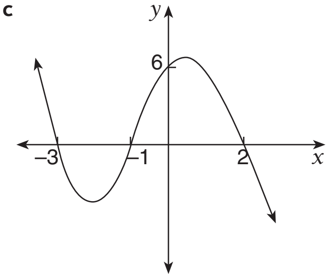
5 a



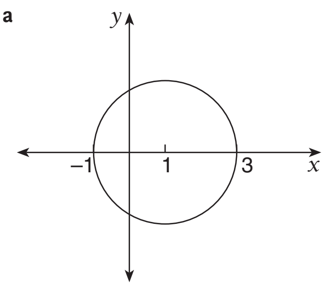
b



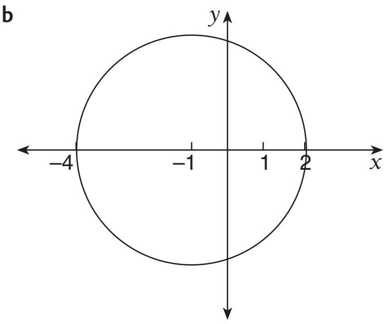
c



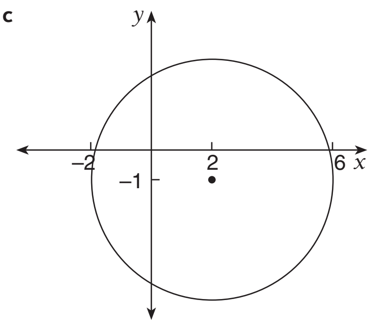
6 a



b



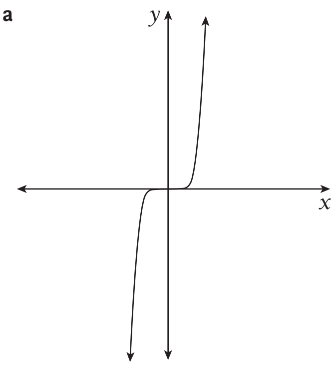
c



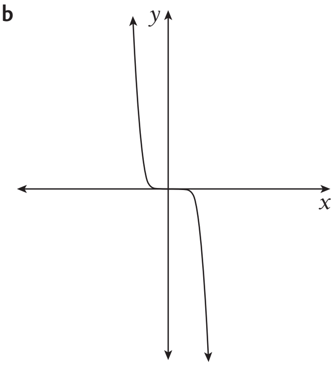
7 a (2, 4) and (−4, −2) **b** no points of intersection **c** (1, 8) and (4, 2)
8 a (3, 9) and (−4, 16) **b** (1, −2) and (2, −1) **c** (1, 3) and (−3, −1)

6A Revision Assignment

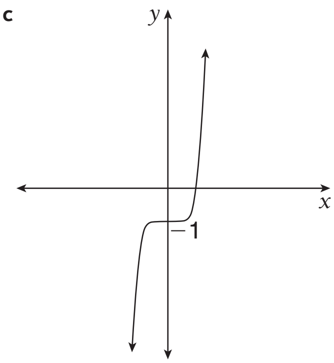
1 a



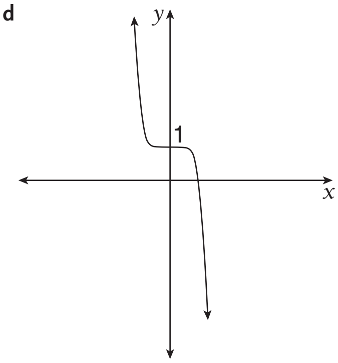
b



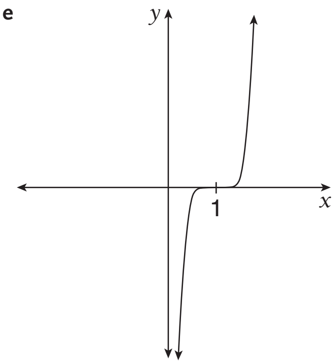
c



d



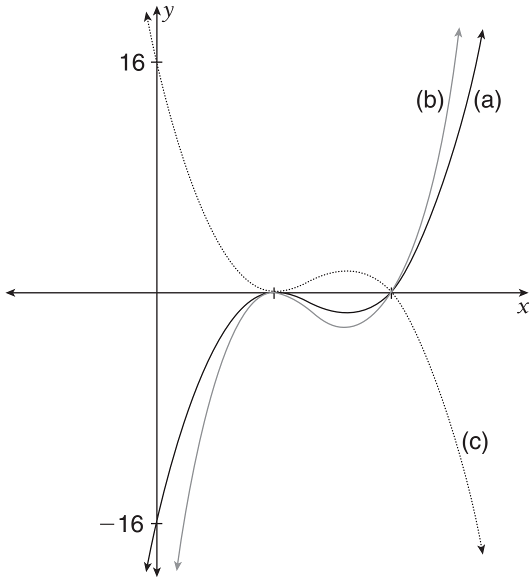
e



2 **a** $y = -2x^8 + 2$ **b** $y = -2x^8 - 2$
c $y = -2(x + 2)^8$ **d** $y = -2(x - 2)^8$

3 **a** **i** $y = -x^5$ **ii** $y = -x^5$
b **i** $y = -x^5 + 2$ **ii** $y = -x^5 - 2$
c **i** $y = -\frac{1}{2}x^6$ **ii** $y = \frac{1}{2}x^6$
d **i** $y = 3(x - 2)^3$ **ii** $y = 3(x + 2)^3$

4



5 **a** $(x + 3)^2 + y^2 = 9$
b $(x - 3)^2 + y^2 = 36$

6 **a** (−1, 1) and (3, 9)
b (2·1, 4·5) and (−2·8, 8·2)
c (2·2, 4·9) and (−2·7, 7·4)

6B Working Mathematically

1 Sketch the curve by looking at the equation $y = ax^n + d$ in three stages. First, consider the curve $y = x^n$. If n was even, it would mean that the curve was like a parabola. If it was odd, it would be like a cubic.
Then look at a. The size of a only affects the steepness of the curve and not the basic shape. The sign of a would indicate whether the curve was the same shape as the curve $y = x^n$ or whether it was upside down. A positive value of a would mean it was the same shape, while a negative value would mean it was upside down.
Finally, the value of d would indicate how far the curve $y = ax^n$ had been translated to produce the curve $y = ax^n + d$. A positive value of d would mean it had been translated d units up, while a negative value would mean it had been translated d units down.

2

$y = -4x$	This is a straight line with a negative slope. It passes through the origin.
$y = 4x^2$	This is a parabola with its vertex at (0, 0). It looks like $y = x^2$ but it is steeper than it.
$y = -4x^2 - 4$	This is an upside-down parabola with its vertex at (0, −4).
$y = (x - 4)^3$	This is the curve that is produced by translating the curve $y = x^3$ four units to the right.
$y = 4^x$	This is an exponential curve. It has a y-intercept of (0, 1).
$y = x^4$	This is shaped like the parabola $y = x^2$.
$x^2 + y^2 = 4$	This is a circle with its centre at (0, 0) and a radius of two units.

3 50 m², 10 m by 5 m **4** $D = \dfrac{C}{\pi} - 2d$ **5** 26 × 26 × 10 × 10 × 26 × 26 = 45 697 600

6 10 full laps = 3000 m; 15 (half laps, *ABNM*) = 3600 m; 15 (half laps, *AOPD*) = 3150 m
Clearly 10 full laps was shorter than 15 half-laps and hence a better deal for the team members.

Chapter 7: Polynomials

Exercise 7:01

1 a C **b** A **c** B **d** A **e** C **2** b, c, g, h, i

3 a 3, 2, 3 **b** 5, 1, −2 **c** 4, 9, 0 **d** 1, −2, 5 **e** 6, 7, 0 **f** 2, 1, 5 **g** 6, −1, 9 **h** 0, 3, 3
i 5, −2, 0 **j** 2, $\sqrt{3}$, 1 **k** 4, $\frac{3}{2}$, 0 **l** 4, $\frac{1}{3}$, $\frac{1}{5}$

4 a 15 **b** 3 **c** 4 **d** 8

5 a **i** 3 **ii** 18 **iii** 4 **b** **i** −3 **ii** 807 **iii** −21 **c** **i** 62 **ii** 15 506 **iii** 8

6 a $4x^2 + 12x + 9$; 2, $4x^2$, 9 **b** $3x + 3$; 1, $3x$, 3 **c** $12x$; 1, $12x$, 0 **d** $3x^4 - x^2 + 2x$; 4, $3x^4$, 0
e $9x^2 - 36x^3 + 36x^4$; 4, $36x^4$, 0 **f** $12x$; 1, $12x$, 0 **g** $x^3 + 6x^2 + 11x + 6$; 3, x^3, 6
h $36x^3 - 15x^2 - 2x + 1$; 3, $36x^3$, 1

7 a $P(x) = 2x^2 + 3$ **b** $P(x) = 2x^3 + 2x^2 + 2x$ **c** $P(x) = x^3 + x + 4$ or $x^3 + \frac{1}{2}x^2 + 4$

Prep Quiz 7:02

1 2 **2** 3 **3** 4 **4** 3 **5** 1 **6** 1 **7** 5 **8** 0 **9** 6 **10** B, C

Exercise 7:02

1 a $x^3 + 3x^2 - x - 1$ **b** $2x^3 + 2x^2 - x$ **c** $x^3 + x^2 + 6x - 3$ **d** $x^3 + x^2 - 7x + 3$
e $2x^2 - 7x + 2$ **f** $-x^3 + x^2 - 1$ **g** $-2x^2 + 7x - 2$ **h** $2x^3 + 3x^2 + 2x - 2$
i $3x^2 - 4x$ **j** $2x^3 + x^2 - 4x + 2$ **k** $x^2 - 10x + 4$ **l** $-x^2 + 10x - 4$

2 a $x^3 - 3x + 10$ **b** $2x^3$ **c** $3x^4 - x^3 + x^2 + x$ **d** $x^4 + 3x^3 + 2x^2 + 3x + 2$
e $x^3 + 6x^2 - 7x + 7$ **f** $x^2 - 2x + 2$ **g** $x^3 + 3x^2 - 11$ **h** $4x^4 + x^3 - 7x^2 + 4x - 6$
i $8x^2 - 2x + 5$ **j** $3x^3 - 4x + 12$ **k** $3x^3 - x^2 - 2x + 11$ **l** $6x^5 - 2x^4 - x^3 + x^2 + 7x + 12$

3 a 3, x^3 **b** 4, $4x^4$ **c** 4, $5x^4$ **d** 2, $3x^2$ **e** 5, $2x^5$ **f** 1, x

4 a 3 **b** 2 **c** 4 **d** 2 **e** 5 **f** 2

5 a $A(x) = x^3 + 2x^2 + x + 4$, $B(x) = x^3 - 3x - 2$ **b** 22, 0, 11, 11 **c** 4, 0, 2, 2

Exercise 7:03

1 a $x^3 + 5x^2 + 7x + 2$ **b** $x^3 + 2x^2 - 5x + 12$ **c** $x^3 + 3x^2 - 14x + 8$ **d** $2x^3 - 5x^2 + 12x - 5$
e $3x^4 - 2x^3 - 3x^2 + 5x - 2$ **f** $-x^4 + 7x^3 - 10x^2 - x + 5$

3 a 4 **b** 2

4 a $x^4 + 6x^3 + 9x^2 + 3x - 4$ **b** $P(1) = 3$; $Q(1) = 5$; $R(1) = 15$ **c** $P(-2) = -3$; $Q(-2) = 2$; $R(-2) = -6$
d $R(a) = P(a) \,.\, Q(a)$

5 a $(x + 1)(x + 1) - 4$ **b** $(x - 3)(x - 2) - 5$ **c** $(x + 3)(3x - 8) + 22$ **d** $(x - 5)(2x + 6) + 33$
e $(x + 1)(x^2 + x) - 7$ **f** $(x - 3)(3x^2 + 9x + 25) + 81$ **g** $(x - 2)(2x^2 + 3x + 7) + 13$
h $(x - 1)(3x^2 + 3x + 5) + 0$

6 a $6x + 17$; 53 **b** $x^2 - 4x + 9$; −17 **c** $x^2 - x + 2$; 1 **d** $3x^2 + 2x - 1$; 0
e $x^4 - 2x^3 + 4x^2 - 5x + 6$; − 3 **f** $2x^5 + 2x^4 + 4x^3 + 4x^2 + x + 1$; 5

7 a $(x + 1)(x^2 + x + 2)$ **b** $(2x + 1)(x^2 - 2x + 3)$ **c** $(x - 1)(2x^2 - x - 1)$ **d** $(x + 1)(2x^2 - x + 1)$

Exercise 7:04

1 a 3 **b** 0 **c** 6 **d** 18 **e** 16 **f** 0

2 a 13 **b** 15 **c** 10 **d** 40 **e** −4 **f** 36 **g** 13 **h** −1

3 a 3 **b** −7 **c** 71 **d** 5 **4 a** 30 **b** 2 **c** 2 or −3

5 a yes **b** no **c** no **d** yes **e** yes **f** yes

6 a $(x - 1)(x + 2)(x + 3)$ **b** $(x + 1)(x - 4)(x - 3)$ **c** $(x + 1)(x + 1)(x + 2)$ **d** $(x - 2)(x - 2)(x - 1)$

7 a $(x - 1)(x - 3)(x + 2)$ **b** $(x - 2)(x - 7)(x + 3)$ **c** $(x + 1)(x - 2)(x - 3)$ **d** $(x - 1)(x + 5)(x - 4)$
e $(x + 1)(x + 6)(x - 3)$ **f** $(x + 1)(2 - 5x)(1 + 3x)$

8 a 4 **b** −10 **c** −6 **d** 4

9 $a = -2$, $b = -13$ **10** $p = -7$, $q = -6$

Exercise 7:05

1 the factors of 12, namely 1, 2, 3, 4, 6, 12 and their opposites (ie −1, −2, −3, −4, −6, −12)

2 Divide $x^3 - 2x^2 - 11x + 12$ by $x - 3$. **3 a** $(x - 4)(x + 3)$ **b** $(2x - 1)(x - 2)$

4 a $x = 3, 2, -1$ **b** $x = -1, 4$ **c** $x = \frac{1}{2}, -4, 1, -2$ **d** $x = 2, -3$

5 a $(x - 2)(x - 4)(x + 2) = 0$; $x = 2, 4, -2$ **b** $(p + 3)(2p - 1)(p + 3) = 0$; $p = -3, \frac{1}{2}$
c $(x - 1)(x - 1)(x + 2)(x + 3) = 0$; $x = 1, -2, -3$ **d** $(p + 2)(p + 2)(p - 3)(2p + 1) = 0$; $p = -2, 3, -\frac{1}{2}$

6 a Substitute $x = -1$ and show that it satisfies the equation. **b** $2x^3 - 9x^2 + 7x + 6$
c Show that $Q(2) = 0$. **d** $2x^4 - 7x^3 - 2x^2 + 13x + 6 = (x + 1)(x - 2)(2x + 1)(x - 3)$ **e** $-1, 2, -\frac{1}{2}, 3$

7 a −1, 2, 3 **b** −2 **c** −1, 1 **d** −2, −3, 3 **e** 1 **f** −3

8 a −1, −2, 3, 4 **b** 1, 4, −2 **c** $x^2(x + 10) = 2000$; 10, 10, 20

Prep Quiz 7:06

1 1 **2** 2 **3** 3 **4** $x^3 + x^2 - 2x$ **5** 0, 1, −2 **6** 0, 1, −2 **7** 0 **8** yes **9** x^4 **10** x^3

Exercise 7:06

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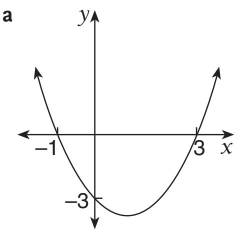

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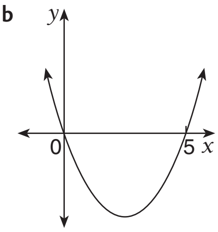

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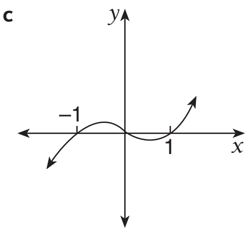

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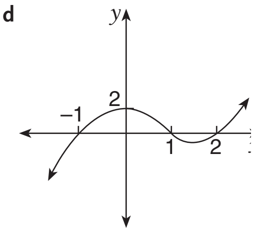

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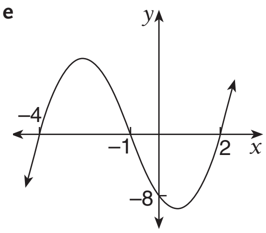

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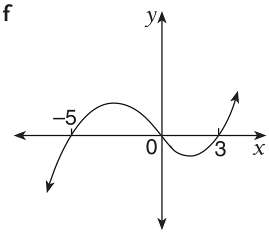

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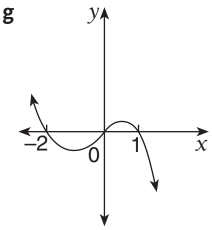

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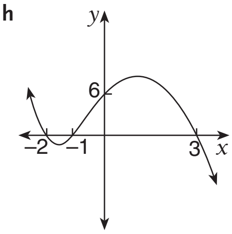

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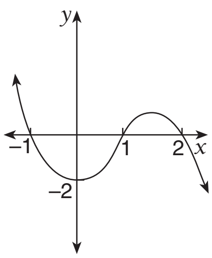

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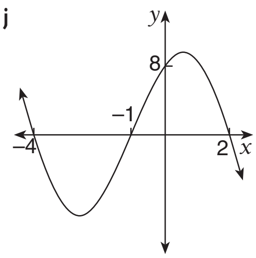

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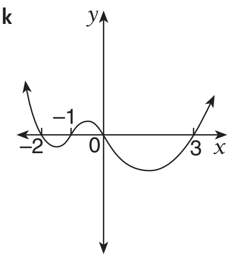

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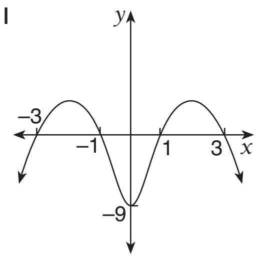

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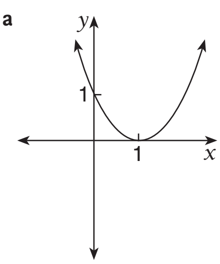

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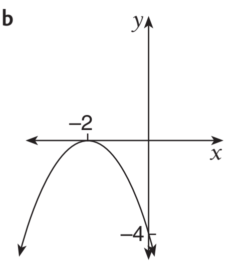

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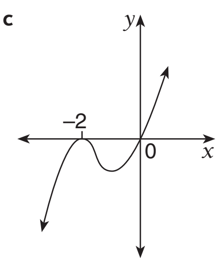

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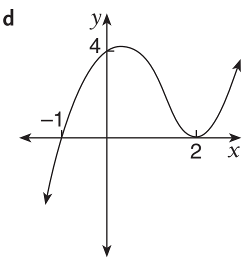

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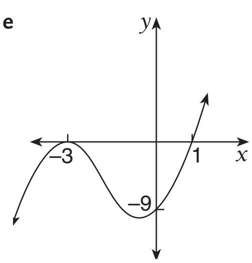

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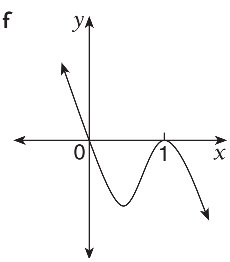

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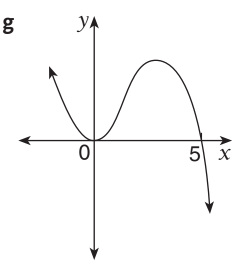

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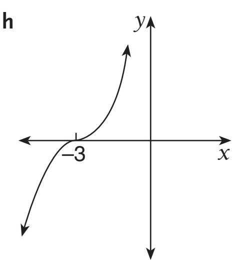

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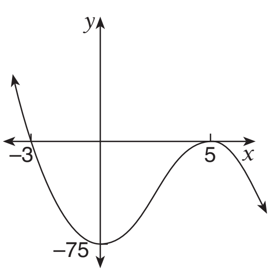

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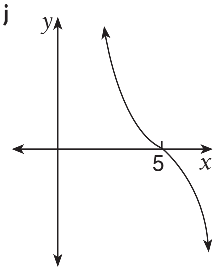

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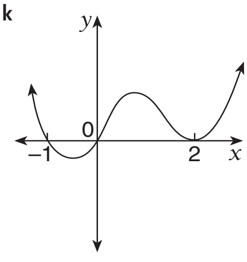

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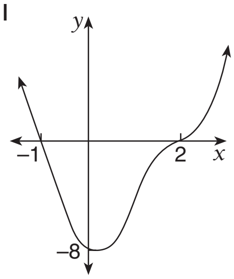

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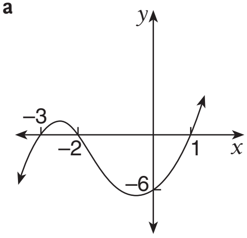

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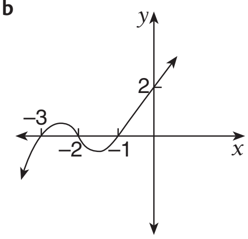

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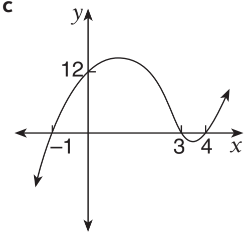

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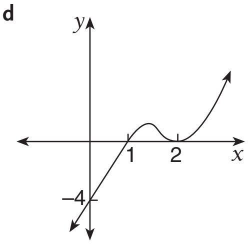

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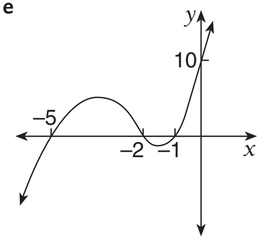

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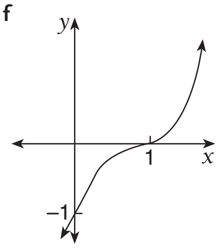

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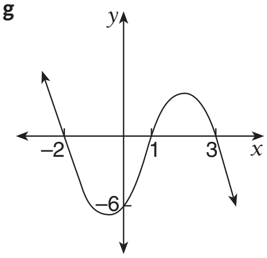

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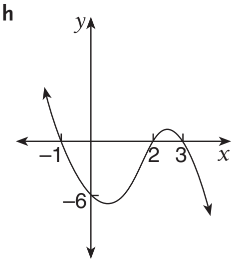

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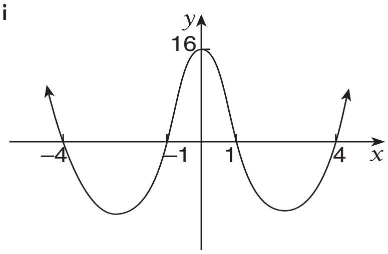

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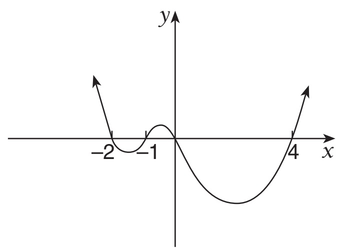


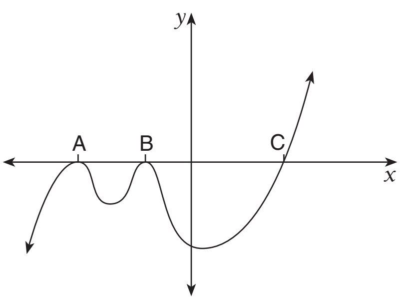
4 a $y = P(x)$ where $P(x) = -(x-2)^2(x+1)$ **b** $y = P(x)$ where $P(x) = -2(x+2)^2(x-1)^3$

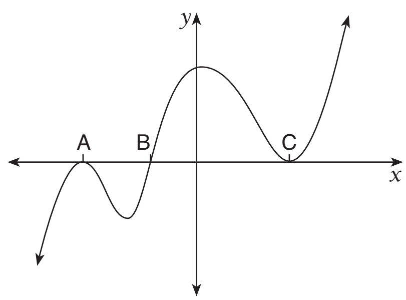
c $y = (x+3)(x+1)(x-4)^2$ or $-(x+3)(x+1)^2(x-4)$ or $-(x+3)^2(x+1)(x-4)$

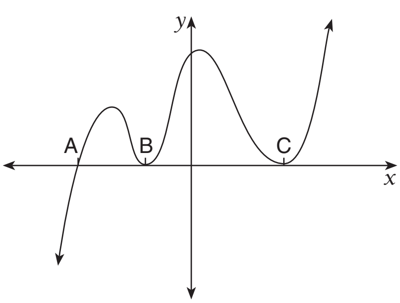
d $y = -(x+3)(x+1)(x-4)^2$ or $(x+3)(x+1)^2(x-4)$ or $(x+3)^2(x+1)(x-4)$

5 no; equations like: $(x-1)^3 = 0$; $(x-1)^2(x+1) = 0$; $(x-1)(x-2)(x-3) = 0$

6



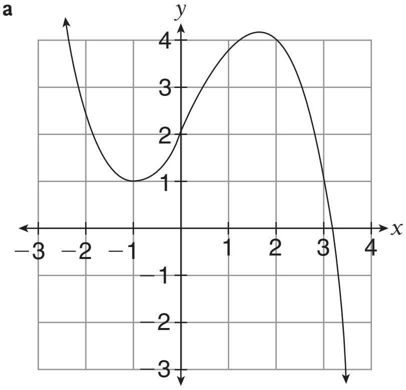




There are three possible shapes. Each curve can be translated horizontally so that the roots occur in different positions.

Exercise 7:07

1 a



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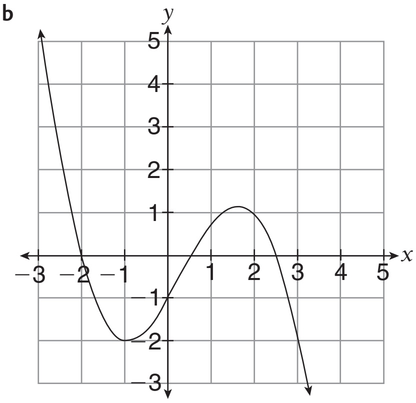
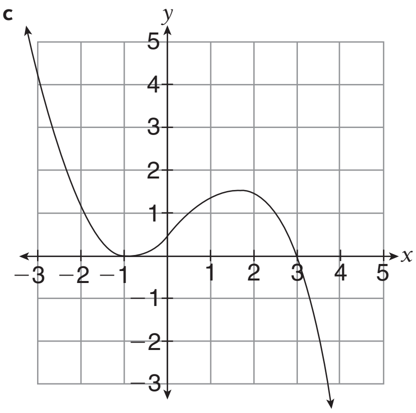
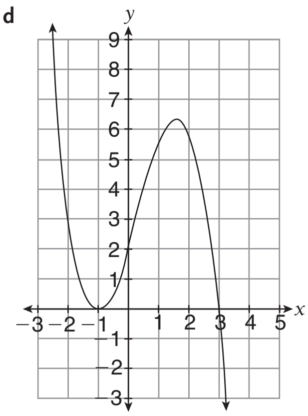


c



d



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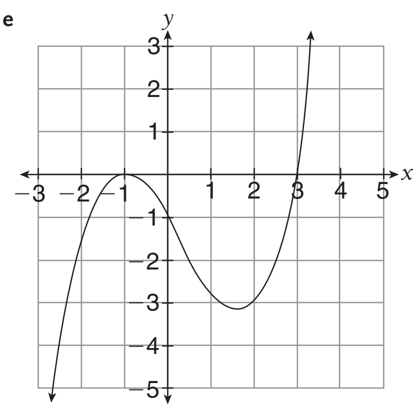

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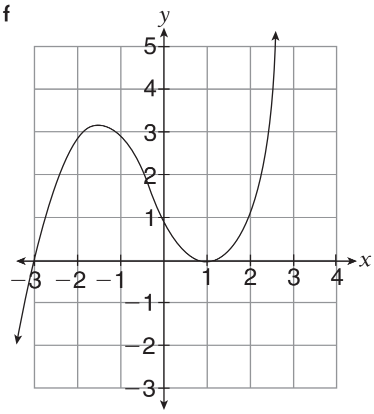

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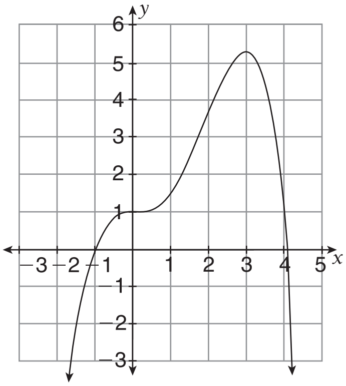

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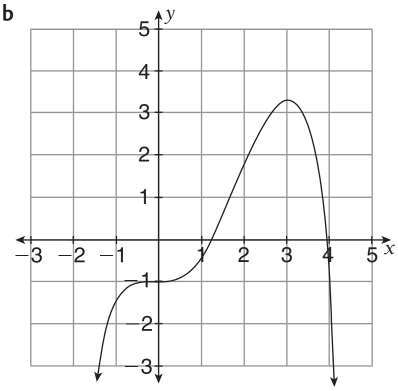

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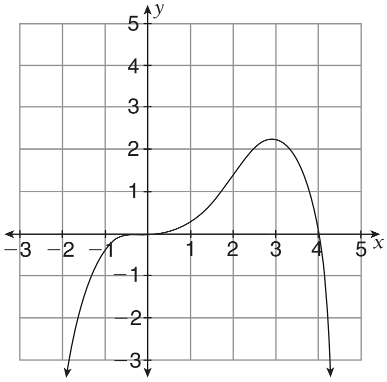

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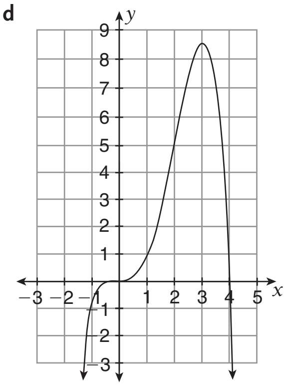

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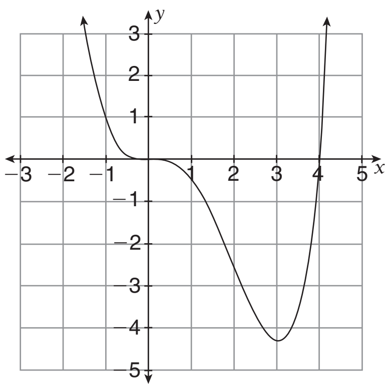

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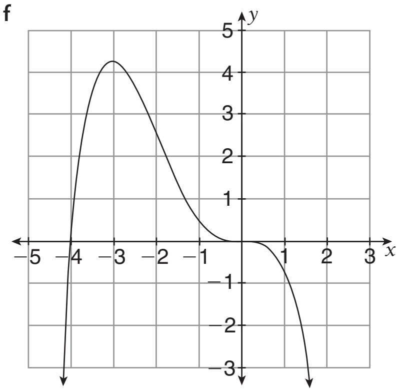

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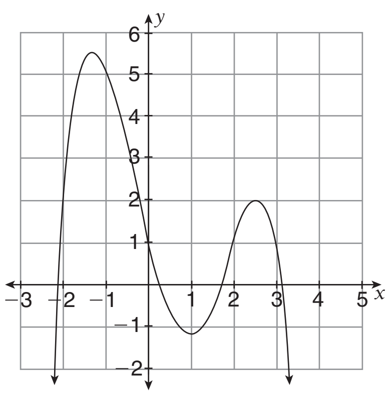

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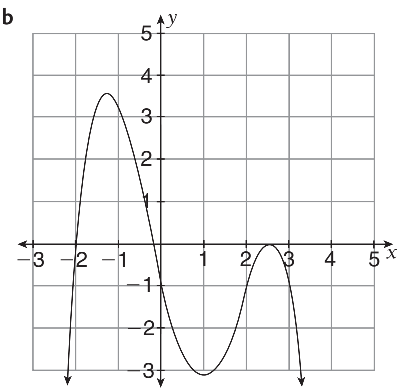

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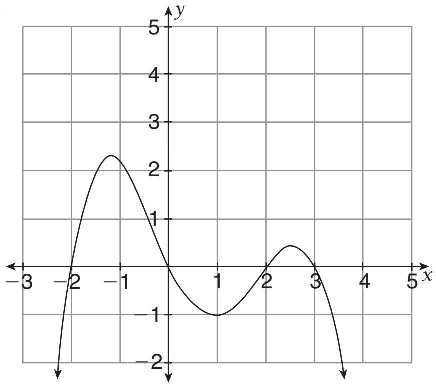
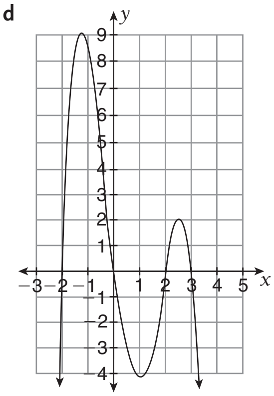

y
5
4
3
2
1
−1
−2
x
−3
−2
−1
1
2
3
4
5

d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9
8
7
6
5
4
3
2
1
−1
−2
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−4
x
−3
−2
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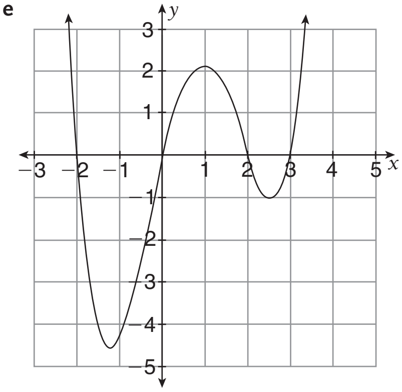
2
3
4
5

e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3
2
1
−1
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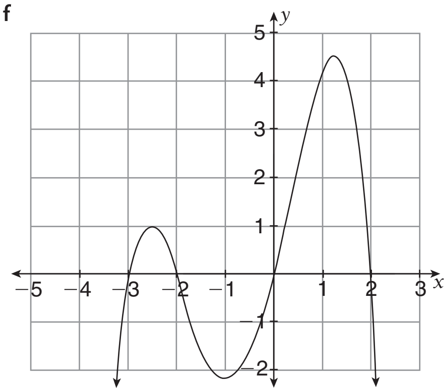
−3
−4
−5
x
−3
−2
−1
1
2
3
4
5

f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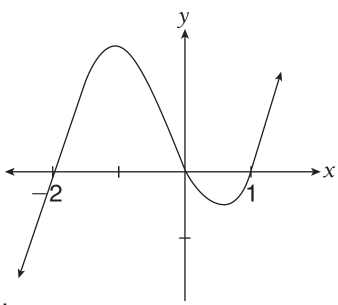
4
3
2
1
−1
−2
x
−5
−4
−3
−2
−1
1
2
3

4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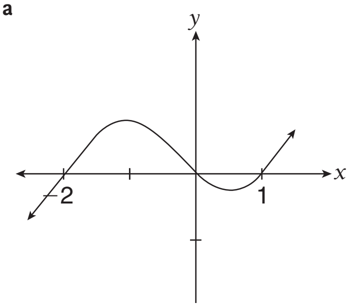
x
−2
1

a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x
−2
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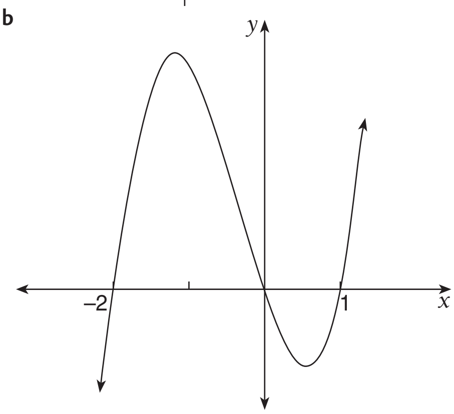
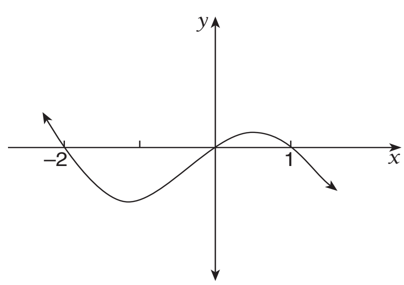
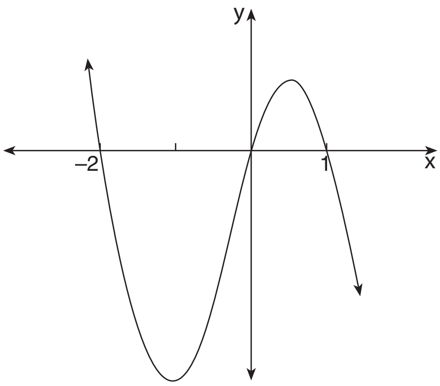
b


y
x
−2
1

c

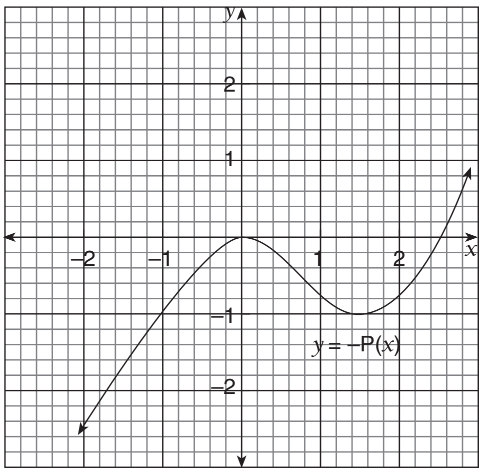

y
x
−2
1

d

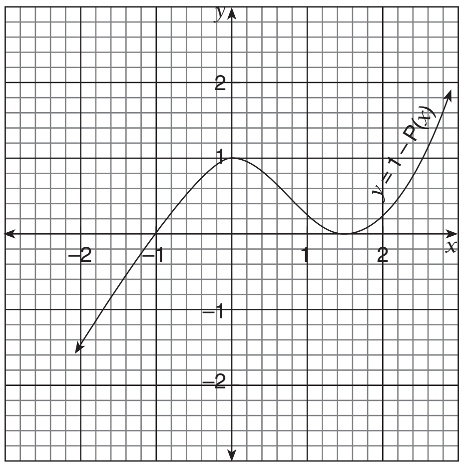


y
x
−2
1

5 a

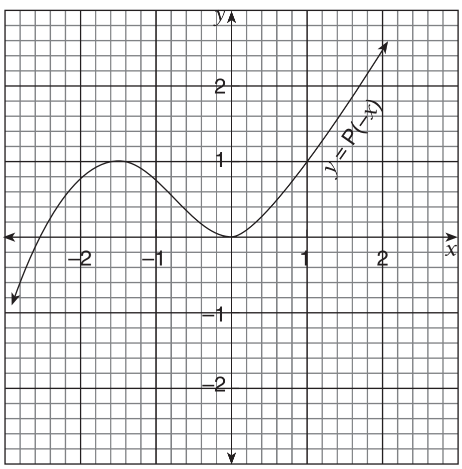


y
x
2
1
−1
−2
−2
−1
1
2
y = −P(x)

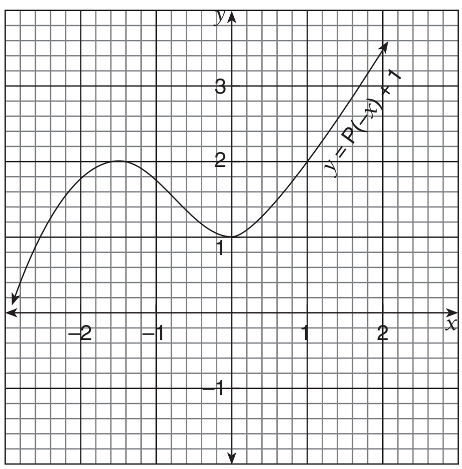


y
x
2
1
−1
−2
−2
−1
1
2
y = 1 − P(x)

5 b

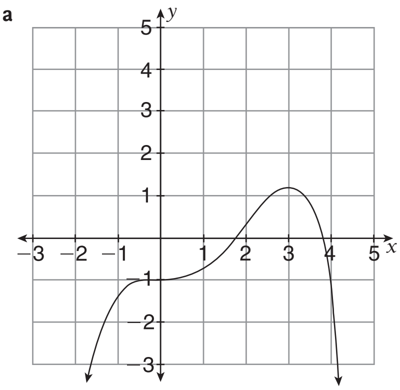


y
x
2
1
−1
−2
−2
−1
1
2
y = P(−x)

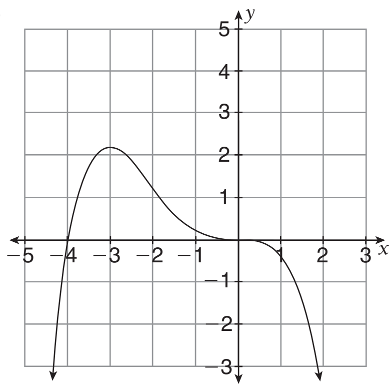


y
x
3
2
1
−1
−2
−1
1
2
y = P(−x) + 1

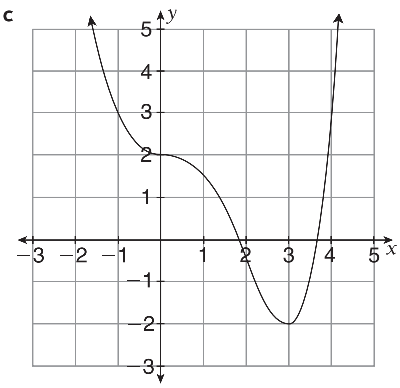
6 a


y
x
5
4
3
2
1
−1
−2
−3
−3
−2
−1
1
2
3
4
5

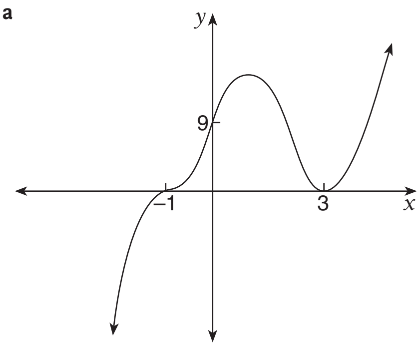
b


y
x
5
4
3
2
1
−1
−2
−3
−5
−4
−3
−2
−1
1
2
3

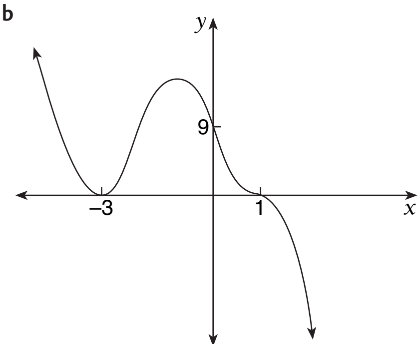
c


y
x
5
4
3
2
1
−1
−2
−3
−3
−2
−1
1
2
3
4
5

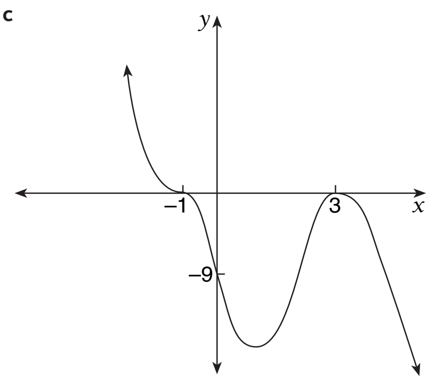
7 a


y
x
9
–1
3

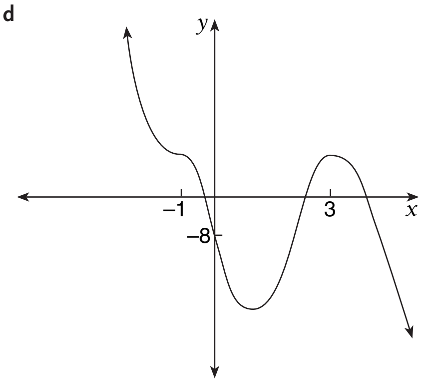
b


y
x
9
–3
1

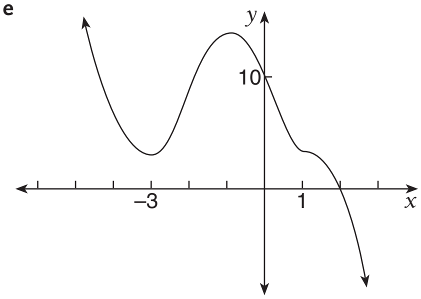
c


y
x
–1
3
–9

d


y
x
–1
3
–8

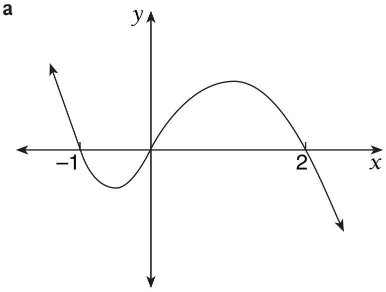
e



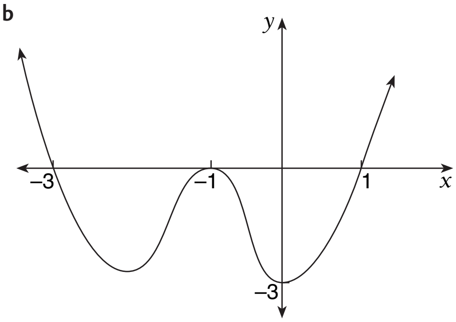
Diagnostic Test 7 Polynomials

1 a 2 **b** 7 **c** 3 **2 a** $-5x^3$ **b** $9x^4$ **c** x^2
3 a B **b** A **c** B **4 a** $2x^3 + 3x^2 - 2x$ **b** $2x^3 + x^2 - 6x$ **c** $2x^3 - x^2 - 8x + 4$
5 a $2x^3 + 7x^2 + 4x - 4$ **b** $2x^4 + 5x^3 - 3x^2 - 8x + 4$ **c** $4x^5 + 8x^4 - 11x^3 - 13x^2 + 16x - 4$
6 a $2x - 1$ **b** $2x^2 - 3x + 1$ **c** $2x^2 + 5x - 1$ **7 a** 9 **b** −16 **c** 226
8 a no **b** yes **c** yes **9 a** 0, 1, −2 **b** −1, 2 **c** −1, 2, 3

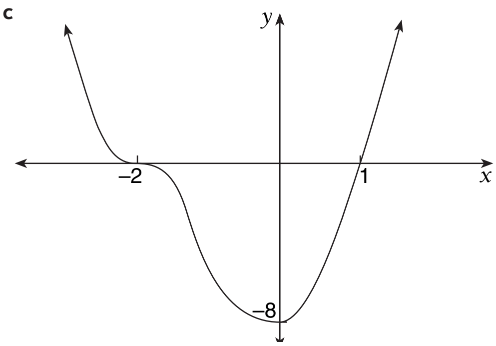
10 a



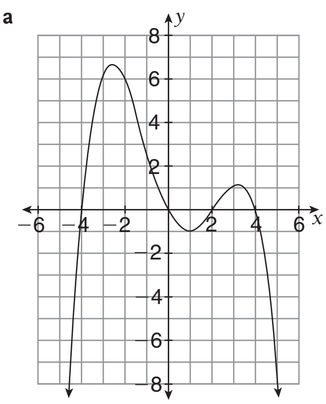
b



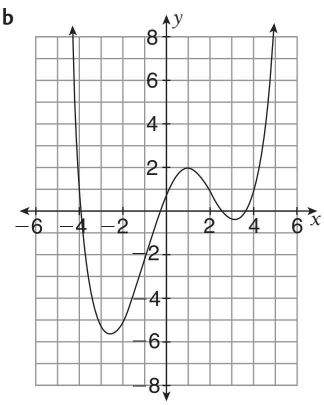
c



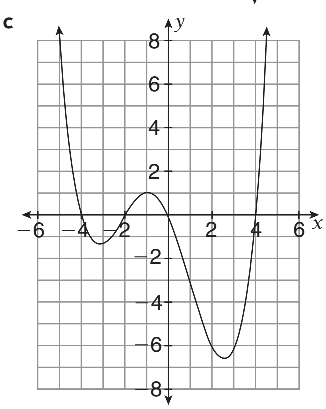
11 a



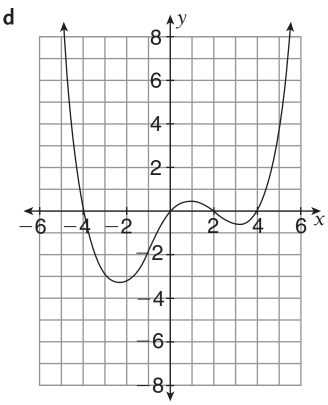
b



c



d



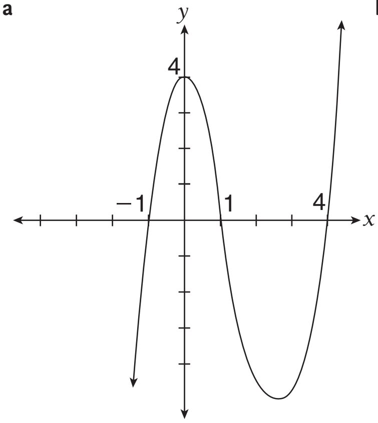
7A Revision Assignment

1 **a** degree = 3, leading term = $-3x^3$, constant term = -3

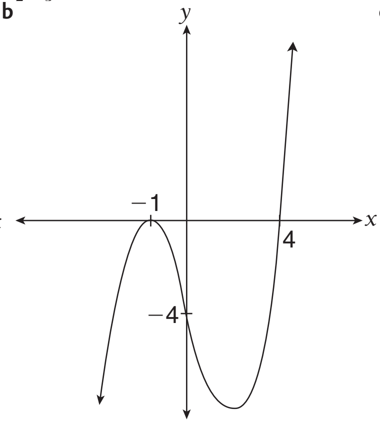
b 0 **c** $a = -\frac{1}{3}$, $b = 2$ **d** 31

2 **a** $x = 0, 3, -4$ **b** $x = 3, -\frac{3}{2}, \frac{1}{3}$ **c** $x = 0, 2$ **d** $x = -2, 2$

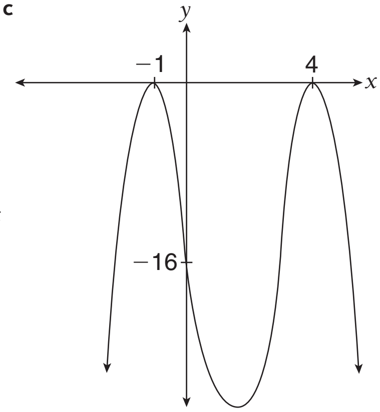
3 **a**



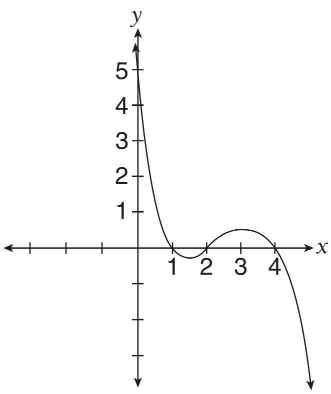
b



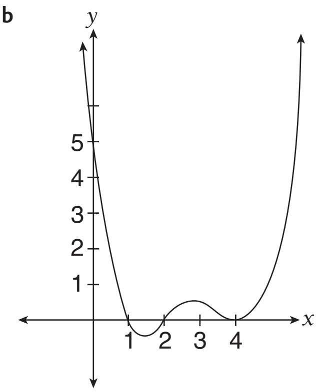
c



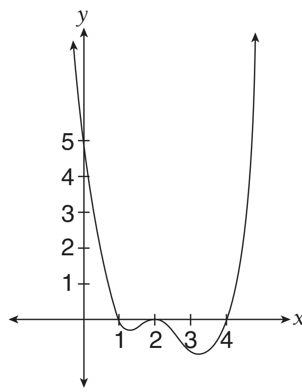
4 **a**



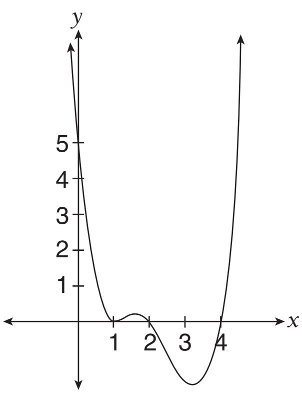
b



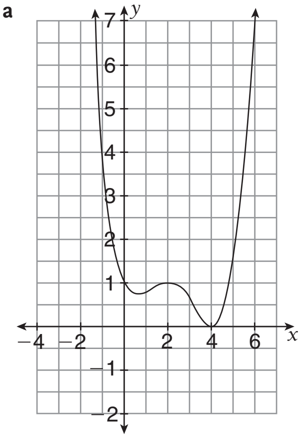
OR



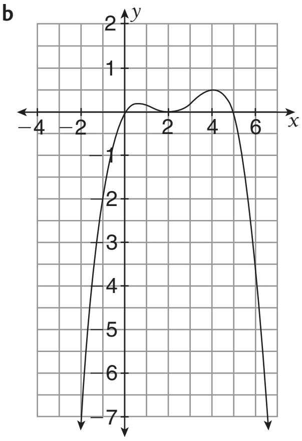
OR



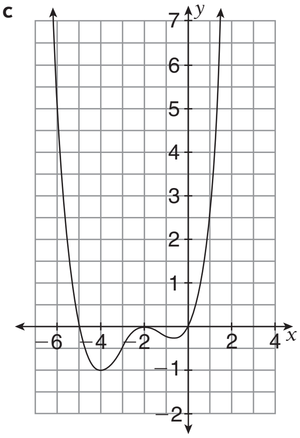
5 a



b



c



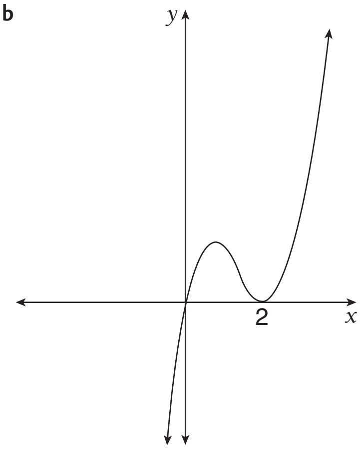
6 a Note: Many other answers are possible.

i $y = x^4 + 1$ **ii** $y = x^4$

iii $y = x^2(x - 1)^2$ **iv** $y = x^2(x - 1)(x + 1)$

v $y = x(x - 1)(x - 2)(x - 3)$

b



i 1 **ii** 3

7B Working Mathematically

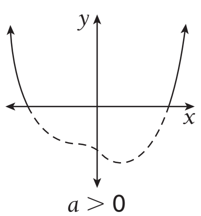
The leading term provides information on how the curve will behave as x becomes very large or very small (ie as x approaches positive or negative infinity). For these values of x, the sign of the polynomial $P(x)$ is the same as the sign of the leading term, ax^n.

If n is even, then ax^n is positive when a is positive and negative when a is negative. This means that the ends of the curve will point in the same direction as shown below.

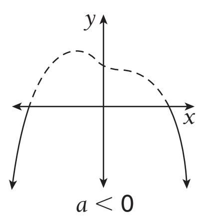
If n is odd and a is positive, then ax^n will be positive when x is positive and negative when x is negative. When a is negative, then ax^n will be negative when x is positive and positive when x is negative.

This means that the ends of the curves will point in opposite directions as shown below.

n is even

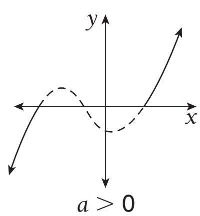


$a > 0$

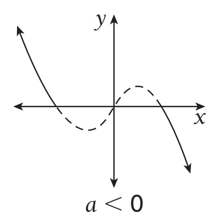


$a < 0$

n is odd



$a > 0$



$a < 0$

2 A is $y = \frac{1}{4}(x - 2)^2 + 1$ or $y = \frac{1}{4}x^2 - x + 2$

B is $y = (x - 2)^2 + 1$ or $y = x^2 - 4x + 5$

C is $y = -\frac{3}{4}(x - 2)^2 + 1$ or $y = -\frac{3}{4}x^2 + 3x - 2$

3 11 **4** $5\frac{1}{4}$ h **5** $\frac{1}{4}$ of the area of each square

6 Rearranging $ax + by + c = 0$ into gradient–intercept form gives $y = -\frac{a}{b}x - \frac{c}{b}$. If the gradient is negative, then $-\frac{a}{b}$ is negative and hence $\frac{a}{b}$ is positive. This means that a and b would be either both positive or both negative.

Chapter 8: Functions and logarithms

Exercise 8:01

1 **a** yes **b** yes **c** yes **d** no **e** yes

2 **a** yes **b** yes **c** no

3 **a** yes **b** no **c** yes

4 **a** –5 **b** –1 **c** –9

5 **a** 3 **b** 2 **c** $1\frac{1}{2}$

6 **a** 2 **b** $2\frac{1}{2}$ **c** $2\frac{1}{12}$

7 **a** 5 **b** –4

8 **a** $4a - 5$ **b** $2a - 3$ **c** $\frac{2}{a} - 5$

9 **a** 8 **b** –4 **c** $W^2 + 4$

10 $2p + h$

11 **a** $f(-x) = (-x)^2 = x^2 = f(x)$ **b** $f(-x) = (-x)^3 = -x^3 = -f(x)$

12 a, b, e, h, i

13 No. If the line is vertical it is not a function.

14 **a** $-3 \leqslant x \leqslant 3, -3 \leqslant y \leqslant 3$ **b** $-2 \leqslant x \leqslant 2, 0 \leqslant y \leqslant 2$ **c** $-2 \leqslant x \leqslant 2, y = 1$ or 2 **d** $-2 \leqslant x \leqslant 2, -2 \leqslant y \leqslant 2$
e $-1 \leqslant x \leqslant 1, 0 \leqslant y \leqslant 1$ **f** $0 \leqslant x \leqslant 9, -3 \leqslant y \leqslant 3$ **g** $-4 \leqslant x \leqslant 4, -4 \leqslant y \leqslant 4$ **h** $-5 \leqslant x \leqslant 5, y = -2$ or 2
i $-2 \leqslant x \leqslant 2, 0 < y \leqslant 1$

15 **a** $y = 4$ **b** $y \leqslant 1$ **c** $y \geqslant -4$

16 **a** x is any real number **b** x is any real number except 0 **c** $x \geqslant 0$

17 **a** x is any real number; y is any real number **b** x is any real number; $y \geqslant 1$
c x is any number except 1; y is any number except 0 **d** x is any real number; $y > 0$
e x is any real number; $y \geqslant 0$

18 **a** **i** the distance fallen in 1·5 seconds **ii** the distance fallen in 1·6 seconds
iii the distance fallen from 1·5 seconds to 1·6 seconds
iv the average speed of the stone in the period from 1·5 to 1·6 seconds
b **i** 2π **ii** $\pi\sqrt{2}$ **iii** the increase in the period when the length is increased from 1 m to 1·5 m

Exercise 8:02

1 Only **a** will have an inverse.

2 **a** yes. {(2, 0), (4, 2), (6, 3), (8, 4)} **b** no

3 **a** $y = \frac{1}{2}x$ **b** $y = \frac{1}{3}(x - 5)$ **c** $y = 3x + 1$

4 **a, c, f** are many-to-one; **b, d, e** are one-to-one.

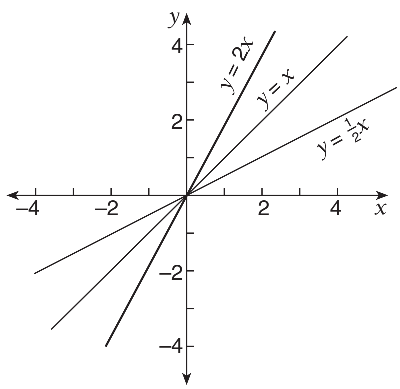
5 Yes, because all straight lines (which are functions) are one-to-one.

6 If any horizontal line drawn through the graph of a function cuts the graph in more than one point, then the function does not have an inverse.

7 **a** $f^{-1}(x) = \frac{1}{4}x$ **b** $f^{-1}(x) = 1 - x$ **c** $f^{-1}(x) = 2 - 3x$

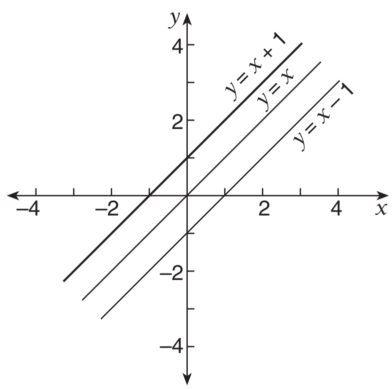
8 **a** $y = x$

b



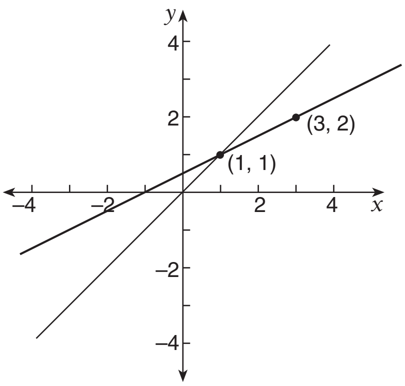
The line $y = 2x$ and $y = \frac{1}{2}x$ are reflections of each other in the line $y = x$.

c



$y = x + 1$ and $y = x - 1$ are reflections of each other in the line $y = x$.

9

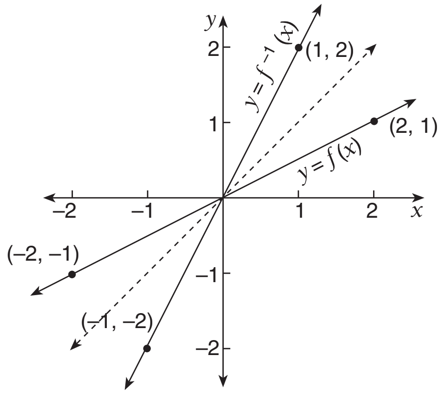


a because they are the points obtained by reversing the x and y coordinates in the points (1, 1) and (3, 2)

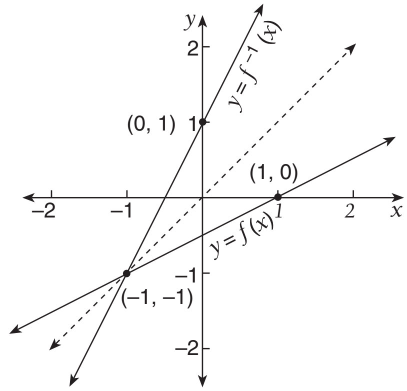
b $y = \frac{1}{2}(x + 1)$; $y = 2x - 1$

c For $y = 2x - 1$, $1 = 2 \times 1 - 1$ and $3 = 2 \times 2 - 1$.
Hence (1, 1) and (2, 3) satisfy the equation $y = 2x - 1$.

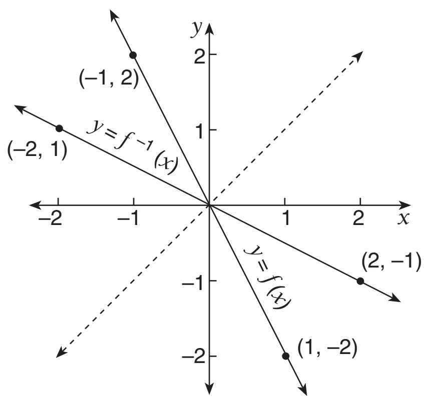
10 a



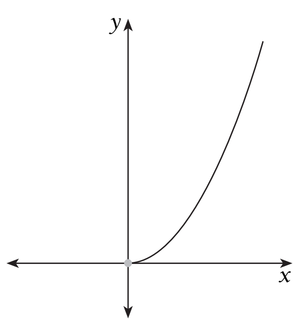
b



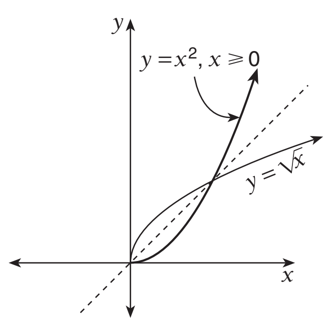
c



11 a

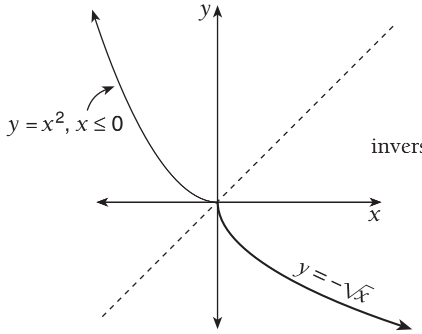


b



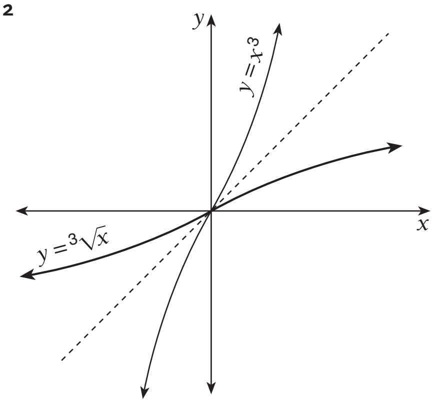
c $y = \sqrt{x}, y \geqslant 0$

d



inverse is $y = -\sqrt{x}$

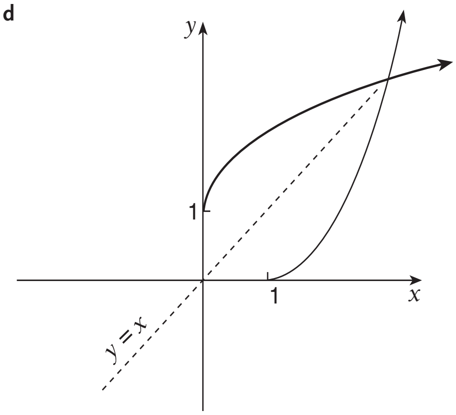
12



Investigation 8:02

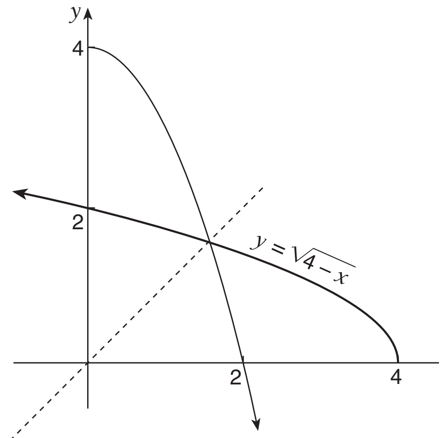
1 a The parabolas are not one-to-one functions.
  b The parabola can be divided into two parts by taking that part where $x \geqslant a$ or the part where $x \leqslant a$.

2 a (1, 0) b $x \geqslant 1$ or $x \leqslant 1$
  c, d



  e $y = 1 + \sqrt{x}, y \geqslant 1$

3 a, b

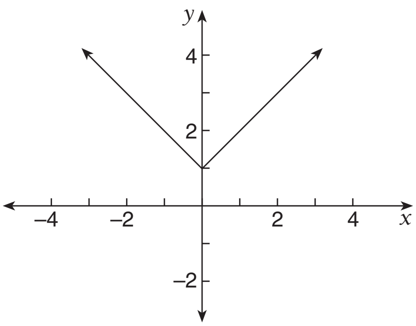


Exercise 8:03

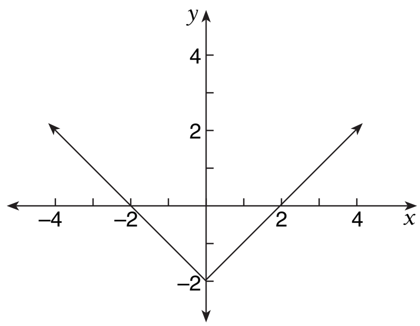
1 a 1 unit to the left b 1 unit up c 1 unit down d 1 unit to the right

2 a $y = 1 + 2^x$ b $y = 2^x - 1$ c $y = 2^{x-1}$ d $y = 2^{x+1}$

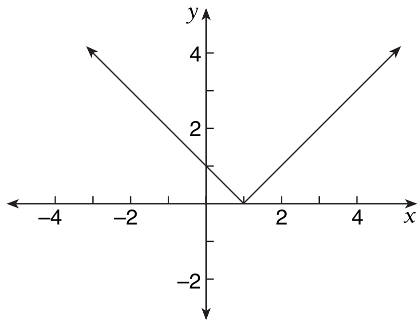
3 a



  b



c



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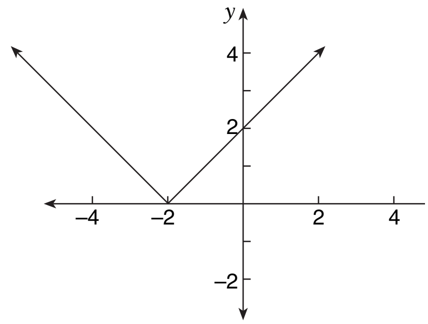
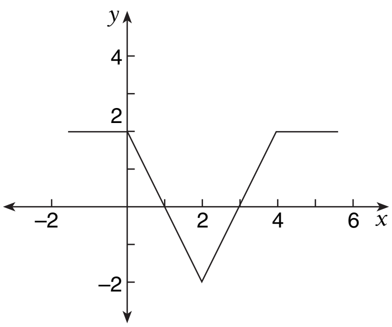
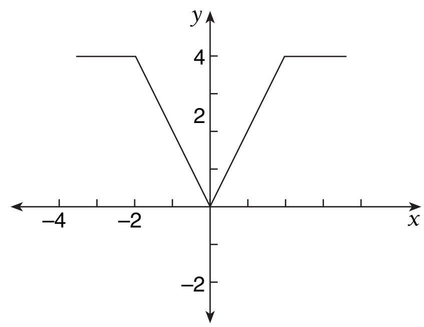


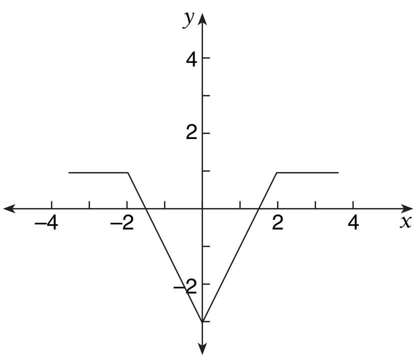
4 a



b



c



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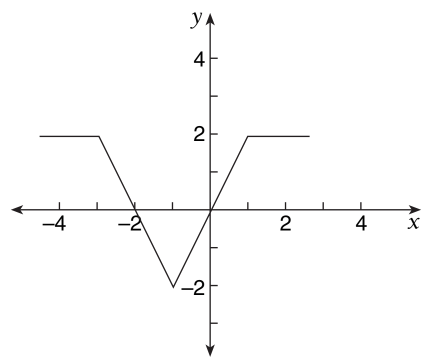
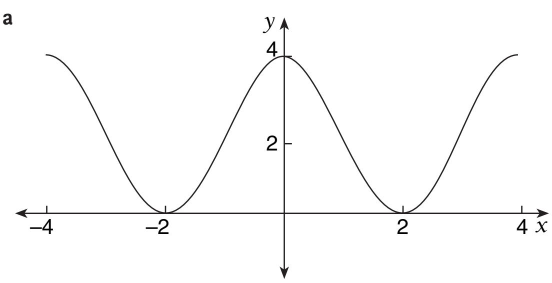


5 a



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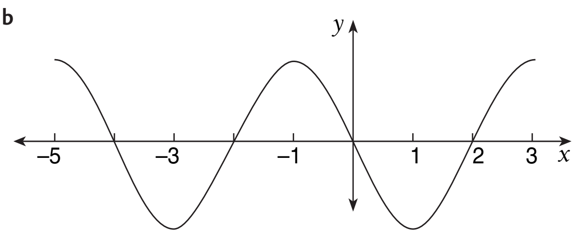

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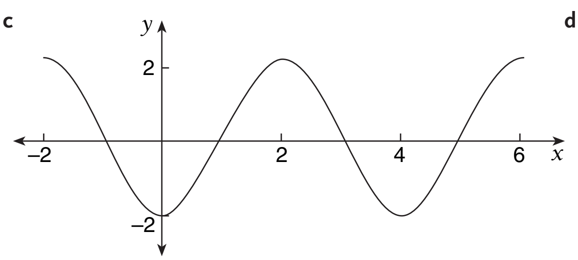

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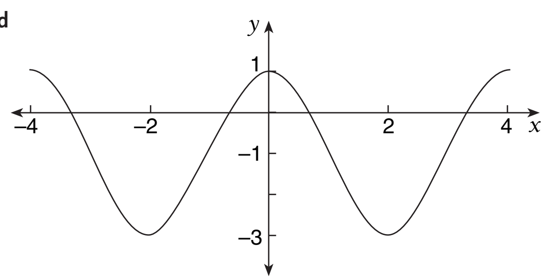
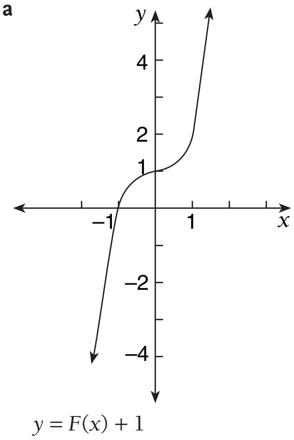
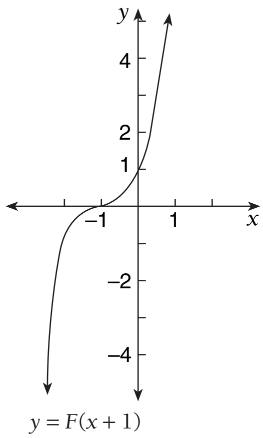
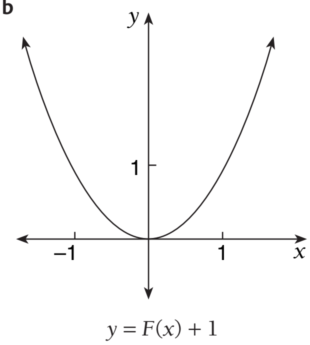


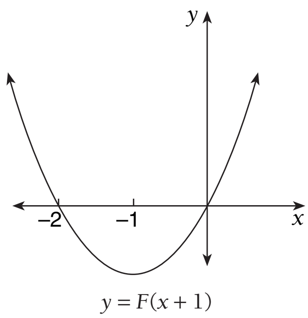
6 a





b





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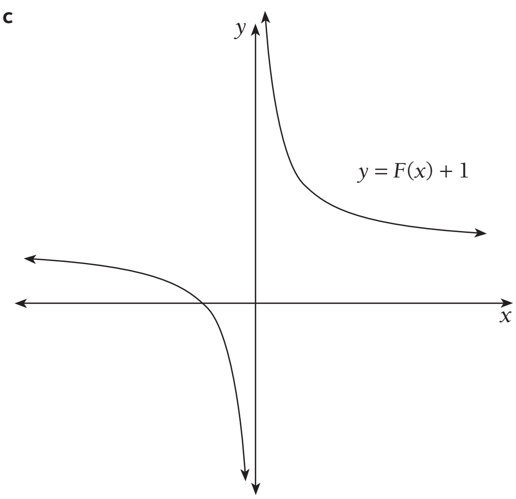
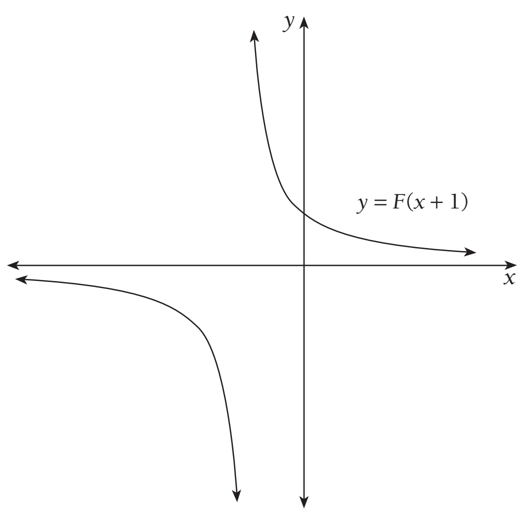




Prep Quiz 8:04

1 64 **2** 64 **3** 64 **4** 2 **5** 3 **6** 6 **7** 2^2 **8** 2^3 **9** 2^{-1} **10** $2^{\frac{1}{2}}$

Exercise 8:04

1 a $\log_2 8 = 3$ **b** $\log_4 16 = 2$ **c** $\log_7 7 = 1$ **d** $\log_2 64 = 6$ **e** $\log_3 81 = 4$ **f** $\log_4 256 = 4$
g $\log_2 32 = 5$ **h** $\log_3 1 = 0$ **i** $\log_2 (\frac{1}{2}) = -1$ **j** $\log_3 (\frac{1}{9}) = -2$ **k** $\log_5 \sqrt{5} = \frac{1}{2}$ **l** $\log_{27} 9 = \frac{2}{3}$
m $\log_{25} 125 = \frac{3}{2}$ **n** $\log_{36} (\frac{1}{6}) = -\frac{1}{2}$ **o** $\log_{16} 8 = \frac{3}{4}$ **p** $\log_8 (\frac{1}{32}) = -\frac{5}{3}$

2 **a** $2^2 = 4$ **b** $3^2 = 9$ **c** $5^0 = 1$ **d** $4^1 = 4$ **e** $10^3 = 1000$ **f** $3^3 = 27$
g $2^4 = 16$ **h** $4^2 = 16$ **i** $7^3 = 243$ **j** $5^4 = 625$ **k** $2^7 = 128$ **l** $6^1 = 6$
m $2^{\frac{1}{2}} = \sqrt{2}$ **n** $3^{-1} = \frac{1}{3}$ **o** $2^{-2} = \frac{1}{4}$ **p** $4^{\frac{3}{2}} = 8$

3 **a** 2 **b** 3 **c** 2 **d** 0 **e** 1 **f** 5 **g** 3 **h** 2
i 3 **j** 0 **k** 4 **l** 4 **m** $\frac{1}{2}$ **n** $\frac{1}{2}$ **o** −1 **p** $\frac{3}{2}$

4 **a** 4 **b** 3 **c** 4 **d** 2 **e** 0 **f** $\frac{1}{2}$ **g** 1 **h** $\frac{1}{2}$
i −1 **j** −2 **k** $\frac{2}{3}$ **l** $\frac{3}{2}$ **m** −1 **n** −2 **o** $\frac{1}{6}$ **p** $\frac{3}{2}$

5 **a** 32 **b** 27 **c** 128 **d** 1 **e** 49 **f** 216 **g** $\frac{1}{3}$ **h** 2
i 27 **j** $\frac{1}{16}$ **k** $\frac{1}{8}$ **l** $\frac{1}{2}$ **m** 125 **n** $\frac{1}{10}$ **o** $\frac{1}{\sqrt{5}}$ **p** 8

6 **a** 2 **b** 3 **c** 5 **d** 3 **e** 5 **f** 10 **g** 2 **h** 3
i 5 **j** 3 **k** $\frac{1}{2}$ **l** 4 **m** 27 **n** 16 **o** 64 **p** 81

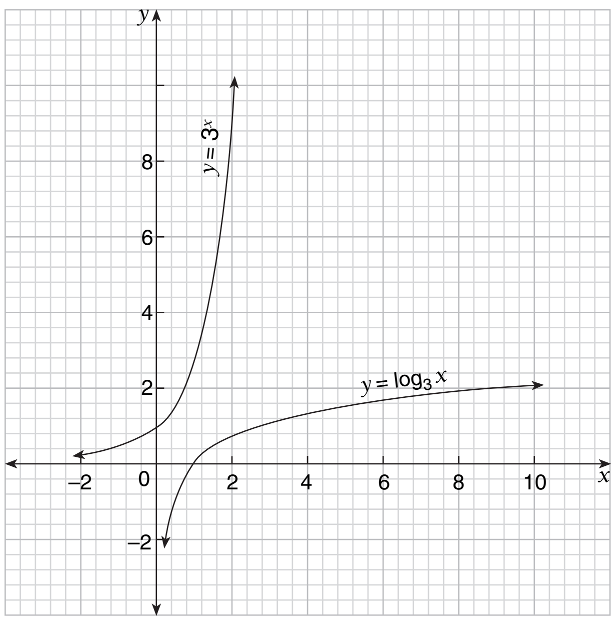
Exercise 8:05

1 $y = 3^x$

x	−2	−1	0	$\frac{1}{2}$	1	$1\frac{1}{2}$	2
y	0·1	0·3	1	1·7	3	5·2	9

$y = \log_3 x$

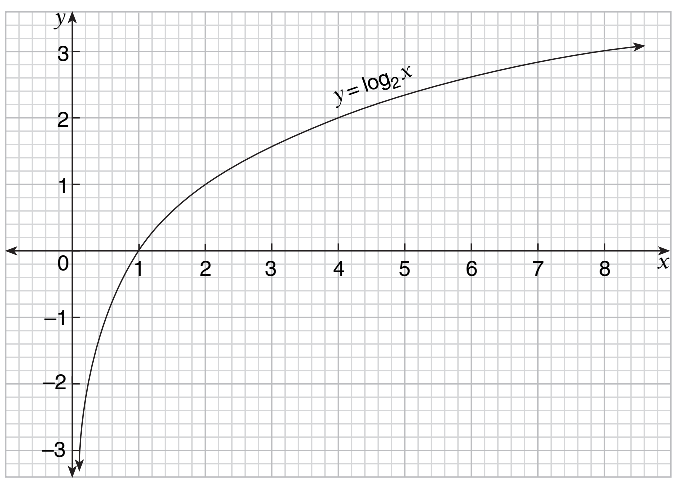
x	0·1	0·3	1	1·7	3	5·2	9
y	−2	−1	0	$\frac{1}{2}$	1	$1\frac{1}{2}$	2



2 **a**

x	0·13	0·18	0·25	0·35	0·5	0·71	1	1·4	2	2·8	4	5·7	8
y	−3	$-2\frac{1}{2}$	−2	$-1\frac{1}{2}$	−1	$-\frac{1}{2}$	0	$\frac{1}{2}$	1	$1\frac{1}{2}$	2	$2\frac{1}{2}$	3

b **i** 1·6 **ii** 2·3 **iii** 0·6 **iv** −0·4

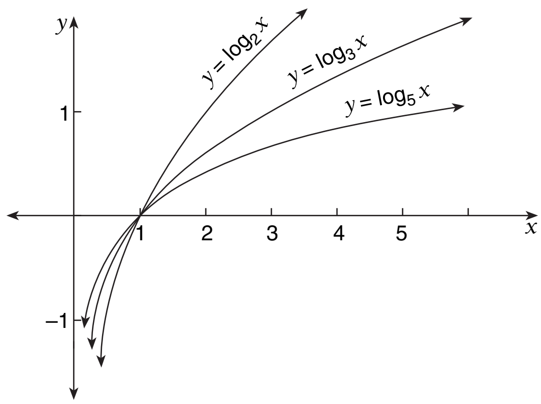


3 **a**

x	0·2	0·4	0·6	0·8	1	2	3	4	5	6	7	8	9	10
y	−·70	−·40	−·22	−·10	0	·30	·48	·60	·70	·78	·85	·90	·95	1

c **i** 5·0 **ii** 1·8 **iii** 0·3

4



Prep Quiz 8:06

1 3 **2** x **3** x **4** y **5** $x + y$ **6** a^{x+y} **7** a^{x-y} **8** a^{mx} **9** x^{-1} **10** 0

Exercise 8:06

1 a 2 **b** 1 **c** 4 **d** 1 **e** 0 **f** 3 **g** 1 **h** 3 **i** 1 **j** 3 **k** 5

2 a 0·778 **b** 0·176 **c** 0·954 **d** –0·301 **e** 0·2385 **f** 1·255 **g** 0·903 **h** 1·380 **i** 1·556 **j** –0·176 **k** –0·602 **l** 0·5395

3 a 3·3 **b** 1·2 **c** 3·0 **d** 0·9 **e** 6·3 **f** 6·6 **g** 4·8 **h** –1·2 **i** 9·0 **j** 2·7 **k** 2·4 **l** 0·3

4 a 2 **b** 1 **c** 0 **d** 5 **e** 0 **f** 2

5 a $\log_a x + \log_a y - \log_a z$ **b** $\log_a x + 1 - 2\log_a z$ **c** $\log_a x - 2\log_a (x+1)$ **d** $\log_a x + \frac{1}{2}\log(x+1)$

e $\frac{1}{2}\log_y x + \frac{1}{2} - \frac{1}{2}\log_y (y+1)$ **f** $\log_y x + \log_y (a+b) - 2$

6 a $\log_a \dfrac{xz^2}{y}$ **b** $\log_a \dfrac{x^2 z}{\sqrt{y}}$ **c** $\log_a \sqrt{xy}$ **d** $\log_a \dfrac{(x+1)(x+2)}{x}$ **e** $\log_y \dfrac{x^2 y^3}{z}$ **f** $\log_a \dfrac{x^2 y}{a^3}$

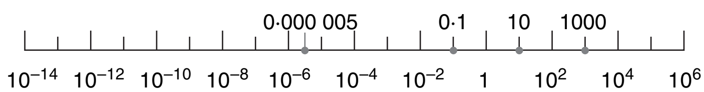
7 a $xy = x + y$ **b** $5x^2 = y$ **c** $x^2 y = 8$ **d** $\dfrac{x}{y} = 125$ **e** $\dfrac{1+x}{1-x} = y$ **f** $y = 5x^7$

8 a 10 **b** 2 **c** 32 **d** 1·5 **e** 4 **f** 2 **g** $\frac{1}{2}$ **h** 12

9 a 50 **b** 24 **c** 1000 **d** 10 **e** $\frac{10}{9}$ **f** $\frac{1}{4}$ **g** 1, –2 **h** $\dfrac{1 \pm \sqrt{7}}{2}$

Investigation 8:06

1 a



b i 99 **ii** 990 000 **iii** 0·99

c The zero is to the left of the numbers on the scale. (It could in fact never be shown on this axis.)

d i 10^{-2} m to 10^{5} m (0·01 m to 100 000 m) **ii** 10^{-7} m to 10^{-6} m

2 a i 10 times **ii** 100 times **b** halfway between 5 and 6 **c** $10^{5\cdot5} \div 10^{5} = 10^{\frac{1}{2}} \doteqdot 3\cdot16$

Prep Quiz 8:07

1 2^2 **2** 2^3 **3** 2^{-1} **4** $2^{\frac{1}{2}}$ **5** 5^3 **6** 3^5 **7** $2^{-\frac{1}{2}}$ **8** yes **9** 4 **10** 5

Exercise 8:07

1 a 4 **b** 7 **c** 4 **d** 0 **e** 3 **f** 1 **g** 3 **h** 9 **i** 5 **j** 4 **k** 4 **l** 3 **m** 4 **n** 3 **o** 5 **p** 4

2 a $\frac{1}{2}$ **b** $\frac{1}{2}$ **c** $\frac{1}{4}$ **d** $\frac{1}{2}$ **e** $\frac{1}{3}$ **f** $\frac{1}{5}$ **g** $\frac{1}{2}$ **h** $\frac{1}{2}$ **i** –1 **j** –2 **k** –1 **l** –2 **m** –2 **n** –3 **o** $-\frac{1}{2}$ **p** $-\frac{1}{2}$

3 a $\frac{3}{2}$ **b** $\frac{5}{2}$ **c** $\frac{2}{3}$ **d** $\frac{4}{3}$ **e** $\frac{3}{2}$ **f** $\frac{2}{3}$ **g** $\frac{6}{7}$ **h** $\frac{3}{5}$ **i** $-\frac{4}{3}$ **j** $-\frac{3}{2}$ **k** $-\frac{2}{3}$ **l** $-\frac{3}{4}$ **m** $-\frac{3}{2}$ **n** –2 **o** $-\frac{3}{2}$ **p** $-\frac{3}{2}$

4 a 5 **b** –2 **c** $\frac{5}{2}$ **d** 2 **e** $\frac{5}{4}$ **f** 8 **g** $\frac{1}{2}$ **h** $\frac{5}{3}$ **i** $\frac{4}{3}$ **j** $\frac{3}{4}$

k $\frac{5}{6}$ **l** $\frac{7}{6}$ **m** $-\frac{11}{4}$ **n** $\frac{13}{12}$ **o** $\frac{7}{2}$ **p** $\frac{5}{4}$ **q** $\frac{4}{3}$ **r** 1 **s** $\frac{2}{7}$ **t** 1

Investigation 8:07

1 3·86 **2** 6·77 **3** 1·02 **4** 4·52

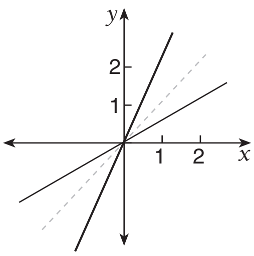
Exercise 8:08

1 a 2·845 **b** 1·619 **c** 4·692 **d** 2·807 **e** 2·861 **f** 0·387
2 a 3·088 **b** 0·292 **c** 4·585 **d** 1·232 **e** 2·048 **f** –0·349
3 a 1·372 **b** 16·610 **c** 1·131
4 a 2·585 **b** 4·660 **c** 0 **d** 7·849 **e** 38·568 **f** 5·547

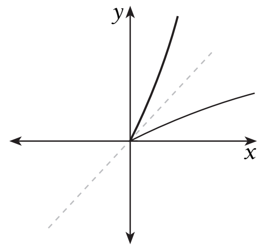
Diagnostic Test 8 Functions and logarithms

1 a 8 **b** 0 **c** $1\frac{3}{4}$ **2 a** $4p - 5$ **b** $2a - 3$ **c** $2x^2 - 5$
3 a, c
4 a x is any real number; y is any real number **b** x is any real number; $y \geqslant -2$ **c** $x \geqslant 0$; $y \leqslant 0$
5 a $y = \frac{1}{2}(x + 1)$ **b** $f^{-1}(x) = \dfrac{3 - x}{2}$ **c** $g^{-1}(x) = \dfrac{2x - 1}{3}$ **6 a, c**

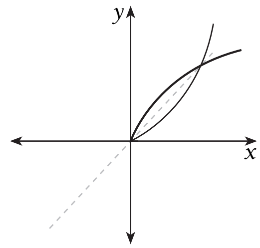
7 a



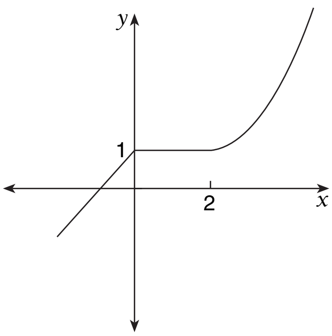
b



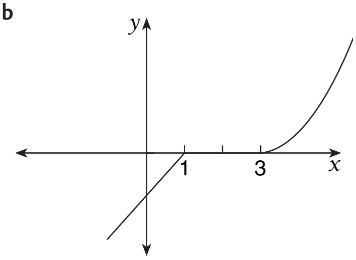
c



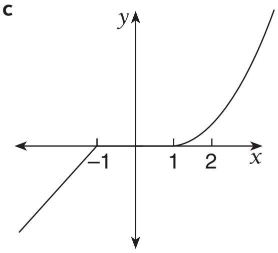
8 a



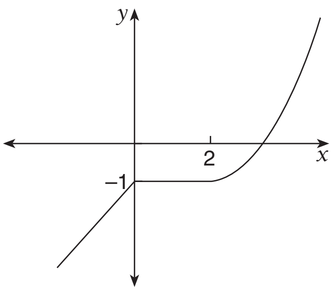
b



c



d



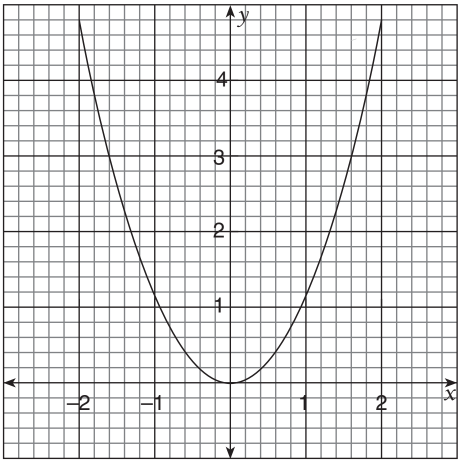
9 a $8 = 2^3$ **b** $9 = 3^2$ **c** $2 = 4^{\frac{1}{2}}$ **d** $1 = 5^0$
10 a $\log_2 32 = 5$ **b** $\log_4 64 = 3$
c $\log_{27} 3 = \frac{1}{3}$ **d** $\log_{10} 0{\cdot}1 = -1$

11 a 1 **b** 2 **c** 3 **d** –3 **12 a** 2 **b** 2 **c** 32 **d** 7
13 a $\log_a 21$ **b** $\log_a 3$ **c** $\log_a 60$ **d** $\log_a 5$ **14 a** $4 \log_a 3$ **b** $2 \log_a 3$ **c** $\frac{1}{2} \log_a 3$ **d** $-\log_a 3$
15 a $\log_a x + 2 \log_a y$ **b** $\frac{1}{2} \log_a x + \frac{1}{2} \log_a y$ **c** $\log_a x - \frac{1}{2} \log_a y$
16 a $\log_a \dfrac{x}{yz}$ **b** $\log_a x^2y\sqrt[3]{z}$ **c** $\log_a x\sqrt{\dfrac{(x-1)}{(x+1)}}$
17 a 7 **b** –2 **c** –2 **d** $\frac{3}{4}$ **18 a** 1·301 **b** 0·431 **c** 0·871

8A Revision Assignment

1 **a** −3 **b** −3 **c** $f^{-1}(x) = \dfrac{x-1}{2}$; $f^{-1}(2) = \frac{1}{2}$

2 $x = \pm\sqrt{2}$; no



3 **a** 4 **b** $\frac{5}{2}$ **c** $\frac{3}{2}$ **d** 4

4 **a** 1·12 **b** 0·28 **c** 1·56 **d** −0·56

5 **a** A(0, 1) B(3, 1) C(1, 0) **b** A(1, 0), B(8, 3), C(0, 3)

6 **a** 1·760 **b** 0·839 **c** 2·771 **d** 3·424
e 4·419 **f** 1·959

7 **a** $\log(x-2)$ **b** 0

8 **a** 12 **b** 3 or −1

9 **a** $\dfrac{x}{y} = x + y$ **b** $x^2 = 2y$ **c** $x^5y^3 = 2$

8B Working Mathematically

1 24

2 **a** $\frac{4}{3}\pi r^3$ **b** $2\pi r^3$ **c** $\frac{2}{3}\pi r^3$ **d** $\frac{2}{3}\pi r^3$ **e** Volume of cylinder = volume of sphere + volume of cone

3 Anna, \$84·75

4 **a** **i** 316 **ii** 187 **iii** 504 **iv** 250 **b** 63% **c** 55%
d *Front left:* 61%, *Front centre:* 0%, *Driver's seat* (front right): 63%, *Rear left:* 46%, *Rear centre:* 11%, *Rear right:* 45%. There were too few occupants in the *front centre* to consider that spot. The safest spot seems to be the *back centre*, followed by the *rear right* and *left*. There isn't much difference between the *driver's seat* and the *front left*.

Chapter 9: Matrices

Exercise 9:01

1 **a** $\begin{bmatrix} 1 & 4 \\ 1 & 4 \\ 0 & 1 \end{bmatrix}$ **b** $\begin{bmatrix} 3 & 2 \\ 6 & 3 \end{bmatrix}$ **c** $\begin{bmatrix} 4 & 2 & 4 \\ 3 & 5 & -1 \end{bmatrix}$ **d** $\begin{bmatrix} 0 & 4 & 3 \\ 3 & 10 & -2 \\ 4 & 8 & 6 \end{bmatrix}$ **e** $\begin{bmatrix} -4 & 6 \\ 12 & 3 \end{bmatrix}$

f $\begin{bmatrix} 1 & -1 & 3 \\ 5 & 2 & 2 \\ 0 & 2 & 5 \\ 2 & 1 & 2 \end{bmatrix}$ **g** $\begin{bmatrix} 1 \\ 7 \\ 2 \\ 1 \end{bmatrix}$ **h** $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ **i** $\begin{bmatrix} 3 & -3 \\ -9 & 9 \end{bmatrix}$ **j** $\begin{bmatrix} 3 & 3 \\ -3 & -4 \end{bmatrix}$

2 **a** $\begin{bmatrix} -1 & -1 \\ 8 & 3 \end{bmatrix}$ **b** $\begin{bmatrix} 3 & 5 \\ -8 & -4 \end{bmatrix}$ **c** $\begin{bmatrix} 7 & -2 \\ 1 & 0 \end{bmatrix}$ **d** $\begin{bmatrix} 2 & 14 \\ -6 & -5 \end{bmatrix}$ **e** $\begin{bmatrix} 5 & -6 \\ 1 & 1 \end{bmatrix}$

3 **i** **a** $\begin{bmatrix} 1 & -1 \\ 6 & 3 \\ 3 & 5 \end{bmatrix}$ **b** $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ **c** $\begin{bmatrix} -6 & -18 & 3 \\ 9 & -9 & 0 \end{bmatrix}$ **d** $\begin{bmatrix} -3 & 1 \\ 4 & -3 \\ -7 & 1 \end{bmatrix}$ **e** $\begin{bmatrix} 7 & -3 \\ -2 & 9 \\ 17 & 3 \end{bmatrix}$

ii **a** $\boldsymbol{P}$ and $\boldsymbol{Q}$ are not of the same order. **b** $\boldsymbol{Q} + \boldsymbol{S}$ gives the null matrix.

4 **a** $x = 2, y = -2$ **b** $a = 3, b = 4, c = 1, d = 0$
c $p = 8, q = -3, r = -5, s = 0, t = -2$ **d** $k = -3, l = 3, m = -3, n = 3$
e $a = 2, b = 0, c = -1, d = 3$ **f** $x = 2\frac{1}{2}, y = -1, z = 0$
g $x = 1, y = -2$ **h** $a = 3, b = -1, c = -4, d = 2\frac{1}{2}$
i $j = -1, k = 2$ **j** $a = 3, b = 6, c = 3, d = 2$

5 $C = \begin{bmatrix} -4 & -10 & 5 \\ -11 & -4 & -5 \\ 7 & -4 & 6 \end{bmatrix}$

Exercise 9:02

1 **a** $\begin{bmatrix} 7 & 5 \\ 8 & 2 \end{bmatrix}$ **b** $\begin{bmatrix} -4 & 6 \\ 1 & 18 \end{bmatrix}$ **c** $[-16]$ **d** $\begin{bmatrix} -16 \\ 19 \end{bmatrix}$ **e** $\begin{bmatrix} -3 & 10 & -6 \\ -2 & 9 & -11 \end{bmatrix}$

f $\begin{bmatrix} 7 & 1 & -1 \\ 10 & -10 & 0 \\ 7 & 17 & -3 \end{bmatrix}$ **g** 0 **h** $\begin{bmatrix} -2 & -1 \\ 3 & 2 \\ 1 & 4 \end{bmatrix}$ **i** $\begin{bmatrix} 2 & 1 \\ -3 & -2 \\ -1 & -4 \end{bmatrix}$ **j** $\begin{bmatrix} 4 & 2 \\ -6 & -4 \\ -2 & -8 \end{bmatrix}$

2 **a** $\begin{bmatrix} 2 & 6 & -2 \\ -11 & -1 & 23 \end{bmatrix}$ **b** $\begin{bmatrix} -3 \\ -10 \end{bmatrix}$ **c** $\begin{bmatrix} 2 & 0 \\ 8 & -8 \\ 16 & 12 \end{bmatrix}$ **d** $\begin{bmatrix} 2 & -9 \\ 10 & 11 \end{bmatrix}$

e Not possible **f** $\begin{bmatrix} 4 & 0 \\ -18 & 16 \end{bmatrix}$ **g** Not possible

3 **a** 2×2 **b** Not possible **c** 3×4 **d** Not possible **e** 2×3
f Not possible **g** 4×2 **h** Not possible **i** 2×4 **j** Not possible

4 **a** $a = 1, b = 2$ **b** $x = 1, y = 0$

5 $X^2 = \begin{bmatrix} 1 & 6 \\ -2 & -3 \end{bmatrix}$, $2X = \begin{bmatrix} 4 & 6 \\ -2 & 0 \end{bmatrix}$, $3I = \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}$

$\therefore X^2 - 2X + 3I = \begin{bmatrix} 1+4-3 & 6-6+0 \\ -2-(-2)+0 & -3-0+3 \end{bmatrix} = 0$

Exercise 9:03

1 **a** $\begin{bmatrix} 6 & -3 \\ 9 & 1 \\ -17 & -8 \end{bmatrix}$ **b** $\begin{bmatrix} 14 & 4 \\ -6 & -1 \end{bmatrix}$ **c** $\begin{bmatrix} 12 & 5 \\ -11 & -4 \end{bmatrix}$ **d** $\begin{bmatrix} 5 \\ 29 \end{bmatrix}$ **e** Not possible **f** $\begin{bmatrix} 12 \\ 11 \\ -13 \end{bmatrix}$

2 **a** $\begin{bmatrix} 18 & -9 \\ -18 & 27 \end{bmatrix}$ **b** $\begin{bmatrix} 8 & -6 \\ -12 & 14 \end{bmatrix}$

3 **a** $\begin{bmatrix} 7 & 6 \\ 2 & 3 \end{bmatrix}$ **b** -3

Exercise 9:04

1 **a** 10, inverse exists **b** 0, inverse does not exist **c** 0, inverse does not exist
d −12, inverse exists **e** 12, inverse exists

2 **a** $\begin{bmatrix} 2 & 1\cdot5 \\ -1 & -0\cdot5 \end{bmatrix}$ **b** No inverse **c** $\begin{bmatrix} -0\cdot5 & -0\cdot25 \\ 0\cdot5 & 0\cdot75 \end{bmatrix}$

d $\begin{bmatrix} 2 & -3 \\ -1\cdot5 & 2\cdot5 \end{bmatrix}$ **e** $\begin{bmatrix} -4 & 5 \\ 5 & -6 \end{bmatrix}$

3 **a** $-\frac{1}{2}\begin{bmatrix} 2 & -4 \\ 4 & 3 \end{bmatrix}$ **b** $-1\begin{bmatrix} 3 & 4 \\ 4 & 5 \end{bmatrix}$ **c** $\frac{1}{4}\begin{bmatrix} 4 & -2 \\ 0 & 1 \end{bmatrix}$ **d** $\frac{1}{6}\begin{bmatrix} 2 & 2 \\ -6 & -3 \end{bmatrix}$ **e** $-\frac{1}{8}\begin{bmatrix} -4 & -5 \\ -4 & -7 \end{bmatrix}$

4 **a** 5 **b** −3 **c** −15 **d** 0 **e** 4 **f** 0

g 20 **h** −12 **i** Not possible **j** $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ **k** $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

5 **a** For any two matrices $\boldsymbol{A}$ and $\boldsymbol{B}$, $\det(\boldsymbol{AB}) = \det(\boldsymbol{A})\det(\boldsymbol{B})$.
b For any matrix $\boldsymbol{A}$, $\boldsymbol{A}\boldsymbol{A}^{-1} = \boldsymbol{A}^{-1}\boldsymbol{A} = \boldsymbol{I}$

6 **a** $a = 2$ **b** $x = 2$ or -2 **c** $p = 2$ or $p = 4$ **d** $x = -3$ or $x = 4$ **e** $x = 0$ or $x = -3$

Exercise 9:05

1 **a** $X = \begin{bmatrix} 16 & 11 \\ -6 & -4 \end{bmatrix}$ **b** $X = \begin{bmatrix} -2 & -\frac{11}{2} \\ 3 & 8 \end{bmatrix}$ **c** $X = \begin{bmatrix} 4 & 9 \\ -5 & -11 \end{bmatrix}$ **d** $X = \begin{bmatrix} -11 & -9 \\ 5 & 4 \end{bmatrix}$ **e** $X = \begin{bmatrix} 1 & 0 \\ -1 & -1 \end{bmatrix}$

2 **a** $x = 2, y = -3$ **b** $x = -2, y = -1$

3 $x = -1, y = 3$

Exercise 9:06

1 **a** $x = 1, y = -2$ **b** $x = -4, y = 3$ **c** $x = -3, y = 2$ **d** $x = 0, y = -2$ **e** $x = 3, y = 2\frac{1}{2}$

2 $x = -2, y = 5$

3 **a** $\begin{bmatrix} 6 & -3 \\ 2 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 \\ -3 \end{bmatrix}$ **b** 0

c The determinant is zero, so there is no inverse to the matrix $\begin{bmatrix} 6 & -3 \\ 2 & -1 \end{bmatrix}$ so the equation cannot be solved for x and y.

This means that the equations have no simultaneous solution, so must be parallel.

4 Express as a matrix product and show that the determinant is zero.

5 **a** $\begin{bmatrix} 2 & 3 & -1 \\ 1 & 2 & -3 \\ 3 & -1 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \\ -4 \end{bmatrix}$ **b** -25 **c** $x = -1, y = 2, z = 1$

Diagnostic Test 9 Matrices

1 **a** $\begin{bmatrix} 5 & 0 & 2 & 2 \\ 7 & 4 & 1 & 2 \\ 3 & 8 & 7 & 5 \\ 1 & 3 & 5 & 0 \end{bmatrix}$ **b** $\begin{bmatrix} 5 & 1 & 5 & 8 \\ 12 & 6 & 1 & 5 \\ 7 & 11 & 12 & 7 \\ 5 & 7 & 12 & 1 \end{bmatrix}$ **c** $\begin{bmatrix} 5 & 1 & 5 & 8 \\ 12 & 6 & 1 & 5 \\ 7 & 11 & 12 & 7 \\ 5 & 7 & 12 & 1 \end{bmatrix}\begin{bmatrix} 2{\cdot}50 \\ 3{\cdot}00 \\ 3{\cdot}50 \\ 2{\cdot}75 \end{bmatrix}$ **d** $\begin{bmatrix} 55{\cdot}00 \\ 65{\cdot}25 \\ 111{\cdot}75 \\ 78{\cdot}25 \end{bmatrix}$

2 **a** $\begin{bmatrix} 19 & 12 \\ -12 & 8 \end{bmatrix}$ **b** $\begin{bmatrix} 15 & 10 & -5 \\ -2 & -8 & 2 \\ 18 & 2 & -4 \end{bmatrix}$ **c** Not possible **d** Not possible

e $\begin{bmatrix} 5 & 10 & -3 \\ 9 & 6 & -3 \end{bmatrix}$ **f** Not possible **g** $\begin{bmatrix} 19 & -10 \\ 15 & -6 \end{bmatrix}$

3 **a** **i** 1 **ii** $\begin{bmatrix} 1 & -2 \\ -1 & 3 \end{bmatrix}$ **b** **i** 0 **ii** No inverse **c** **i** 2 **ii** $\begin{bmatrix} 2 & -3 \\ -1{\cdot}5 & 2{\cdot}5 \end{bmatrix}$

d **i** -1 **ii** $\begin{bmatrix} -3 & 2 \\ 5 & -3 \end{bmatrix}$ **e** **i** 0 **ii** No inverse

4 **a** $x = 5$ **b** $a = \pm 2$ **c** $y = 1$ **d** $z = 2\frac{2}{3}$

5 **a** $\begin{bmatrix} 7 & 3 \\ -9 & -4 \end{bmatrix}$ **b** $\begin{bmatrix} 4 & 1 \\ -3 & -1 \end{bmatrix}$ **c** $\begin{bmatrix} 38 & 31 \\ -22 & -18 \end{bmatrix}$ **d** $\begin{bmatrix} -14 & -11 \\ 38 & 30 \end{bmatrix}$ **e** $\begin{bmatrix} -1{\cdot}5 & 2{\cdot}5 \\ 2 & -3 \end{bmatrix}$

6 **a** $x = -2$ and $y = 3$ **b** $x = 4$ and $y = -6$

9A Revision Assignment

1 **a** $A + B = \begin{bmatrix} 4 & 2 \\ 1 & 0 \end{bmatrix}$ **b** $-3A = \begin{bmatrix} -3 & -6 \\ -9 & 3 \end{bmatrix}$ **c** $AB = \begin{bmatrix} -1 & 2 \\ 11 & -1 \end{bmatrix}$

2 **a** $A^{-1} = \begin{bmatrix} \frac{2}{3} & -\frac{1}{3} \\ -\frac{7}{3} & \frac{5}{3} \end{bmatrix}$ **b** $D = \begin{bmatrix} 7 & -11 \\ 11 & -13 \end{bmatrix}$ **c** $X = \begin{bmatrix} 1 & -3 \\ 2 & 4 \end{bmatrix}$

3 $B = \begin{bmatrix} 1 & 3 \\ 4 & 12 \end{bmatrix}$

4 a $\det M = -4$ **b** $M^{-1} = -\frac{1}{4}\begin{bmatrix} -1 & -1 \\ -2 & 2 \end{bmatrix} = \begin{bmatrix} \begin{bmatrix} \frac{1}{4} & \frac{1}{4} \\ \frac{1}{2} & -\frac{1}{2} \end{bmatrix} \end{bmatrix}$ **c** $X = \begin{bmatrix} 3 \\ -2 \end{bmatrix}$ $(x = 3, y = -2)$

5 a $2A - B \begin{bmatrix} 4 & 2 \\ 2k-1 & 5 \end{bmatrix}$ **b** $\det(2A - B) = 22 - 4k$

6 a $6C + 3D = 163{\cdot}17$
$9C + 2D = 200{\cdot}53$ **b** D = \$17.69 (\$17.7) **c** \$14.85

9B Working Mathematically

1 a $A + B = \begin{bmatrix} a+1 & b \\ c+d & e \end{bmatrix}$ **b** $AB = \begin{bmatrix} a & b \\ c & 0 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ d & e \end{bmatrix}$

2 a i $A^{-1} = \begin{bmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{bmatrix}$ **ii** $A^2 = \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix}$

b $p = 2, q = 3$ **c** $A^{-1}B = \begin{bmatrix} 0 & \frac{3}{2} \\ 1 & 1 \end{bmatrix}$ **d** $X = \begin{bmatrix} 0 & \frac{3}{2} \\ 1 & 1 \end{bmatrix}$

3 a i $a = 5$ **ii** $b = -5$ **b** $q = 3$

4 $a = \frac{28}{33}, b = \frac{59}{33}, c = \frac{20}{33}, d = \frac{28}{33}$

5 $p = -7$ or $p = 1$

6 a $50b + 20c = 260$ **b** $12b + 6c = 66$ **c** Solve to get $b = 4$

Chapter 10: Surface area and volume

Exercise 10:01

1 a 562 cm^2 **b** 544 cm^2 **c** 1721·6 cm^2 **d** 792 cm^2
2 a 527·8 cm^2 **b** 274·9 cm^2 **c** 227·9 cm^2 **3 a** $x = 10{\cdot}82$ cm, 363·84 cm^2 **b** $x = 8{\cdot}49$, 622·86 cm^2
4 a 730 m^2 **b** 346 m^2 **c** 792 m^2 **5 a** 500 cm^2 **b** 312 cm^2 **c** $608 + 204\pi \doteqdot 1248{\cdot}9$ cm^2

Prep Quiz 10:02

1 yes **2** no **3** 3 cm **4** 160 $unit^2$ **5** 5 **6** $\sqrt{281}$
7 6 **8** $\sqrt{292}$ **9** $6\sqrt{281} \doteqdot 100$ $unit^2$ **10** $5\sqrt{292} \doteqdot 85$ $unit^2$

Exercise 10:02

1 a 108 cm^2 **b** 1281·48 cm^2 **2 a** 156 cm^2 **b** 198·75 cm^2 **c** $4\sqrt{17} + 2\sqrt{20} + 8 \doteqdot 33{\cdot}4$ cm^2
3 a 360 cm^2 **b** $(100\sqrt{5} + 100) \doteqdot 324$ cm^2 **4 a** 308·4 cm^2 **b** 258·8 cm^2 **c** 148·4 cm^2
5 a 106 cm^2 **b** 160·12 cm^2 **6 a** 104 m^2 **b** 228 cm^2 **c** 62 cm^2
7 265 cm^2 (to the nearest cm^2) **8 a** $x = 40$ **b** $h = 10\sqrt{15}$

Prep Quiz 10:03

1 πs^2 **2** $2\pi s$ **3** $\frac{r}{s}$ **4** πrs **5** $\frac{1}{4}$ **6** $\frac{1}{3}$ **7** $\frac{1}{3}$ **8** $\frac{1}{2}$ **9** $\frac{r}{s}$ **10** 71·5 (correct to 1 dec. pl.)

Exercise 10:03

1 a 80π cm^2 **b** 50π cm^2 **c** 96π cm^2 **d** 800π cm^2 **e** $0{\cdot}48\pi$ m^2
2 a 180π cm^2 **b** 75π cm^2 **c** 160π cm^2 **d** 1200π cm^2 **e** $0{\cdot}64\pi$ m^2
3 a 144π cm^2 **b** $5{\cdot}76\pi$ cm^2 **c** 200π cm^2 **d** $0{\cdot}9\pi$ m^2
4 a 2092 cm^2 **b** 267·0 cm^2 **c** 1013 cm^2 **d** 2·730 m^2 **e** 23·56 m^2
5 a 1005·3 cm^2 **b** 712·6 cm^2 **c** 226·5 cm^2
6 a 15 cm **b** 5 cm **c** $10\sqrt{2} \doteqdot 14{\cdot}1$ cm
7 a $\sqrt{1200} \doteqdot 34{\cdot}6$ cm **b** 23·2 cm

Exercise 10:04

1 a 100π cm^2, 314 cm^2 **b** $231{\cdot}04\pi$ cm^2, 726 cm^2 **c** $40{\cdot}96\pi$ m^2, 129 m^2 **d** 324π cm^2, 1020 cm^2
e $2{\cdot}56\pi$ m^2, 8·04 m^2 **f** $6{\cdot}4 \times 10^7\pi$ km^2, $2{\cdot}01 \times 10^8$ km^2
2 a 432π cm^2 **b** 108π cm^2
3 a 804·25 cm^2 **b** 1182·37 cm^2 **c** 1140·40 m^2
4 23·12 m^2 **5** 4·0 cm **6 a** 333 cm^2 **b** 855 cm^2 **c** 103 m^2

Investigation 10:05

Volume of a pyramid = $\frac{1}{3}$ of the volume of a prism.

Exercise 10:05

1 a 480 cm^3 **b** 193·83 cm^3 **2 a** 22·18 m^3 **b** 234·9 cm^3
3 a 3·6 m^3 **b** 31 cm^3
4 a 60 cm **b** $10\frac{5}{12} \doteqdot 10{\cdot}4$ cm **c** area = 1600 cm^2, side length = 40 cm
d length = 20 cm, width = 10 cm
5 a 300·12 cm^3 **b** 324 cm^3 **6 a** 0·289 m^3 **b** 0·608 m^3
7 a 115·7 m^3 **b** 960 cm^3 **c** 3500 cm^3 **d** 2110 cm^3 **e** 185·625 m^3
8 a $\frac{64\sqrt{193}}{3} \doteqdot 296{\cdot}4$ cm^3 **b** $144\sqrt{3}$ cm^3 **9** $18\sqrt{2}$ cm^3

Exercise 10:06

1 a 89·22 m^3 **b** 17·67 m^3 **c** 1707·75 cm^3 **2 a** 220 cm^3 **b** 636 cm^3 **c** 0·7 m^3
3 a $h = 5{\cdot}011$ m; $V = 32{\cdot}8$ m^3 **b** $h = 15{\cdot}263$ cm; $V = 117$ cm^3 **c** $h = 3{\cdot}175$ m; $V = 43{\cdot}1$ m^3
4 a 100π cm^3 **b** 240π cm^3
5 a 9·5 cm **b** 8·5 cm **c** $r = 7{\cdot}8$ cm, $h = 15{\cdot}6$ cm **d** $13\frac{1}{3}$ cm
6 a 2080 cm^3 **b** 868 m^3 **c** 401 cm^3 **7** 600 cm^3

Exercise 10:07

1 a 36π cm^3 **b** $\frac{32\pi}{3}$ cm^3 **c** $62{\cdot}208\pi$ cm^3 **d** $18{\cdot}432\pi$ cm^3
2 a 7·24 m^3 **b** 8780 cm^3 **c** 65·5 cm^3 **d** 3·62 m^3
3 a 1570 cm^3 **b** 1690 cm^3 **c** 83·8 cm^3 **4** $\frac{364\pi}{3}$ cm^3
5 a 7·8 cm **b** 11 cm **c** 0·63 m

Exercise 10:08

1 a \$6984 **b** 8640 L **2 a** 34·72 m^3 **b** 41·664 t **c** 54·02 m^2
3 a 7238 m^3 **b** 7238 kL **c** 1206·4 kL **d** 1709·0 m^2
4 a $1{\cdot}1 \times 10^{21}$ m^3 **b** $5{\cdot}9 \times 10^{21}$ t **c** $3{\cdot}6 \times 10^8$ km^2 **5** 107 m^3
6 33·93 m^2 **7** 16 099·1 kg **8** 17·7 m^3; 44·25 t **9** 213·1 m^3 **10** 80·784 m^3 **11** 789 m^2
12 a i 1·8 m **ii** 27·57 m^3 (correct to 2 dec. pl.) **b** 65·9 m^3

Diagnostic Test 10 Surface area and volume

1 a 384 cm^2 **b** 222·78 cm^2 **c** 360 cm^2 **2 a** 94·3 cm^2 **b** 27·1 cm^2 **c** 51·3 m^2
3 a 314·16 cm^2 **b** 865·70 cm^2 **c** 84·82 cm^2 **4 a** 44·712 m^3 **b** 80·136 m^3 **c** 180·728 cm^3
5 a 46·0 cm^3 **b** 118·1 cm^3 **c** 21·8 cm^3 **6 a** 523·6 cm^3 **b** 333·0 cm^3 **c** 883·6 cm^3

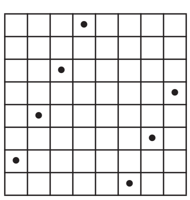
10A Revision Assignment

1 SA = 216 cm^2, vol. = 162 cm^3 **2** 200π cm^2 **3** 600 cm^2 **4** 2·879 m **5** 476 cm^3
6 739 cm^2 **7** 12·3 m^3 **8** SA = $72\sqrt{3}$ cm^2; vol. = $72\sqrt{2}$ cm^3

10B Working Mathematically

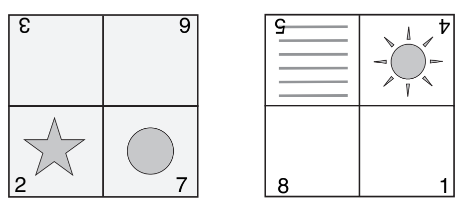
1 \$417 605 **2 a** $\frac{a\sqrt{3}}{3}$ **b** $\frac{a\sqrt{3}}{6}$ **3** $AX = \sqrt{12}$ units, vol. = 36 unit^2 **4** 18, 42, 105

5



6 more queens can be placed.
An example is shown to the left.

6



Chapter 11: Similarity

Exercise 11:01

1 a B: Its angles are not right angles and hence matching angles are not equal.
b C: It is not an equilateral triangle and hence its matching angles are not equal to those of triangle A or B.
c B: Matching sides are not in the same ratio.
d A: Matching sides are not in the same ratio.
2 a 14·4 cm **b** 18 cm by 12 cm
c As squares have all angles equal to 90° and all sides equal, then any two squares will have matching angles equal and the ratio of matching sides will be equal. In an equilateral triangle, all angles equal 60° and all sides are equal. Hence, in any two equilateral triangles, matching angles will be equal and the ratios of matching sides will also be equal.
3 a 8 cm **b** it is tripled **c** no
4 a A: 3 cm long, 1 cm wide; B: 6 cm long, 2 cm wide; C: 2·5 cm long, 1·5 cm wide; D: 1·5 cm long, 1 cm wide; E: 5 cm long, 3 cm wide; F: 3 cm long, 2 cm wide
b A and B; C and E; D and F
5 a G **b** F **c** J **d** H **e** I
6 a P **b** F **c** T **d** W **e** L
7 a enlarg. factor = 1·8, $x = 3·6$ **b** red. factor = $\frac{5}{12}$, $x = 8·75$ **c** enlarg. factor = 2·4, $y = 28·8$
d red. factor = 0·8, $y = 14·4$. $a = 21·6$ **e** red. factor = 0·8, $y = 3·2$, $x = 5$, $a = 1·6$
f enlarg. factor = 1·25, $x = 12·5$, $y = 15$

Prep Quiz 11:02

1 no **2** no **3** false **4** false **5** *GH* **6** $<J$ **8** 120° **8** $\frac{1}{2}$ **9** 0·75 **10** 1·25

Exercise 11:02A

1 a yes **b** yes **2 a** $x = \sqrt{8}, y = \sqrt{18}$ **b** $\frac{6}{4} = \frac{\sqrt{18}}{\sqrt{8}}$; yes **c** yes
3 a i yes **ii** *DE*, *EF*, *DF* **iii** yes **b i** $<G$ and $<D$; $<H$ and $<E$; $<I$ and $<F$; yes
ii All ratios are equal to $\frac{3}{2}$. **iii** yes **c i** yes **ii** yes **iii** yes
4 $\frac{DE}{AB} = \frac{47}{26} = 1·8, \frac{DF}{AC} = \frac{35}{20} = 1·8, \frac{EF}{BC} = \frac{54}{30} = 1·8$
5 a B and D **b** A and D **c** B and C
6 a W **b** L **c** N **d** M **e** P **f** U **g** R **h** Q **i** V **j** T

Exercise 11:02B

1 a yes **b** They are matching angles of congruent triangles.
2 a i $\frac{3}{2}$ **ii** $\frac{3}{2}$ **iii** $\frac{3}{2}$ **b** AAS **c** yes
3 a yes **b** All the small triangles have sides 8, 10 and 12 units in length. Hence they are congruent (SSS). The three angle pairs are equal because they are matching angles of congruent triangles.
4 a A and B **b** A and C **c** B and C **d** A and C
5 a M **b** P **c** N **d** K **e** L **f** Q **g** R **h** S **i** T **j** W
6 a $\frac{DF}{AC} = 2$ or $<A = <D$ or $<C = <F$. **b i** They are matching angles of congruent triangles. **ii** yes
7 a B and C **b** A and B **c** A and B **d** A and B
8 a ΔABC and ΔADE (3 angles) **b** ΔABC and ΔEDC (3 angles)
c ΔABE and ΔACD (sides about an equal angle in same ratio)
d ΔABC and ΔEDC (matching sides in the same ratio)
e ΔADE and ΔACB (3 angles) **f** ΔABC and ΔEDC (3 angles)

Prep Quiz 11:03

1 largest **2** smallest **3** true **4** true **5** true **6** *DE* **7** $\frac{3}{4}$ **8** *FE* **9** *DE* **10** $\frac{4}{3}$

Exercise 11:03

1 a *DE*, *EF*, *DF*
Enlargement factor = $\frac{2·5}{1·5}$
$\therefore x = 3 \times \frac{2·5}{1·5}$
$= 5$ m

b *TR*, *TS*, *RS*
Reduction factor = $\frac{25}{40}$
$\therefore y = 32·6 \times \frac{25}{40}$
$= 20·375$ m

2 **a** $x = 10{\cdot}7$ **b** $y = 10{\cdot}4$ **c** $a = 30$ **d** $y = 40$
3 **a** $a = 30{\cdot}8$ **b** $x = 7{\cdot}1$ **c** $x = 5{\cdot}0$
4 **a** $x = 7{\cdot}5$ **b** $x = 5{\cdot}4$ **c** $a = 10$ **d** $x = 2\frac{2}{9}$
5 **a** $x = 4{\cdot}8$ **b** $h = 2{\cdot}4$ **c** $x = 13{\cdot}5$ **d** $x = 7$
6 **a** $x = 8, y = 5\frac{1}{3}$ **b** $a = 8, b = 16$ **c** $a = 40, b = 28{\cdot}8$ **d** $x = 28, y = 26{\cdot}25$ **e** $y = 10$ **f** $a = 2{\cdot}4$
7 **a** $x = 12, y = 6$ **b** $x = 12{\cdot}8$ **c** $x = 16{\cdot}8, y = 22$

Exercise 11:04

1 **a** In Δs ABC and DBE
(1) $\angle CAB = \angle EDB$ (data)
(2) $\angle ABC = \angle BDE$ (vert. opp. $\angle$s)
$\therefore \Delta ABC \mathrel{|||} \Delta DBE$ (2 pairs of equal $\angle$s)
$\therefore \frac{x}{4} = \frac{6}{5}$ (ratio of matching sides are equal)
$x = 4{\cdot}8$

b In Δs ABC and DEC
(1) $\angle ACB = \angle DCE$ (vert. opp. $\angle$s)
(2) $\frac{AC}{DC} = \frac{BC}{EC} = \frac{26}{16}$ (data)
$\therefore \Delta ABC \mathrel{|||} \Delta DEC$
(1 equal angle and sides about the angle are in same ratio)
$\therefore \frac{x}{44} = \frac{16}{26}$ (ratio of matching sides)
$\therefore x = 27{\cdot}1$ (to 1 dec. pl.)

c In Δs EFG and JHG
$\frac{JG}{EG} = \frac{JH}{EF} = \frac{GH}{GF} = \frac{7}{5}$
$\therefore \Delta EFG \mathrel{|||} \Delta JHG$ (matching sides in same ratio)
$\therefore x = 83°$ (matching $\angle$s of sim. Δs)

d In Δs ABC and AED
$\frac{AD}{AC} = \frac{AE}{AB} = \frac{DE}{CB} = \frac{4}{3}$
$\therefore \Delta ABC \mathrel{|||} \Delta AED$ (matching sides in same ratio)
$\therefore a = 53°$ (matching $\angle$s of sim. Δs)

2 **a** 2 pairs of equal $\angle$s
$e = 3{\cdot}1$
$f = 1{\cdot}6$

b 2 pairs of equal $\angle$s
$x = 23{\cdot}3$

c 1 equal angle and sides about the angle are in the same ratio
$x = 28{\cdot}8$

d matching sides in same ratio
$h = 42$

e 2 pairs of equal angles
$a = 150$
$b = 282$

f 2 pairs of equal angles
$x = 28{\cdot}3$
$y = 11{\cdot}7$

3 $\frac{EC}{AC} = \frac{DC}{BC} = \frac{1}{2}$
$\angle ECD = \angle ACD$
$\therefore \Delta ABC \mathrel{|||} \Delta EDC$ (1 equal angle and sides about the angle are in the same ratio)
$\therefore \angle CED = \angle CAB, \angle CDE = \angle CBA$ $\therefore ED \parallel AB$ and $\frac{ED}{AB} = \frac{CE}{CA} = \frac{CD}{CB} = \frac{1}{2}$
$\therefore ED = \frac{1}{2}AB$

4 **a** $x = 19$ **b** $x = 21$ **c** $x = 15, y = 24$

5 **a** **i** $\angle BED = \angle BAC$ (corresponding $\angle$s)
$\angle EBD = \angle ABC$
$\therefore \Delta EBD \mathrel{|||} \Delta ABC$ (2 pairs of equal $\angle$s)
ii $BD : BC = 4 : 9$
iii $BE = 15\frac{5}{9} = 15{\cdot}56$ mm
iv $EA = 19\frac{4}{9} = 19{\cdot}44$ mm
v $BE : EA = 4 : 5$
vi $\frac{BE}{EA} = \frac{4}{5} = \frac{BD}{DC}$

b **i** $DE : DF = 3 : 5$
ii $DH : DG = 3 : 5$
iii $\angle DEH = \angle FDG$
$\therefore \Delta EDH \mathrel{|||} \Delta FDG$ (1 equal angle and sides about the angle are in the same ratio)
iv yes, matching $\angle$s of similar Δs
v yes, matching $\angle$s of similar Δs
vi Yes, if a line divides two sides of a triangle in the same ratio it must be parallel to the third side.

6 **a** $a = 18{\cdot}57$ **b** $b = 20{\cdot}36$ **c** $c = 25{\cdot}2$

7 **a** $\angle AED = \angle CEB$ (vert. opp $\angle$s)
$\angle DAE = \angle BCE$ (data)
$\therefore \Delta ADE \parallel\!| \Delta CBE$ (2 pairs of equal $\angle$s)
$\therefore \dfrac{AE}{CE} = \dfrac{ED}{EB}$ (ratio of matching sides are equal)
$\Rightarrow AE \times EB = CE \times ED$

b $\Delta ABC \parallel\!| \Delta DAC$ (1 equal angle and sides about the angle are in the same ratio)
$\therefore \dfrac{a}{b} = \dfrac{b}{c}$ (ratio of matching sides are equal)
$\Rightarrow ac = b^2$

c $\angle ACB = \angle DCA$
$\therefore \Delta ACB \parallel\!| \Delta DCA$ (2 pairs of equal $\angle$s)
since $\angle ADC = 90°$, $AC = \sqrt{b^2 + c^2}$ and $AB = \sqrt{a^2 + c^2}$
since $\angle BAC = 90°$, $b^2 + c^2 + a^2 + c^2 = (a + b)^2$
$\Rightarrow a^2 + 2c^2 + b^2 = a^2 + 2ab + b^2$
$\Rightarrow 2c^2 = 2ab$
$\Rightarrow c^2 = ab$

Prep Quiz 11:05

1 5 : 3 **2** 25 : 9 **3** $a : b$ **4** $a^2 : b^2$ **5** $R : r$ **6** $R^2 : r^2$ **7** $k : 1$ **8** $k^2 : 1$ **9** 3 : 2 **10** 9 : 4

Exercise 11:05

1 **a** 4 : 9 **b** 36 : 49 **c** 25 : 64 **d** 81 : 100 **e** 16 : 25 **f** 49 : 225
2 **a** 2·6 m^2 **b** 270 cm^2 **c** 15·625 cm^2 **d** $200(1 + \sqrt{2})$ m^2 **e** $\dfrac{25\pi}{6}$ cm^2 **f** 78·1 m^2
3 **a** 3 : 5, $x = 15$ **b** 3 : 4, $y = 24$ **c** 11 : 20, $x = 30$ **d** 4 : 5, $y = 2·5$ **e** 2 : 3, $x = 15$
4 **a** Increased by a factor of 9. **b** 4 : 5 **c** 22 500 : 1 **d** 4 : 7
5 0·681 m^2 (correct to 3 dec. pl.)
6 **a** 418 cm^2 **b** 32 cm **7** 40 kg
8 **a** 2 : 3 **b** 5 : 3 **c** 9 : 25 **d** 9 : 16 **e** 25 : 16

Prep Quiz 11:06

1 $x : y$ **2** $x : y$ **3** $x^2 : y^2$ **4** $x^2 : y^2$ **5** $x^3 : y^3$ **6** $a : b$ **7** $a : b$ **8** $a^2 : b^2$ **9** $a^2 : b^2$ **10** $a^3 : b^3$

Exercise 11:06

1 **a** **i** 4 : 9 **ii** 8 : 27 **b** **i** 16 : 25 **ii** 64 : 125 **c** **i** 25 : 36 **ii** 125 : 216 **d** **i** 49 : 64 **ii** 343 : 512
e **i** 9 : 25 **ii** 27 : 125 **f** **i** 81 : 100 **ii** 729 : 1000
2 **a** 1755 m^2 **b** 9056 m^3 **c** 19·6875 m^3 **d** 140·625 cm^3 **e** 306·25 m^2 **f** 75 m^2
g 37·5 m^3 **h** 183 cm^2 **i** 209 cm^3
3 **a** 15 : 16, 225 : 256 **b** 1 : 2, 1 : 4 **c** 3 : 4, 9 : 16
4 **a** 55·37 cm^2 **b** 1·4 m **c** 1·26 m^2 **d** 21·33 m **e** 128π m^2
5 **a** 8859·375 cm^3 **b** 225 mL
6 **a** 6 750 000 cm^3 = 6·75 m^3 **b** 125 000 mL = 125 L
7 **a** 16 units **b** 1125 cm^3
8 90 glasses
9 **a** 81 **b** 729
10 **a** $\dfrac{1}{1024}$ **b** $\dfrac{1}{32\,768}$ **c** 2·75 cm^3

Diagnostic Test 11 Similarity

1 **a** yes **b** no **c** yes **2** **a** no **b** no **c** no
3 **a** yes **b** no **c** yes **4** **a** 21 **b** 25 **c** 8·8
5 **a** 18 **b** 21·6 **c** 50
6 **a** $\angle ADE = \angle BAD$ (alt. $\angle$s, $BA \parallel DE$)
$\angle DAE = \angle ACB$ (alt. $\angle$s, $BC \parallel AE$)
$\therefore \Delta AED \parallel\!| \Delta CBA$ (equiangular)

b $\dfrac{AB}{DB} = \dfrac{AD}{BC} = \dfrac{DB}{DC} = \dfrac{3}{2}$ (data)
$\therefore \Delta ABD \parallel\!| \Delta BDC$ (corresp. sides in prop'n)

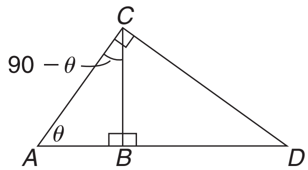
c $\angle DAC = \angle EAB$ (common)
$\dfrac{DA}{EA} = \dfrac{CA}{BA} = \dfrac{2}{1}$ (data)
$\therefore \Delta DAC \parallel\!| \Delta EAB$ (sides about an equal angle are in prop'n)

7 **a** **i** 100 : 49 **ii** 550 m^2 **b** **i** 25 : 64 **ii** 200 m^2 **c** **i** 9 : 10 **ii** 20 mm
8 **a** **i** 9 : 4 **ii** 27 : 8 **iii** 6750 m^3 **b** **i** 1 : 2 **ii** 1 : 8 **iii** 145·6 cm^3 **c** 114·8175 m^3

11A Revision Assignment

1 **a** $x = 9, y = 15$ **b** $a = 16, y = 20$ **c** $a = 32, y = 30$
2 **a** $x = 25·2, y = 19·6$ **b** $x = 21$ **c** $x = 5·7, y = 6·9$
3 triangle C: 25·6 cm, 38·4 cm, 56·32 cm triangle F: 10·8 cm, 16·2 cm, 23·76 cm

4



$\angle ACB = 90 - \theta$ (comp. ∠s, ΔABC)
$\angle ADC = 90 - \theta$ (comp. ∠s, ΔACD)
∴ $\angle BDC = 90 - \theta$ ($\angle ADC = \angle BDC$)
In Δs ABC and CBD,
1 $\angle ADC = \angle CBD$ (both right ∠s)
2 $\angle ACB = \angle BDC$ (proved above)
∴ ΔABC ||| ΔCBD (equal angles)
$\therefore \frac{BD}{BC} = \frac{CD}{AC} = \frac{CB}{AB}$ (matching sides in same ratio)
$\therefore \frac{x}{4} = \frac{y}{5} = \frac{4}{3}$
$\therefore x = \frac{16}{3}, y = \frac{20}{3}$
$= 5\frac{1}{3}$ $= 6\frac{2}{3}$

5 59·73 m^3

11B Working Mathematically

1 23, 29, 31, 35, 37, 41, 43, 44, 46, 47, 49, 52, 53, 55, 56, 58, 59 **2** 70·9% (correct to 1 dec. pl.) **3** 220 g
4 **a** 4 **b** 6 **c** 36 **d** 16 **5** $17\frac{7}{9}$
6 **a** the percentage change for Australia as a whole
b **i** about 1% increase **ii** about 23% decrease **iii** about 32% decrease
c There has been an average drop in the number of fatalities of approximately 16% throughout Australia. All states and territories showed a reduction in the number of fatalities except for Western Australia.

Chapter 12: Further trigonometry

Exercise 12:01

1 **a** 0·94 **b** −0·34 **c** 0·34 **d** −0·94 **2** **a** −2·8 **b** −0·4
3 **a** 0·97 ≑ sin 75° = sin 105° **b** 0·99 ≑ sin 80° = sin 100° **c** 0·34 ≑ cos 70° = −cos 110°
d 0·26 ≑ cos 75° = −cos 105°
4 **a** 60° **b** 70° **c** 20° **d** 75° **e** 90° **f** 105° **g** 98° **h** 110° **i** 134°
5 **a** 20°, 160° **b** 40°, 140°
6 **a** 30° and 150° **b** 15° and 165° **c** 70° and 110° **d** 125° **e** 45° **f** 145°
g 40° or 140° **h** 66° or 114° **i** 18° or 162° **j** 0° or 180° **k** 33° or 147° **l** 90°
7 **a** 0·469 **b** −0·616 **c** −5·671 **d** −0·766 **e** −3·732 **f** 0·996
8 $\tan 90° = \frac{\sin 90°}{\cos 90°} = \frac{1}{0}$. However, a fraction cannot have a zero denominator. **9** 1 when $\theta = 90°$

Prep Quiz 12:02

1 2 **2** 1 **3** 1 **4** 90° **5** 0° **6** 0°, 180° **7** 90° **8** 180° **9** 0·6428 **10** 0·6428

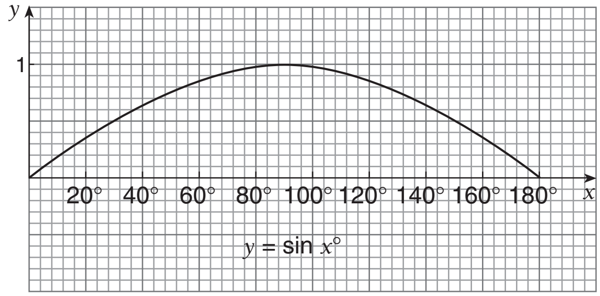
Exercise 12:02

1 **a** 0·216 **b** −0·770 **c** −0·570
2 **a** sin 30° **b** −cos 80° **c** −tan 40° **d** sin 55° **e** sin 39°28′ **f** −cos 12°29′
g −tan 78°57′ **h** −cos 87°10′
3 **a** 44°16′ **b** 135°44′ **c** 83°4′ **d** 96°56′ **e** 120° **f** 135° **g** 25°3′ **h** 110°
i 62°0′ **j** 130° **k** 165°1′ **l** 52°59′ **m** 142°59′ **n** 139°28′ **o** 73°15′ **p** 175°34′

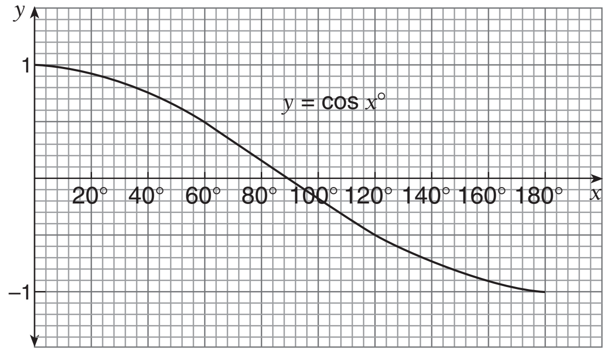
4 **a** 30°, 150° **b** 46°53′, 133°7′ **c** 21°6′, 158°54′ **d** 26°56′, 153°4′ **e** 81°53′, 98°7′ **f** 21°30′, 158°30′
g 45°6′, 134°54′ **h** 3°15′, 176°45′

5

$x°$	0	20°	40°	60°	80°	90°	100°	120°	140°	160°	180°
$\sin x°$	0	0·34	0·64	0·87	0·98	1	0·98	0·87	0·64	0·34	0



6



Exercise 12:03

1 **a** 22·6 **b** 5·2 **c** 4·7 **d** 25·3 (all answers correct to 1 dec. pl.)

2 **a** $\frac{x}{\sin 55°} = \frac{9}{\sin 60°}$, $x = 8·5$ **b** $\frac{x}{\sin 36°} = \frac{12}{\sin 71°}$, $x = 7·5$ **c** $\frac{x}{\sin 120°} = \frac{6}{\sin 27°}$, $\therefore x = 11·4$

3 **a** 5·3 **b** 5·8 **c** 7·7

4 **a** **i** 46° **ii** 45°52′ **b** **i** 55° **ii** 54°44′ **c** **i** 12° **ii** 11°32′ **d** **i** 59° **ii** 59°26′
e **i** 39° **ii** 38°50′

5 **a** $\frac{\sin\theta}{9} = \frac{\sin 63°}{10}$, $\theta = 53°$ **b** $\frac{\sin\theta}{3·6} = \frac{\sin 39°}{4·2}$, $\theta = 33°$ **c** $\frac{\sin\theta}{4·6} = \frac{\sin 66°}{8·9}$, $\theta = 28°$

6 **a** 55°41′ **b** 63°16′ **c** 53°30′ **7** **a** 127° **b** 129° **c** 144°

8 **a** 14·2 cm **b** 8·1 cm **c** 15°31′ **d** 18°33′ **9** **a** 35° **b** 77° **c** 8·4 cm

10 **a** 66° **b** 74° **c** 28 km **11** 37 m

12 *P* is 585 m from the plane; *Q* is 864 m from the plane.

Exercise 12:04

1 **a** 59°, 121° **b** 12°, 168° **c** 41°, 139° **d** 20°, 160°

2 The obtuse angle is 139°. The acute angle is 41°.

3 **a** In ΔBDC,

$$\frac{\sin x°}{4} = \frac{\sin 45°}{2·9}$$

$$\sin x° = \frac{4 \sin 45°}{2·9}$$

In ΔABC,

$$\frac{\sin y°}{4} = \frac{\sin 45°}{2·9}$$

$$\sin y° = \frac{4 \sin 45°}{2·9}$$

b $x° = 103°$
$y° = 77°$

4 **a** 64°9′ or 115°51′ **b** 24°59′ or 155°1′ **c** 37°9′ or 142°51′ **d** 31°45′ or 148°15′

5 **a** 59°0′ or 121°0′ **b** 46°12′ **c** 73°15′ or 106°45′

Exercise 12:05

1 **a** 11·1 **b** 14·1 **c** 16·6 **2** **a** 53° **b** 127° **c** 53° **d** 117°
3 **a** 4·3 **b** 10·0 **c** 6·6 **d** 8·6 **e** 7·2 **f** 13·1
4 **a** 50°59′ **b** 62°58′ **c** 134°37′ **d** 31°55′ **e** 124°2′ **f** 47°17′
5 **a** 9·1 cm **b** 2·8 m **c** 17·4 cm **d** 73°24′ **e** 46°33′
6 18 km **7** 20° **8** 21·9 cm **9** 17·3 km **10** 492 m **11** 19·4 m
12 r = 731 m, s = 510 m (to the nearest metre)

Exercise 12:06

1 **a** 18·0 m^2 **b** 84·0 m^2 **c** 38·7 m^2 **d** 24·3 m^2 **e** 22·9 m^2 **f** 50·9 m^2
2 **a** 21 cm^2 **b** 69 cm^2
3 **a** 26 units^2 **b** 120 units^2 **c** 4 units^2 **d** 1292 units^2 **e** 249 units^2 **f** 2033 units^2
4 **a** 3064 cm^2 **b** 65·0 cm^2 **c** $\frac{325\sqrt{3}}{2}$ cm^2 **5** 17·2 cm

Exercise 12:07

1 **a** sine rule; 8·1 **b** cosine rule; 5·4 **c** sine rule; 8·2 **d** sine rule; 16·6
e cosine rule; 11·7
2 **a** 25°13′ **b** 46°51′ **c** 23°52′ **d** 67°58′ **e** 48°15′ **f** 98°34′
3 **a** 58°40′ **b** 7·7 **c** 1·1 **d** 6·9 **e** 13·2 **f** 60°0′
4 60°21′ **5** 1431 m **6** 139 km (to nearest km) **7** 20 238 m^2
8 **a** 138·8 cm^2 **b** 7·7 cm, 19·8 cm **9** S20°58′W **10** 94°7′; 8·2 m

Exercise 12:08

1 **a** 3·8 km **b** 55° **2** **a** 53·61 m **b** 29·79 m **c** 23·8 m
3 **a** 19·9 m **b** 33·7 m **c** 53·6 m **4** **a** 28·0 m **b** 16·0 m
5 **a** 21·2 m **b** 25·1 m **c** 39·1 m
6 79°6′ **7** 56 km **8** 5·4 km **9** 59·8 m
10 a = 4·6, b = 9·2, c = 3·4, d = 2·2, e = 5·1, f = 1·9

Diagnostic Test 12 Further trigonometry

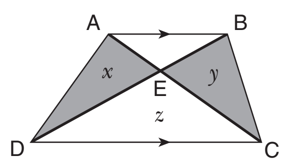
1 **a** 150° **b** 130° **c** 100° **2** **a** sin 40° **b** −cos 40° **c** −tan 40°
4 **a** 128°41′ **b** 14°54′ **c** 91°2′ **4** **a** 11·2 **b** 5·7 **c** 15·4
5 **a** 55° **b** 74° **c** 20° **6** **a** 7·2 **b** 14·5 **c** 28·0
7 **a** 75°31′ **b** 108°3′ **c** 119°10′ **8** **a** 26 cm^2 **b** 35 cm^2 **c** 4 cm^2

12A Revision Assignment

1 **a** 150° **b** 120° **c** 135°
2 **a** 11·9 cm (correct to 1 dec. pl.) **b** 64°17′ (to the nearest minute)
3 **i** θ = 43°, ϕ = 117°, x = 15·6 cm **ii** θ = 137°, ϕ = 23°, x = 6·9 cm
4 **a** 85·6 cm^2 (correct to 1 dec. pl.) **b** 61·9 cm^2 (correct to 1 dec. pl.)
5 **a** 83° (to the nearest degree) **b** 56° or 124°
6 11·5 km

12B Working Mathematically

1 x = 32·5 mm, y = 65 mm
2 (1) *AD* is bisected at *E* by *BC*. *AB* is parallel to *CD*.
(2) *AB* is parallel to *CD* and *BC* bisects *AD* at *E*.
(3) In Δ*ABE*, *AE* is produced to *D* where *AE* = *ED*. From *D*, a line is drawn parallel to *BA*. This line meets *BE* produced at *C*.
3 area = 10 unit^2 **4** 400 cm^2
5

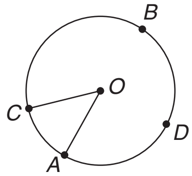


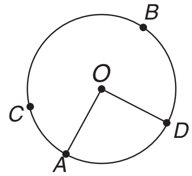
Proof: Let the area of Δ*ADE* = x unit^2
Let the area of Δ*BEC* = y unit^2
Let the area of Δ*DEC* = z unit^2
Now, area of Δ*ADC* = ($x + z$) unit^2
area of Δ*BDC* = ($y + z$) unit^2
But area of Δ*ADC* = area of Δ*BDC*
(Both Δs have same base and perp. ht.)
$\therefore x + z = y + z$
$\therefore x = y$
∴ area of Δ*ADE* = area Δ*BEC*.

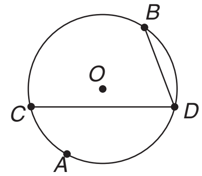
6 58·7 unit^2 (correct to 1 dec. pl.)

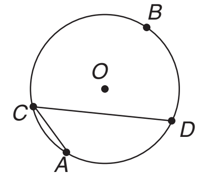
Chapter 13: Circle geometry

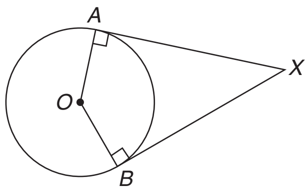
Exercise 13:01

1 a


b


c


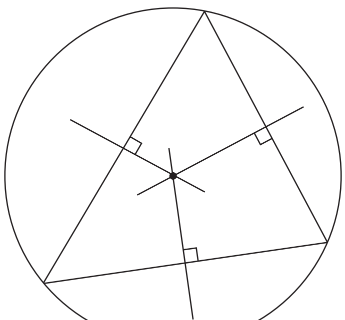
d


2 a i ∠*AYX* **ii** ∠*BXY* **b i** ∠*XAY*, ∠*XBY* **ii** ∠*AXB*, ∠*AYB*
3 a ∠*ADE*, ∠*ABE* **b** ∠*AOC*, ∠*ADC*, ∠*ABC* **c** 3 **d** ∠*AED* **e** *AC*
4 a i ∠*BDC*, ∠*BFC* **ii** ∠*BAC*, ∠*BEC* **b i** ∠*BAC*, ∠*BEC* **ii** ∠*BDC*, ∠*BFC*
5 a 90° **b**


6 See text.

Exercise 13:02

1 a $MD = \frac{1}{2}CD$ (perp. from *O* bisects chord) ∴ $MD = 17{\cdot}5$ **b** $MF = EM$ (perp. from *O* bisects chord) ∴ $MF = 27$ cm
c $GH = 2GM$ (perp. from *O* bisects chord) ∴ $GH = 36$ m **d** $MH = 9$ cm, $JK = 18$ cm
e ∠*RMO* = 90° **f** $TM = 9{\cdot}3$ m, $MU = 9{\cdot}3$ m
2 a Both *MP* and *NP* are perpendicular bisectors of chords; *P* is the centre as both *MP* and *NP* pass through the centre, *P*.
b See question **2a**. **c** See question **2a**.

3


4 a 3 cm, 6 cm **b** 5·3 m, 10·6 m **c** 5 mm, 10 mm
d 10 cm, 16 cm **e** 7 cm, 7·6 cm **f** 9 m, 4·4 m
g 5·3 cm **h** 6·4 cm
5 a i 6·5 cm **ii** 6·7 cm **iii** 13·0 cm
b 12·2 cm **c** 9·2 cm

6 a In Δs *POC* and *QOC*
PO = *QO* (equal radii)
PC = *QC* (equal radii)
OC is common.
∴ Δ*POC* ≡ Δ*QOC* (SSS)

b ∴ ∠*POC* = ∠*QOC* (corresp. ∠s in cong. Δs)

c In Δs *PON* and *QON*
∠*POC* = ∠*QOC* (proved in **b**)
ON is common.
PO = *QO* (equal radii)
∴ Δ*PON* ≡ Δ*QON* (SAS)

d ∴ *PN* = *QN* (corresp. sides in cong. Δs)
∴ *N* bisects *PN*.
∠*PNO* = ∠*QNO* (corresp. ∠s in cong. Δs)
Since ∠*PNO* + ∠*QNO* = 180° (straight angle)
Then ∠*PNO* = ∠*QNO* = 90°
∴ *PQ* ⊥ *OC*

Investigation 13:02

1 B **2** G **3** H **4** J

Exercise 13:03

1 a $OM = 8$ m (equal chords are the same distance from the centre) **b** $x = 8{\cdot}5$ (as in **a**)
c $ON = 3{\cdot}5$ cm (as in **a**) **d** $CD = 11{\cdot}5$ m (chords equidistant from the centre are equal)
e $FG = 20$ m (as in **d**) **f** $KL = 13{\cdot}8$ m (as in **d**)

2 **a** $\angle COD = 55°$ (equal chords subtend equal angles at centre) **b** $FE = 8$ m (as in **a**)
c $\angle JOK = 51°$ (as in **a**) **d** $WX = 12$ m (perp. from O bisects WX) $\therefore ON = 5$ m (as in **a**)
e $QP = 14$ m (as in **1d**) $\therefore QM = 7$ m (perp. from O bisects QP) **f** $\angle BOD = 125°$ (as in **a**)
3 **a** 120° **b** 90° **c** 60° **d** $51\frac{3}{7}°$ **e** 45° **f** 36°
4 Your answers will look like **3a**, **3b**, **3c** and **3e**.
5 See text.

Prep Quiz 13:04

1 30 **2** 120 **3** 60 **4** 50 **5** 32 **6** 50 **7** 40 **8** 94 **9** 50 **10** 2

Exercise 13:04

1 **a** 44° **b** 25° **c** 34° **d** 39° **e** 39° **f** 39° **g** 96 **h** 118 **i** 150
2 **a** $a = 33, b = 33$ **b** $c = 62, d = 62$ **c** $e = 59, f = 59$ **d** $g = 39$ **e** $h = 58, i = 58$ **f** $k = 98$
g $m = 90, n = 90$ **h** $p = 180$ **i** $t = 90$
3 **a** **i** 30° **ii** 60° **iii** 35° **iv** 70° **v** 65° **vi** 130°
b **i** 30° **ii** 60° **iii** 35° **iv** 70° **v** 65° **vi** 130° **vii** yes
c 120 **d** 125 **e** 204

Exercise 13:05

1 **a** $a = 90$ (angle in a semicircle) **b** $b = 90$ (angle in a semicircle) **c** $c = 90$ (angle in a semicircle)
d $\angle ABC = 90°$ (angle in a semicircle) $\therefore d + 147 = 180$ (angle sum of Δ) $\therefore d = 33$ **e** $e = 20$
f $\angle JHG = 90°$ (angle in a semicircle) $\therefore f + 162 = 180$ (angle sum of Δ) $\therefore f = 18$
g $g + 76 = 180$ (cyclic quad.) $\therefore g = 104$
h $h + 91 = 180$ (cyclic quad.) $\therefore h = 89$, $i + 78 = 180$ (cyclic quad.) $\therefore i = 102$
i $j + 82 = 180$ (cyclic quad.) $\therefore j = 98$, $k + 80 = 180$ (cyclic quad.) $\therefore k = 100$
2 **a** 53 **b** 127 **c** $a = 53, b = 127$ **d** $c = 70, d = 110$ **e** $e = 75, f = 105$
f $g = 45, h = 135$ **g** $m = 50, n = 40$ **h** $m = 46, n = 44$
3 **a** yes (opp. $\angle$s supplementary) **b** no **c** yes (as in **a**) **d** no **e** yes (as in **a**) **f** no
4 **a** $a = 110$ (cyclic quad.), $b = 70$ (adj. supp. angles) **b** $c = 70$ (cyclic quad.), $d = 110$ (adj. supp. angles)
c $e = 85$ (cyclic quad.), $f = 95$ (adj. supp. angles) **d** $g = 86$ (cyclic quad.), $h = 94$ (adj. supp. angles)
e $j = 60$ (cyclic quad.), $k = 120$ (adj. supp. angles)

Prep Quiz 13:06

1 6 m **2** AT **3** OP **4** DT (or DE) **5** radius **6** 75 **7** 56 **8** 20 **9** 4 **10** 13

Exercise 13:06

1 **a** $a = 90$ (radius $OT \perp$ tangent TP) **b** $b = 14$ (equal tangents from P)
c $\angle OPT = 90°$ (radius $OT \perp$ tangent TP) $\therefore c + 143 = 180$ (angle sum of Δ) $\therefore c = 37$
d $d = e$ ($TP = WP$, equal tangents from P) $\therefore 2e + 66 = 180$ (angle sum of Δ) $\therefore e = d = 57$
e $f + 36 = 90$ (radius $OT \perp$ tangent TP) $\therefore f = 54$
f $h = 90$ (radius $OT \perp$ tangent TP), $g = 25$ (corresponding to $\angle OTP$ about axis of symmetry OP)
g $k = 90$ (radius $OT \perp$ tangent TP), $m = 90$ (radius $OW \perp$ tangent PW), $n = 132$ (angle sum of a quadrilateral)
h $p = 8{\cdot}2$ (equal tangents from P), $q = 64$ ($\Delta\, PTW$ is isosceles)
2 **a** $x = 39$ **b** $y = 120$ **c** $a = 48$ **d** $e = 105, f = 105$
3 **a** $a = 90, b = 25, c = 25, d = 65$ **b** $e = 57, f = 57, g = 123$ **c** $g = 35, h = 35, k = 110$
d $k = 90, m = 65$ **e** $n = 90, r = 91, v = 62$ **f** $w = 34, x = 34, y = 112, z = 56$
g $a = 58, b = 64, c = 32$ **h** $d = 67, e = 23$ **i** $f = 38, g = 52$
4 **a** $a = 8$ **b** $b = 15$ **c** $c = 90,\ d = \sqrt{51},\ e = \sqrt{89}$ **d** $f = 8$ **e** $t = 5$
5 **a** Let $\angle BTP$ be $x°$.
$\therefore \angle BAT = x°$ (angle in the alternate segment)
$\angle BOT = 2x°$ (angle at centre is twice angle at circum. on same arc)
$\therefore \angle BOT = 2\ \angle BTP$ (Q.E.D.)

b $\angle ABT + \angle RBA + \angle PBT = 180°$
($\angle RBP$ is a straight angle)
But $\angle ATQ = \angle ABT$ (angle in the alternate segment)
$\therefore \angle ATQ + \angle RBA + \angle PBT = 180°$
(Q.E.D.)

c Let $\angle BAT$ be $a°$.
$\therefore \angle BTP = a°$ (angle in the alternate segment)
$\angle PBT = a°$ (angle in the alternate segment)
$\therefore \angle BPT + a° + a° = 180°$ (angle sum of a triangle)
$\therefore \angle BPT = 180° - 2a°$
$= 180° - 2\ \angle BAT$ (Q.E.D.)

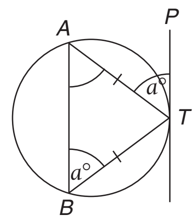
6 Let $\angle PTA$ be $a°$

$\therefore \angle ABT = a°$ (angle in the alternate segment)

$\angle BAT = a°$ ($\angle BAT$ and $\angle ABT$ are base angles of isos. ΔBAT)

$\therefore \angle PTA = \angle BAT$ (both $a°$)

$\therefore AB \parallel PT$ (alternate angles are equal) (Q.E.D.)



7 **a** $\angle ATO = 90°$ (angle between tangent and radius)

$\angle ATC = 90°$ (angle between tangent and radius)

$\therefore \angle OTC = 180°$ ($\angle ATO + \angle ATC$)

$\therefore OC \perp AB$

b (first two lines the same as **a**)

$\therefore \angle ATO = \angle ATC$ (same angle)

$\therefore OC \perp AB$

8 $PT = PA$ (equal tangents from P)

$PT = PB$ (equal tangents from P)

$\therefore PA = PB$

9 $PA = PC$ (equal tangents from P)

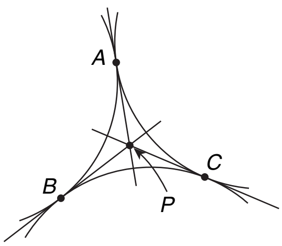
$\angle APO = \angle CPO$ (given)

$\therefore PO$ is axis of symmetry passing through the centre O.

Similarly for QO and RO

$\therefore$ Angle bisectors are concurrent at the centre O.

10



The tangents from A and B will meet at a point P with $AP = BP$ (equal tangents from an external point).
But from P, AP will also equal CP and CP will equal BP.
Since P is the common external point, the three common tangents are concurrent.

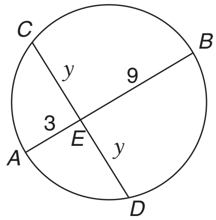
Exercise 13:07

1 **a** $b = 18$ **b** $c = 12$ **c** $d = 16$ **d** $e = 6$ **e** $f = 1$ **f** $h = 4$ **g** $k = 6$ **h** $r = 12$

i $t = 12$ **j** $a = 15,\ b = 10\frac{2}{3}$ **k** $c = 15,\ d = 13\frac{1}{3}$ **l** $x = \sqrt{32} = 4\sqrt{2}$

2 **a** 6 m **b** 6·9 cm **c** 11·8 m **d** 37·5 m **e** $19\frac{1}{15}$ m **f** 11·2 cm **g** 13·4 cm

3



$y^2 = 3 \times 9$

$y = \sqrt{27}$ or $3\sqrt{3}$

$\therefore CE = 3\sqrt{3}$ m

(Note: Consider only positive solutions as lengths cannot be negative.)

4 $PC^2 = AC \,.\, CB$ (square of the tangent equals the product of the intercepts)

$TC^2 = AC \,.\, CB$ (as for line 1)

$\therefore PC^2 = TC^2$

$\therefore PC = TC$ (lengths are positive)

Prep Quiz 13:08

1 complementary angles **2** supplementary angles **3** vertically opposite angles **4** 360°
5 corresponding angles **6** alternate angles **7** co-interior angles **8** alternate angles, lines parallel
9 isosceles Δ since radii are equal **10** angle sum of a Δ

Exercise 13:08

1 **a** $\angle OBC = 43°$ (alternate to $\angle AOB$, $AO \parallel BC$)

$\angle OCB = 43°$ (ΔOBC is isosceles)

$a + 86 = 180$ (angle sum of ΔOBC)

$\therefore a = 94$

b $\angle AOB = 50°$ (alternate to $\angle OBC$, $AO \parallel BC$)

$\angle ACB = 25°$ (angle at circumference is half angle at centre)

$\therefore b = 25$

c $\angle TAO = 30°$ (ΔOTA is isosceles)

$\angle OTP = 90°$ (radius $OT \perp$ tangent TP)

$\therefore \angle ATP = 120°$

$d + 150 = 180$ (angle sum of ΔATP)

$\therefore d = 30$

d $\angle COB = 118°$ (angle at centre is twice angle at circumference)

$\angle CBO = e°$ (ΔOBC is isosceles)

$2e + 118 = 180$ (angle sum of ΔOBC)

$\therefore e = 31$

e $\angle ACB = 35°$ (angle at circumference is half angle at centre)
$e + 35 = 180$ ($\angle ACD$ is a straight angle)
$\therefore e = 145$

f $\angle CDB = 90°$ (only the bisector of AB which is perpendicular to AB will pass through O)
$f + 150 = 180$ (angle sum of ΔBCD)
$\therefore f = 30$

g $\angle TOW = 120°$ (angle at centre is twice angle at circumf.)
$\angle OTP = 90°$ (radius $OT \perp$ tangent TP)
$\angle OWP = 90°$ (radius $OW \perp$ tangent WP)
$g + 300 = 360$ (angle sum of quad. $WOTP$)
$\therefore g = 60$

h $\angle OTP = 90°$ (radius $OT \perp$ tangent TP)
$\therefore \angle OTA = 30°$ ($120° - 90°$)
$\therefore \angle OAT = 30°$ (isosceles ΔOAT)
$h + 60 = 180$ (angle sum of ΔOAT)
$\therefore h = 120$

i $\angle DOC = 50°$ (angles subtended by equal chords at the centre are equal)
$\therefore k + 140 = 360$ (angles at a point)
$\therefore k = 220$

j $\angle FJH = 63°$ (adjacent supplementary angle to $\angle FJK$)
$\angle FGH = 117°$ (cyclic quad. $FGHJ$)
$\therefore m = 117$

k $\angle NQL = 108°$ (cyclic quad. $LMNQ$)
$\angle NQR = 72°$ (adjacent supplementary angles)
$\therefore n = 72$

l $\angle ABC = 90°$ (angle in a semicircle)
$\therefore \angle BCD = 90°$ (cointerior to $\angle ABC$, $BA \parallel CD$)
$\therefore r = 90$

m $\angle PTW = \angle TWP$ (isosceles ΔTPW formed by equal tangents from P)
$\therefore 2a + 44 = 180$ (angle sum of ΔPTW)
$\therefore a = 68$
$\angle AWP = 90°$ (radius $OW \perp$ tangent PW)
$\therefore b = 22$ ($90° - 68° = 22°$)
$\angle ATW = 90°$ (angle in a semicircle)
$\therefore x + 112 = 180$ (angle sum of ΔATW)
$\therefore x = 68$

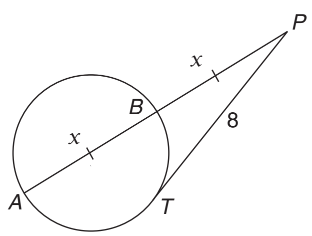
n $\angle OTP = 90°$ (radius $OT \perp$ tangent TP)
$\therefore a = 40$ ($90° - 50° = 40°$)
$\angle OBT = 40°$ (ΔOBT is isosceles)
$b + 80 = 180$ (angle sum of ΔOBT)
$\therefore b = 100$
$\angle BAT = 50°$ (angle at circumference is half angle at centre)
$\therefore y = 50$

2 a $\angle THF = e°$ (angle in alternate segment)
$\angle THG = e° + 40°$
$\angle GFQ = \angle THG$ (ext. angle of cyclic quad.)
$\therefore 85 = e + 40$
$\therefore e = 45$

b $AE = x$ cm (perp. from centre bisects chord AB)
$\therefore x^2 = 8 \times 2$ (products of intersecting chords)
$x = 4$
In ΔOEB, $OE^2 + EB^2 = OB^2$ (Pythagoras' theorem)
$\therefore 5^2 = y^2 + x^2$
$25 = y^2 + 16$
$\therefore y = 3$

c $\angle TAC = 30°$ (angle in alternate segment)
$\angle CAB = 65°$ (angle at circum. is half the angle subtended at centre on same arc)
$\angle BAT = \angle TAC + \angle CAB$
$\therefore x = 30 + 65$
$= 95$

3 Let AB and BP be x cm.
$\therefore 2x \times x = 64$
$\therefore x = \sqrt{32}$ (since $x > 0$)
$= 4\sqrt{2}$



4 a $\angle PTN = 90°$ (radius $NT \perp$ tangent PT)
$\angle OSN = 90°$ (corresp. to $\angle PTN$, $OS \parallel PT$)

b $\angle PTS = 90°$ ($\angle PTN = 90°$, $\angle NTS = 180°$)
$\angle OPT = 90°$ (radius $OP \perp$ tangent PT)
$\angle SOP = 90°$ (corresp. to $\angle OPT$, $OS \parallel PT$)
$\therefore$ $OPTS$ is a rectangle (all angles 90°)

c ΔOSN is a right-angled Δ.
$SN = 11$ cm, $ON = 15$ cm
$OS^2 + SN^2 = ON^2$
$OS^2 + 121 = 225$
$\therefore OS = \sqrt{104}$
$\therefore PT = \sqrt{104}$ (PT and OS are opposite sides of a rectangle)

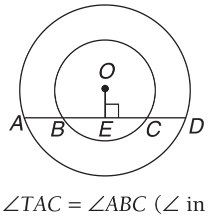
5 Let $\angle CTQ = x°$.
$\therefore \angle CAT = x°$ (angle in alternate segment)
$\angle PTD = x°$ (vert. opp. to $\angle CTQ$)
$\therefore \angle DBT = x°$ (angle in alternate segment)
$\therefore \angle CAT = \angle DBT$
$\therefore AC \parallel DB$ (alternate angles are equal) (Q.E.D.)

6 Let BT be x m.
$\therefore BP = x$ m (equal tangents)
Join OT and OP.
$OT \perp AB$ (radius perp. to tangent)
$\therefore AT = x$ m (perp. from centre to chord bisects the chord)
$\therefore AB = 2x$ m
Similarly $CB = 2x$ m
$\therefore AB = CB$ (Q.E.D.)

7 $\angle PSR = \angle PTR$ (opp. $\angle$s in parallelogram are equal)
$\angle PSR = \angle RQT$ (ext. angle of a cyclic quad.)
$\therefore \angle RQT = \angle QTR$
$\therefore \Delta RQT$ is isosceles.

8 $\angle ECA = \angle ABC$ ($\angle$ in alt. seg't)
$\angle ABC = \angle BAC$ (base $\angle$s in isos. Δ)
$\therefore \angle BAC = \angle ECA$
$\therefore AB \parallel ED$ (equal alt. $\angle$s)

9



Let OE be perp. bisector of both chords.
Then $AE = DE$
and $BE = CE$
$\therefore AE - BE = DE - CE$
$\therefore AB = CD$

10 a In Δs AEB and ACD **i** $\angle A$ is common.
ii $\angle AEB = \angle ACD$ (ext. angle of cyclic quad. equals interior opp. angle). $\Delta AEB \,|||\, \Delta ACD$ (equiangular) (Q.E.D.)
b But $\angle AEB = 90°$ (angle in a semicircle)
$\therefore \angle ACD = 90°$ (corresponding angles of similar triangles)

11 $\angle TAC = \angle ABC$ ($\angle$ in alt. seg't)
$\angle TAC = \angle BPC$ (alt. $\angle$s, $AT \parallel BP$)
$\therefore$ In Δs ABC and ABP
$\angle ABC = \angle BPC$ (shown above)
$\angle A$ is common.
$\therefore \angle ABP = \angle ACB$ (angle sum of Δ)

Fun Spot 13:08

Dots	*Chords*	*Sections*
5	10	16
6	15	31

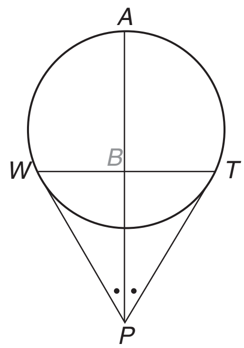
Diagnostic Test 13 Circle geometry

1 a $\angle AOB$ **b** $\angle ADB$ or $\angle ACB$ **c** $\angle BDC$
2 a $AM = 4{\cdot}5$ cm (perp. from O bisects AB)
b The perp. bisector of a chord passes through the centre (as it is an axis of symmetry).
c $CD = 36$ m (perp. from O bisects CD)
3 a $OM = 10$ m **b** $OE = 15$ cm **c** $FG = 60$ m
4 a $ON = 8$ m (equal chords are the same distance from the centre)
b $GH = 11$ km (chords equidistant from the centre are equal) $\therefore GN = 5{\cdot}5$ km (perp. from centre bisects the chord)
c $YZ = 14$ m (chords equidistant from the centre are equal)
5 a $\angle AOB = 69°$ (angles subtended at centre by equal chords)
b $EF = 1{\cdot}1$ m (equal angles at the centre are subtended by equal chords at circumference)
c $\angle BOC = 72°$, $\angle BOD = 144°$
6 a $a = 108$ **b** $b = 86$ **c** $c = 110$ **7 a** $a = 36$ **b** $b = 50$, $c = 50$ **c** $d = 99$
8 a $a = 90$ **b** $b = 37$ **c** $c = 90$, $d = 90$
9 a $a = 93$, $b = 110$ **b** $c = 104$, $d = 115$ **c** $e = 100$, $f = 78$, $g = 102$
10 a $a = 90$ **b** $c = 24$ **c** $e = 41$ **11 a** $a = 8$, $b = 67$ **b** $c = 67$, $e = 23$ **c** $d = 17$
12 a $a = 80$ **b** $x = 64$ **c** $y = 62$ **13 a** $x = 4$ **b** $f = 19$ **c** $r = 5$

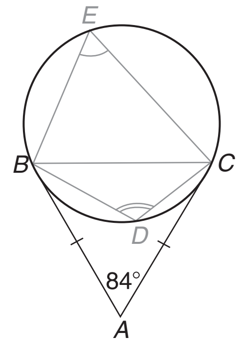
13A Revision Assignment

1 a $a = 31$, $b = 59$ **b** $c = 142$, $d = 109$ **c** $e = 65$, $f = 53$ **d** $x = \sqrt{85}$ **e** $y = 7{\cdot}2$
f $r = 6{\cdot}89$, $t = 6{\cdot}6$ **g** $v = 45$, $w = 35$, $x = 100$ **h** $a = 0{\cdot}9$, $b = 110$ **i** $c = 65$, $d = 65$, $e = 50$
j $f = 45$, $g = 90$, $i = 45$, $j = 45$ **k** $k = 152$ **l** $n = 41$, $m = 68$ **m** $x = \sqrt{32} = 4\sqrt{2}$
n $m = 154$, $n = 77$
2 a i $WP^2 = AP \,.\, PB$ (square of tangent = product of intercepts)
$PT^2 = AP \,.\, PB$ (same reason)
$\therefore WP = PT$
ii $AB = 11{\cdot}2$ (to 3 sig. figs.)

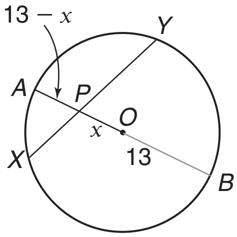
b Join WT to meet PA at B.
In Δs WPB and TPB
i BP is common
ii $\angle WPB = \angle TPB$ (PB bisects $\angle WPT$)
iii $WP = PT$ (equal tangents)
$\therefore \Delta WPB \equiv \Delta TPB$ (SAS)
$\therefore WP = BT$ (corres. sides)
and $\angle WBP = \angle TBP$ (corres. angles of congruent Δs)
$\therefore \angle WBP = 90°$ ($\angle WBT = 180°$)
$\therefore PA$ is the perp. bisector of chord WT.
$\therefore PA$ passes through the centre of the circle. (Q.E.D.)



c $\angle ABC = 48°$
(isos. ΔABC)
$\angle BEC = 48°$
(angle in the alternate segment)
$\angle BDC = 132°$
(opp. angles of cyclic quad. are suppl.)



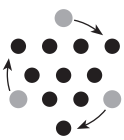
d



Let OP be x cm.
$\therefore AP = (13 - x)$ cm
and $PB = (13 + x)$ cm
$AP \,.\, PB = XP \,.\, PY$
$(13 - x)(13 + x) = 25$
$\therefore x = 12$
$\therefore OP$ is 12 cm

13B Working Mathematically

1

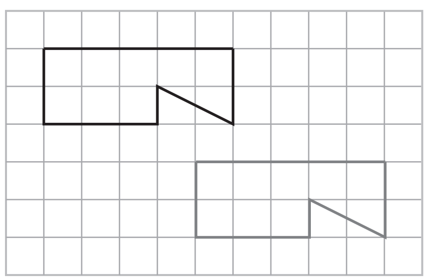


2 4
3 2 cm, 6 cm, 7 cm
4 10
5 126 984
6 **a** 91 **b** 1392 **c** 18 893 **d** $(n - 99) \times 3 + 90 \times 2 + 9$

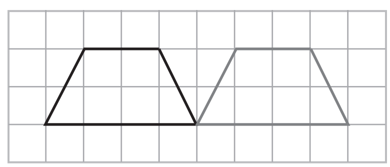
Chapter 14: Transformations and Matrices

Exercise 14:01

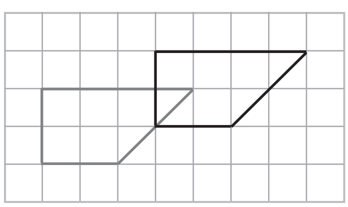
1 **a**



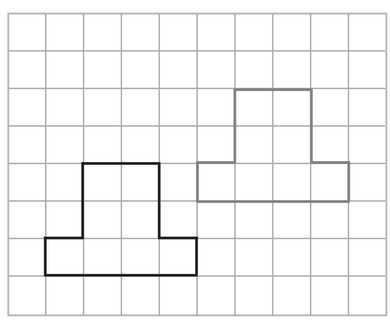
b



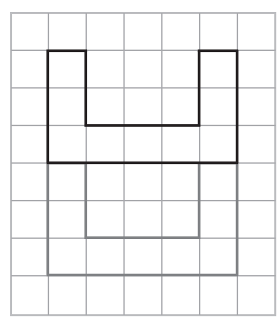
c



d



e



1 **a** Horizontal translation of 5 units, $\begin{pmatrix} 5 \\ 0 \end{pmatrix}$
b Horizontal translation of 3 units and vertical translation of −3 units, $\begin{pmatrix} 3 \\ -3 \end{pmatrix}$

c Horizontal translation of −4 units and vertical translation of −3 units, $\begin{pmatrix}-4\\-3\end{pmatrix}$

d Horizontal translation of 3 units and vertical translation of −3 units, $\begin{pmatrix}3\\-3\end{pmatrix}$

e Horizontal translation of −5 units, $\begin{pmatrix}-5\\0\end{pmatrix}$

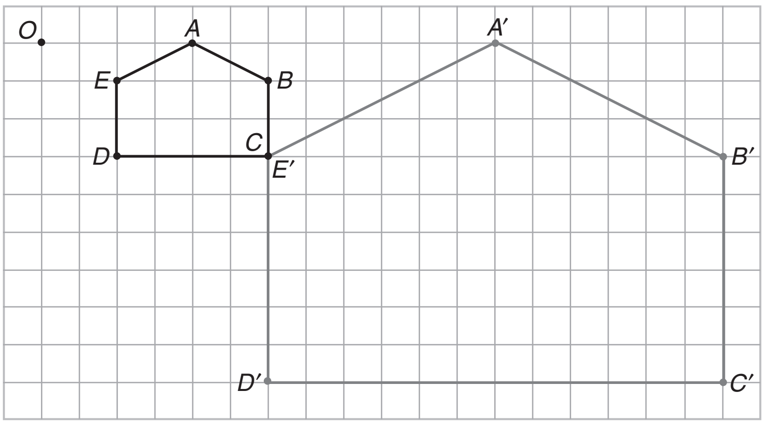
3 $A(-1, 6)$, $B(-1, 9)$, $C(1, 11)$ and $D(2, 4)$

4 a $P(-1, 0)$, $Q(1, -4)$, $R(2, -7)$, $S(4,-2)$, $T(3, 2)$ **b** $\begin{pmatrix}-1\\5\end{pmatrix}$

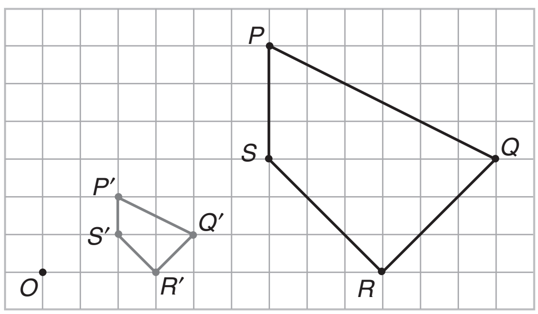
5 $\begin{pmatrix}-9\\-2\end{pmatrix}$

Exercise 14:02

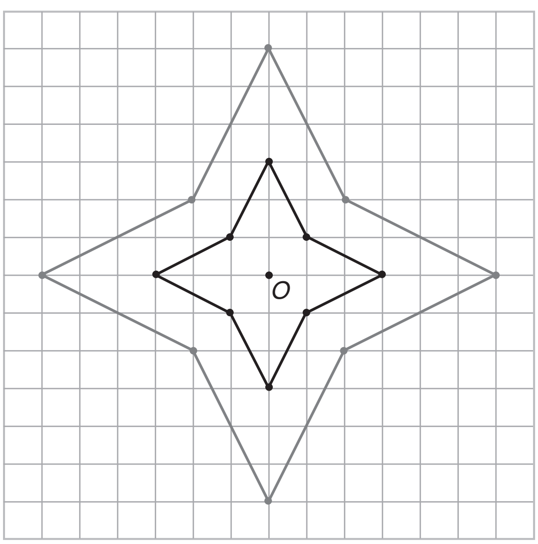
1 a



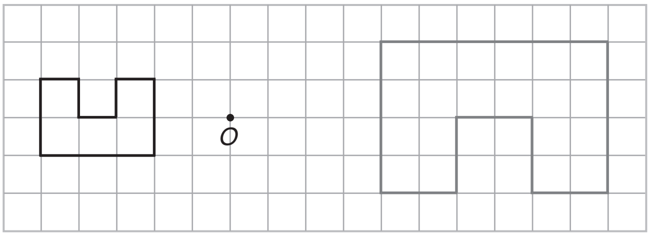
b



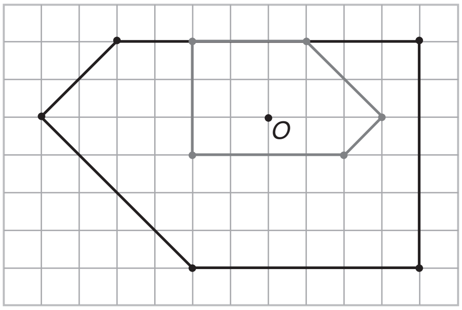
c



d



e



2 **a** 2 **b** −2 **c** $\frac{1}{3}$ **d** −3 **e** $\frac{1}{3}$

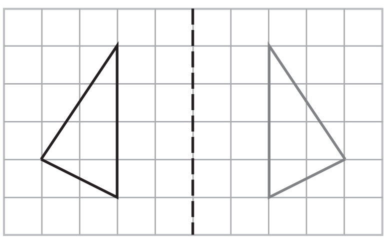
3 **a** (0, 0), 2 **b** (1, 1), −1 **c** (0, 3), 3 **d** (1, 3), −2 **e** (2, 2), $1\frac{1}{2}$

4 $P'(-8, 2)$ $Q'(4, 2)$ $R'(7, -10)$ $S'(-14, -10)$

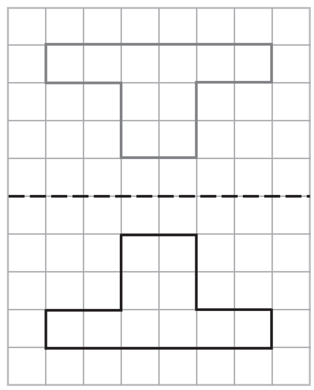
5 (−6, 11)

Exercise 14:03

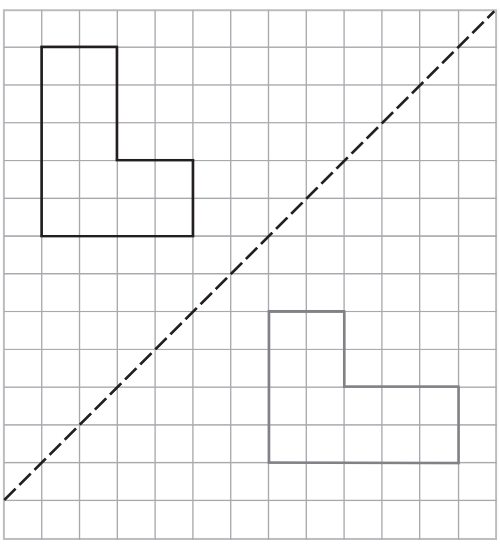
1 **a**



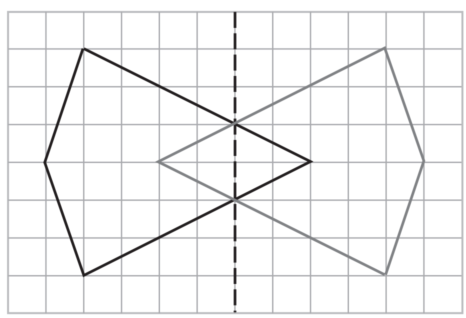
b



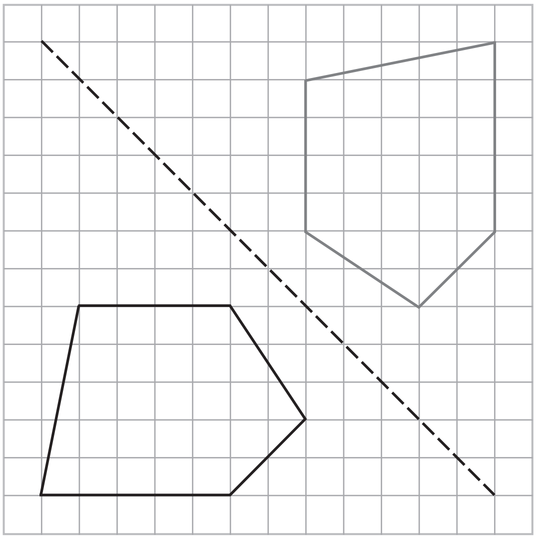
c



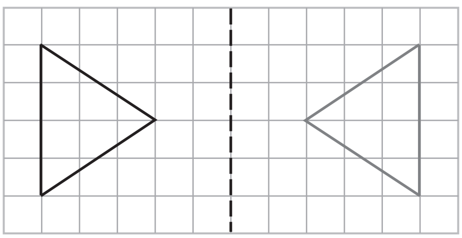
d



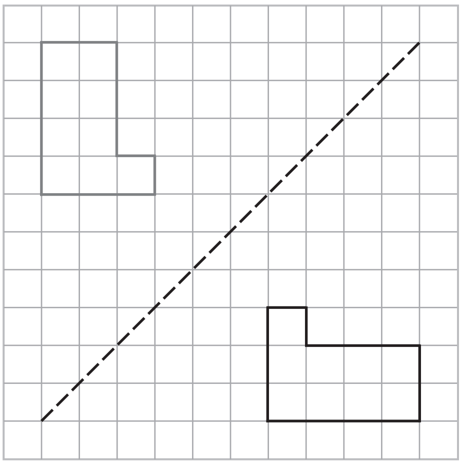
e



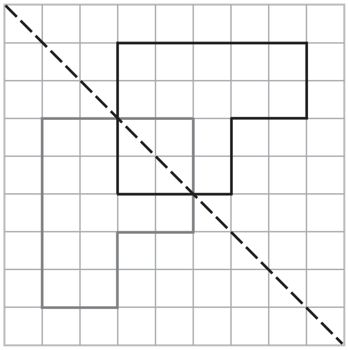
2 a



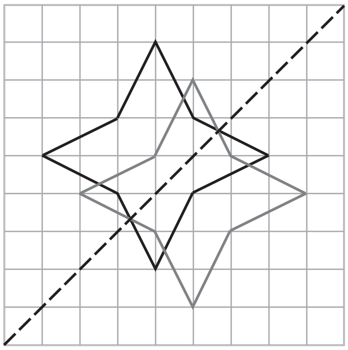
b



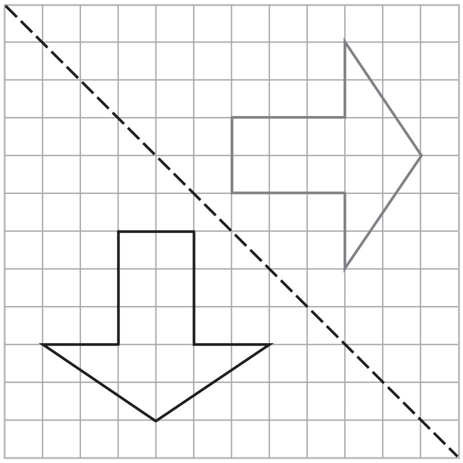
c



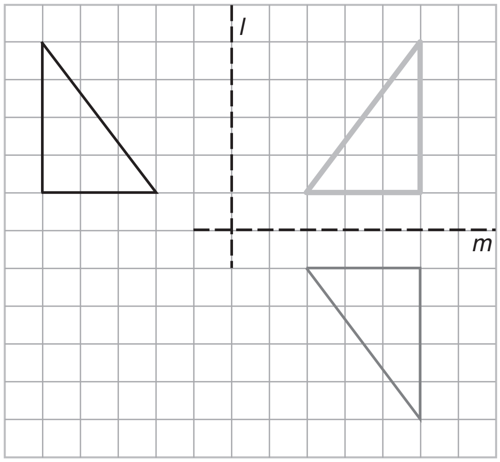
d



e

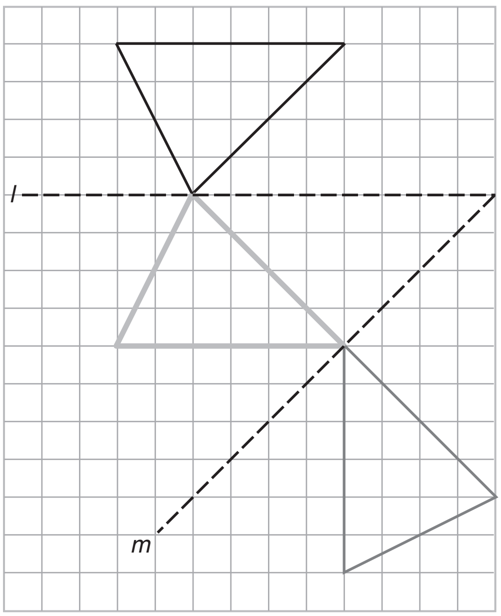


3 **a**



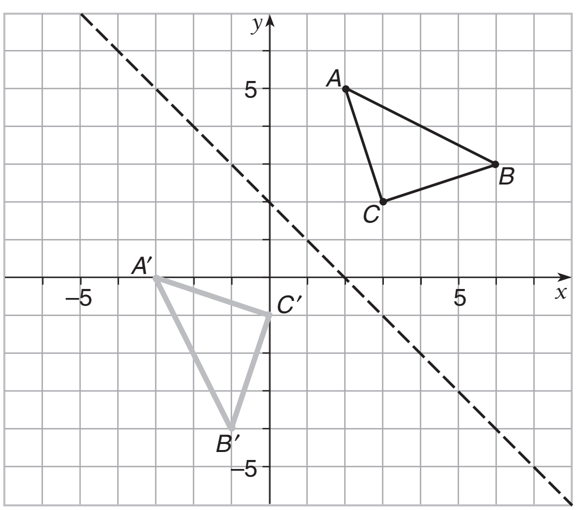
b The original shape would be obtained.

4 **a**

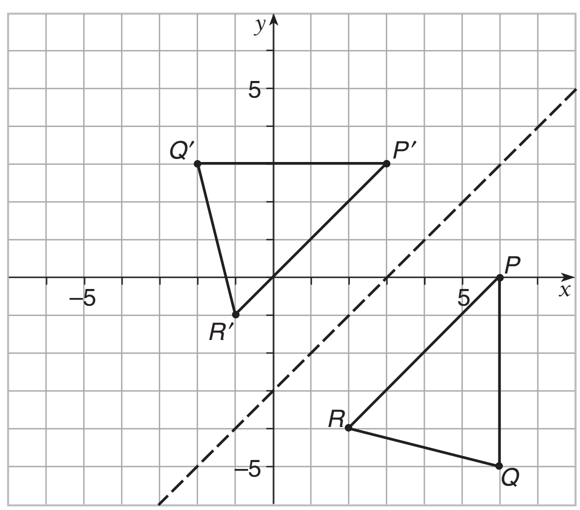


b No

5 $A'(-5, 0)$, $B'(-1, -4)$, $C'(0, -1)$

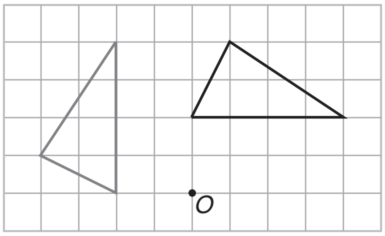


6 $y = x - 3$

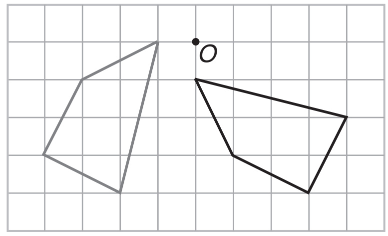


Exercise 14:04

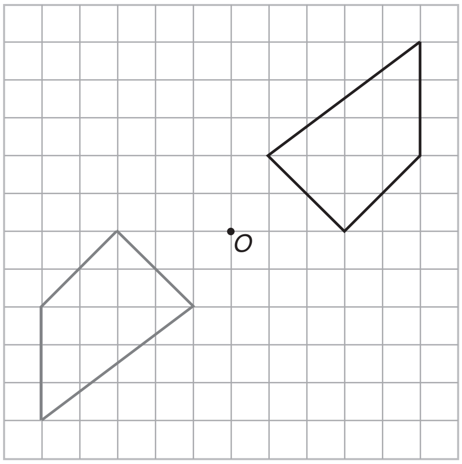
1 a



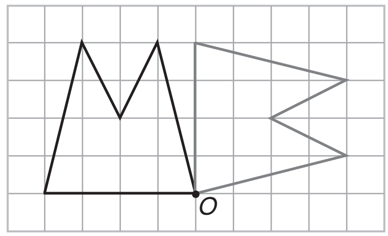
b



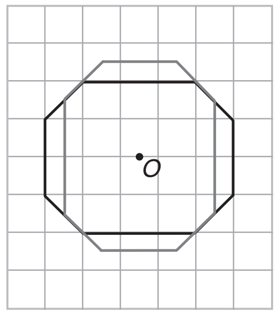
c



d



e



2 a −90° about *O* **b** 180° about *O* **c** −90° about *O* **d** 90° about *O* **e** −135° about *O*

3 a 4 **b** 2 **c** 5 **d** 8 **e** 3 **f** 1

4 a A′(−1, 4), B′(−4, 3), C′(−2, 1) **b** A′(−2, 3), B′(−5, 2), C′(−3, 0)

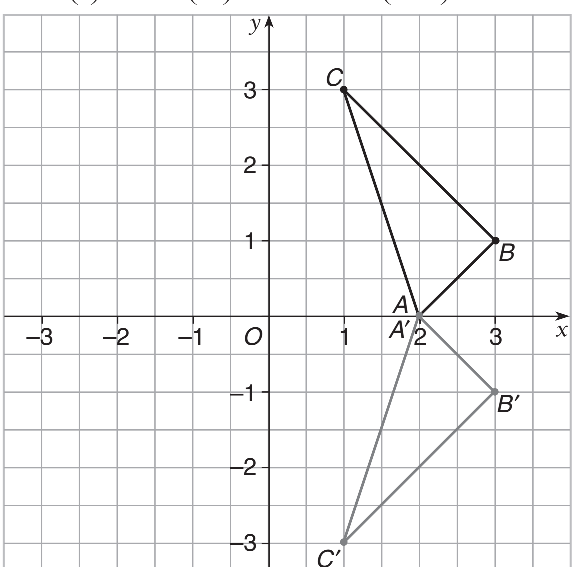
5 a −90° **b** (1, 1), it is the centre of rotation

6 a 90° **b** (3, 1), it is the centre of rotation

Exercise 14:05

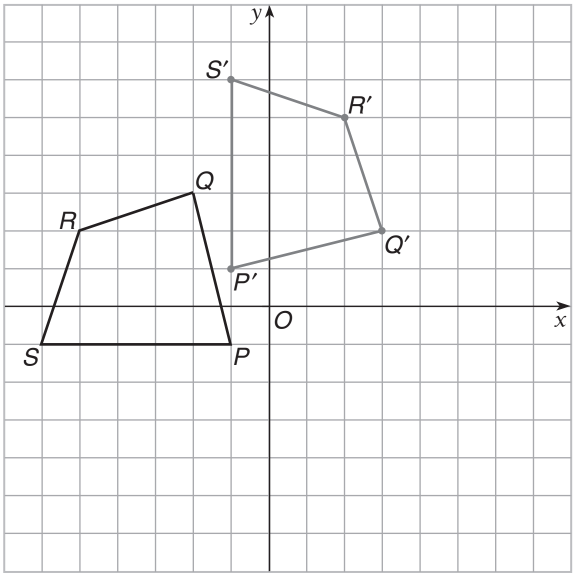
1 a $\overrightarrow{OX} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $\overrightarrow{OY} = \begin{pmatrix} 0 \\ -1 \end{pmatrix}$ **b** $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ **c** $A'(2, 0)$, $B'(3, -2)$, $C'(1, -3)$

d



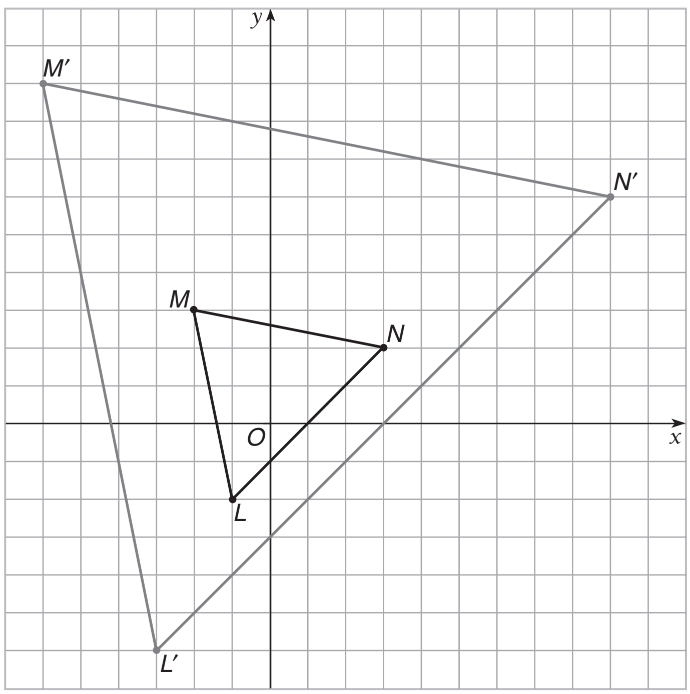
2 **a** $\overrightarrow{OX} = \begin{pmatrix} 0 \\ -1 \end{pmatrix}$, $\overrightarrow{OY} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ **b** $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ **c** $P'(-1, 1)$, $Q'(3, 2)$, $R'(2, 5)$, $S'(-1, 6)$

d



3 **a** $\overrightarrow{OX} = \begin{pmatrix} 3 \\ 0 \end{pmatrix}$, $\overrightarrow{OY} = \begin{pmatrix} 0 \\ 3 \end{pmatrix}$ **b** $\begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix}$ **c** $L'(-3, -6)$, $M'(-6, 9)$, $N'(9, 6)$

d

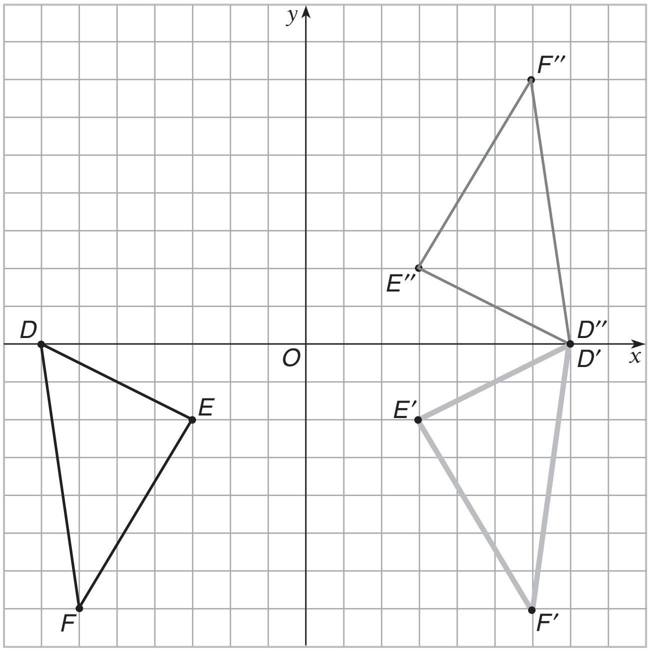


4 **a** **i** $\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$ **ii** $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ **iii** $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$

b **i** $D'(7, 0)$, $E'(3, -2)$ and $F'(6, -7)$ **ii** $D''(7, 0)$, $E''(3, 2)$ and $F''(6, 7)$ **iii** $D'''(7, 0)$, $E'''(3, 2)$ and $F'''(6, 7)$

c a rotation of 180° about O

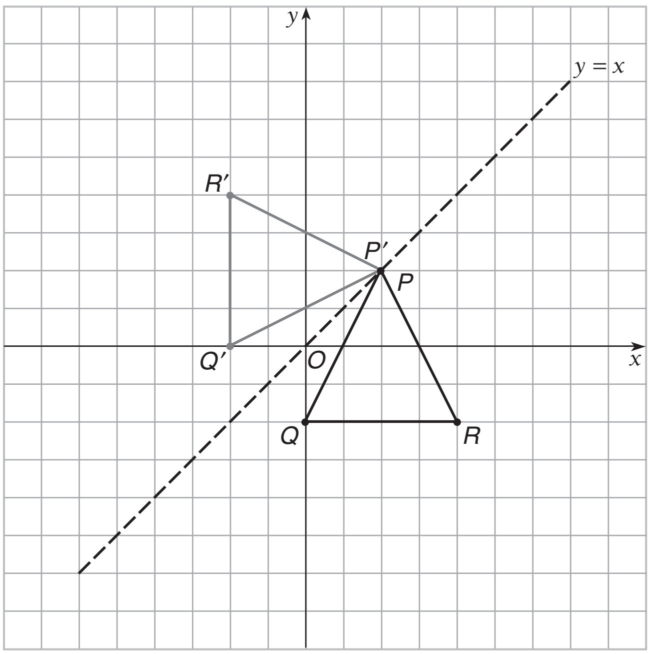
d



5 a $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

b P′(2, 2), Q'(−2, 0) and R′(−2, 4)

c

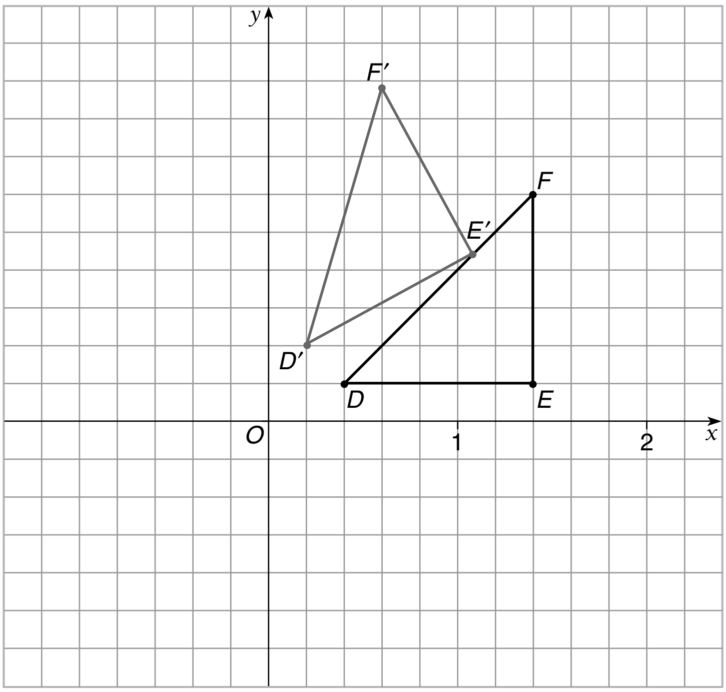


6 a Rotation of −90° about O and then an enlargement by a factor of $\frac{1}{2}$ centre O

7 a No solution given as the answer is in the question (show).

b $\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$, $D'(1.23, 1.87)$ $E'(5.56, 4.37)$ $F'(3.06, 8.70)$

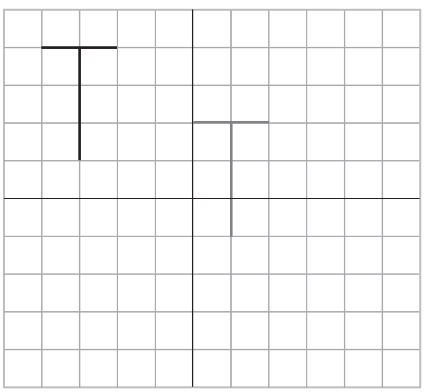
c



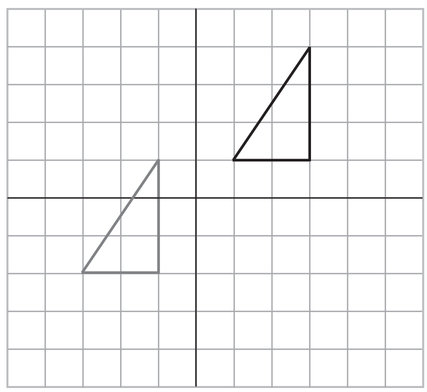
d $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$

Diagnostic Test 14 Transformations and Matrices

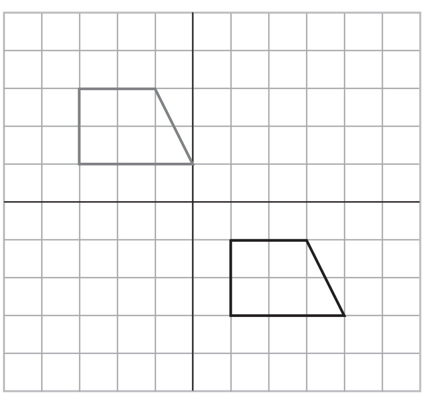
1 a



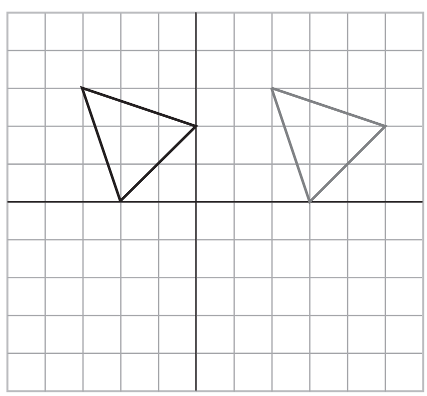
b



c



d



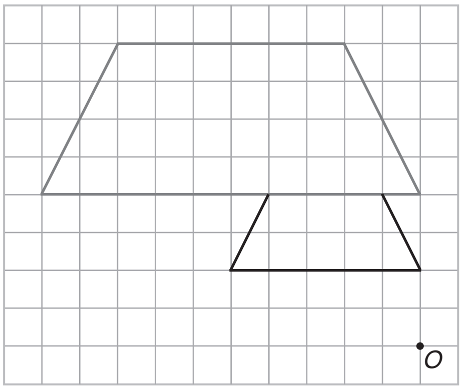
2 **a** Horizontal translation of 1 unit and a vertical translation of –4 units, $\begin{pmatrix} 1 \\ -4 \end{pmatrix}$

b Vertical translation of –4 units, $\begin{pmatrix} 0 \\ -4 \end{pmatrix}$

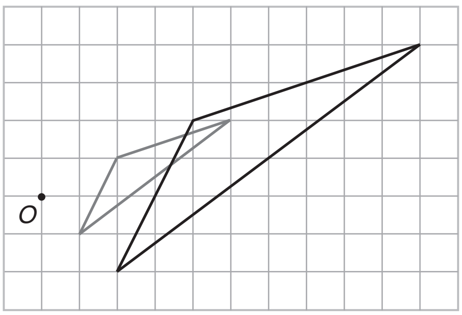
c Horizontal translation of 3 units, and a vertical translation of –4 units, $\begin{pmatrix} 3 \\ 0 \end{pmatrix}$

d Horizontal translation of 2 units and a vertical translation of –4 units, $\begin{pmatrix} 2 \\ -4 \end{pmatrix}$

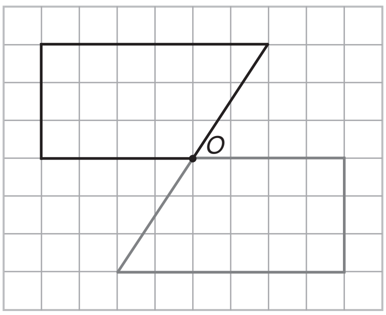
3 **a**



b



c



4 Enlargement of $-\frac{3}{2}$ centre *A*

5 **a** Reflection in the *x* axis **b** Reflection in the line $y = 1$

c Rotation of –90° about *O* **d** Reflection in the line $y = x$

6 **a** $\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$ **b** $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ **c** $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ **d** $\begin{pmatrix} 4 & 0 \\ 0 & 4 \end{pmatrix}$

7 *A*′(4.87, 6.43), *B*′(0.87, 0.5), *C*′(–0.1, 5.8)

14A Revision Assignment

1 $\begin{pmatrix} 6 \\ -4 \end{pmatrix}$

2 *A*′ (6, 4), B′ (3, 16), *C*′ (12, 1)

3 *M*′ (3, 4), *N*′ (0, 0)

4 *P*′ (–2, –1), *Q*′ (1, –2), *R*′ (–1, –4)

14B Working Mathematically

1 $a = -3$, $b = -8$

2 (6, 13)

3 $y = x - 3$

4 **a** $\overrightarrow{OX} = \begin{pmatrix} -1 \\ 0 \end{pmatrix}$ and $\overrightarrow{OY} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

b $\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$

c *A*′ (–2, 0), *B*′ (–3, 1), *C*′ (–1, 3)

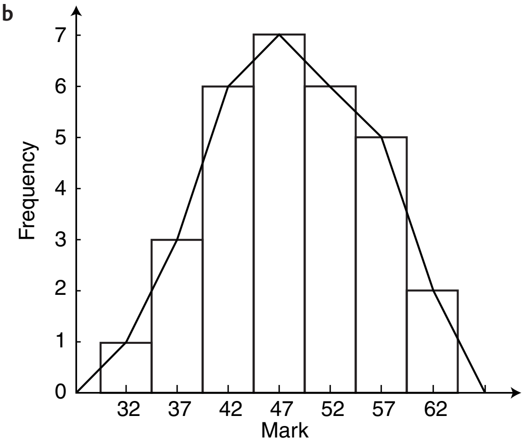
Chapter 15: Statistics

Exercise 15:01A

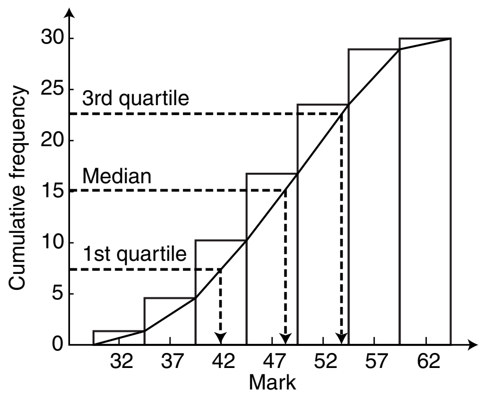
1 a

Class	*Class centre*	*Frequency*	*Cumulative frequency*
30–34	32	1	1
35–39	37	3	4
40–44	42	6	10
45–49	47	7	17
50–54	52	6	23
55–59	57	5	28
60–64	62	2	30

b

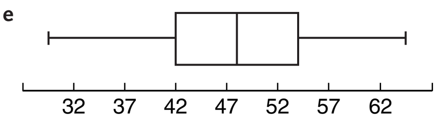


c



d **i** 48 **ii** 42 **iii** 53 **iv** 11

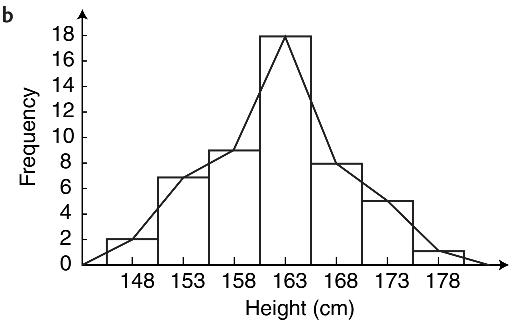
e



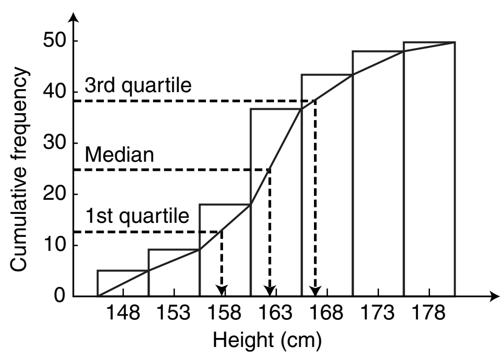
2 a

Height (cm)	*Class centre*	*Frequency*	*Cumulative frequency*
146–150	148	2	2
151–155	153	7	9
156–160	158	9	18
161–165	163	18	36
166–170	168	8	44
171–175	173	5	49
176–180	178	1	50

b

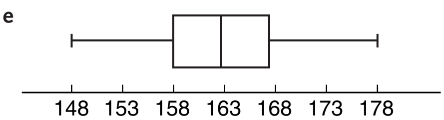


c



d **i** 162 **ii** 158 **iii** 166 **iv** 12

e



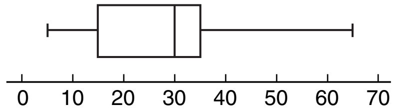
3 **a** 120 **b**

Time (min)	*Class centre*	*Frequency*	*Cumulative Frequency*
$5 \le x < 15$	10	30	30
$15 \le x < 25$	20	10	40
$25 \le x < 35$	30	50	90
$35 \le x < 45$	40	0	90
$45 \le x < 55$	50	10	100
$55 \le x < 65$	60	20	120

c This data is continuous whereas the data in questions 1 and 2 is discrete.

d **i** 29 **ii** 15 **iii** 35 **iv** 20

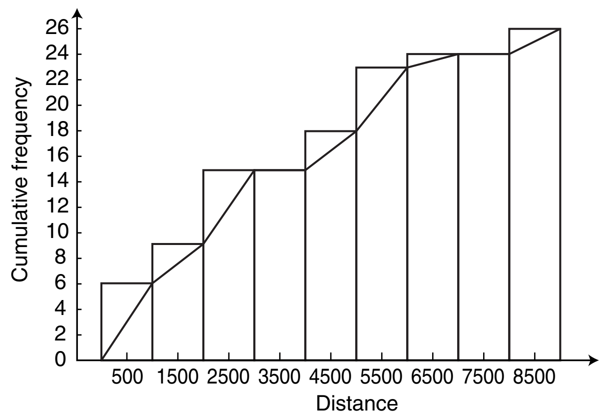
e



4 **a**

Distance (km)	*Class centre*	*Frequency*	*Cumulative frequency*
0–	500	6	6
1000–	1500	3	9
2000–	2500	6	15
3000–	3500	0	15
4000–	4500	3	18
5000–	5500	5	23
6000–	6500	1	24
7000–	7500	0	24
8000–	8500	2	26

b



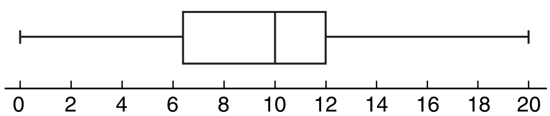
i 2700 km **ii** 1250 km
iii 5250 km **iv** 4000 km

5 a

Length of call (min)	*Class centre*	*Frequency*	*Cumulative frequency*
$0 \le x < 2$	1	25	25
$2 \le x < 4$	3	50	75
$4 \le x < 6$	5	25	100
$6 \le x < 8$	7	100	200
$8 \le x < 10$	9	50	250
$10 \le x < 12$	11	125	375
$12 \le x < 14$	13	25	400
$14 \le x < 16$	15	0	400
$16 \le x < 18$	17	50	450
$18 \le x < 20$	19	50	500

b **i** 10 minutes
ii 6.5 minutes
iii 12 minutes
iv 5.5 minutes

c



Exercise 15:01B

1 a 48·2 **b** 45 – 49 **c** 64 – 30 = 34 **d** 7·6
2 a 162·2 **b** 161 – 165 **c** 180 – 146 = 34 **d** 6·73
3 a 30·8 **b** $25 \le x < 35$ **c** 65 – 5 = 60 **d** 17·1
4 a 3346 **b** $0 \le x < 1000$, $2000 \le x < 3000$
c 8500 – 0 = 8500 **d** 2428·9
5 a 9·9 **b** $10 \le x < 12$ **c** 20 – 0 = 20 **d** 5·08

Exercise 15:02

1 a

	Ted	*Anna*
Mean	25	27·4
Maximum time	22·8	23·2
Minimum time	28·9	30·3
Standard deviation	2·2	1·93

b Ted: 6·1s Anna: 7·1s
c Anna, because she has the lowest standard deviation in her times.
d For Anna, the 23·2 time is an outlier and has affected the range, giving an unrealistic range.

2 a

	Mattheus	*Emil*
Mean	4	4
Standard deviation	2	1·65

b Emil
c Although Emil is more consistent, most of Mattheus low numbers were in the early games of the season. He has performed better later in the season. However, you need to see all the numbers, not just the statistics to make this decision.
3 a Max: 7, 2 Sam: 7, 2
b No. They are the same according to these.
c Max: 1·4 Sam: 1·1 So Sam's has the most regular customers.
4 a Hillary 4·3 Bron 4·8 **b** Bron has a lower range **c** Hillary: 1·37 Bron: 1·34
5 a Estuary: Mean = 5·6, St Dev = 1·93 Beach: Mean = 5·1, St Dev = 1·61
b the beach
c Because he caught 67 crabs in the estuary in total compared to 61 at the beach.

Exercise 15:03

1 i a 95% **b** 49·85% **c** 16% **ii** 2·5%
2 i a 47·5% **b** 2·35% **c** 13·5% **ii** 10
3 a 0·95 metres **b** 1·55 metres **c** 0·35 metres **d** 1·35 metres
4 a 15 min **b** 2 h 45 min **c** 1 h 15 min **d** 1 h 45 min and 2 h 15 min
5 a 3 cm **b** 80 **c** 82 cm and 88 cm **d** 16%

Exercise 15:04

1 a $r = 0.99$, very strong positive correlation **b** $r = 0.38$, weak positive correlation
c $r = -0.77$, strong negative correlation **d** $r = 0.08$, no correlation
e $r = -0.90$, strong negative correlation

2 a $y = 5.05x + 3.36$ **b** $y = 0.284x + 56.3$ **c** $y = -0.134x + 5.28$
d no correlation **e** $y = -0.703x + 8.86$

3 a 0·715 **b** a strong positive correlation **c** $y = 6.63x + 244$
d 456·05 *t* **e** 224 *t*

4 a −0·952 **b** very strong negative correlation **c** $y = -2.31x + 69.7$
d −11·15 (does not make any sense)
e The line of best fit is used as an estimate in some cases. Clearly here there cannot be a negative number of pests. Its use is restricted.

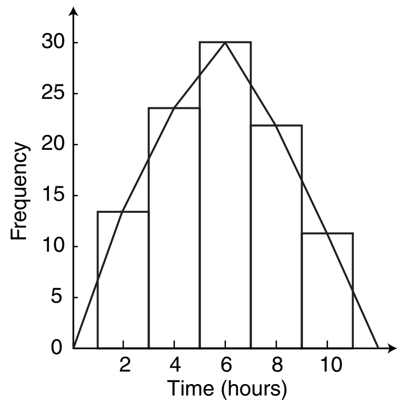
5 a −0·95 **b** Very strong negative correlation **c** $y = -0.158x + 54$
d 54 litres **e** 5 hours 42 minutes
f The driver may have stopped at times and there may have been different terrain to drive over.

Diagnostic Test 15 Statistics

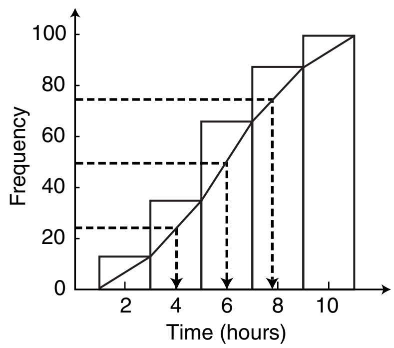
1 a

Time (hours)	*Class centre*	*Frequency*	*Cumulative frequency*
$1 \le x < 3$	2	13	13
$3 \le x < 5$	4	23	36
$5 \le x < 7$	6	30	66
$7 \le x < 9$	8	22	88
$9 \le x < 11$	10	12	100

b

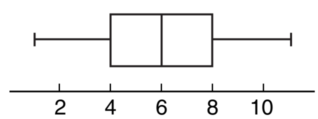


c



d **i** 6 **ii** 4 **iii** 7·8 **iv** 3·8

e



2 a **i** 5·94, 5·92 **ii** $5 \le x < 7$, $5 \le x < 7$ **iii** 2·41, 2·64
b There is a larger spread of times for the second school so people live at more diverse distances.

3 a 68% **b** **i** 2·35% **ii** 47·5% **iii** 0·15% **c** 129

4 a 2 **b** 48 and 52 **c** 13·5% d 44 and 46

5 a $r = 0.977$ very strong positive correlation
b $y = 1.3x - 39$ **c** 221 m

15A Revision Assignment

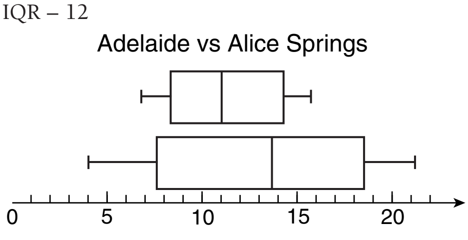
1 a Adelaide: mean – 11·15; standard deviation – 3·26
Alice Springs: mean – 13·125; standard deviation – 6·4

b

Frequencies		
Class	*Adelaide (X)*	*Alice Springs (Y)*
0–<5	0	1
5–<10	5	4
10–<15	5	2
15–<20	2	2
20–<25	0	3

c Adelaide: median – 11; Q1 – 8.05, Q3 – 14.3; IQR – 5.25
Alice Springs: median – 13.65; Q1 – 7.05, Q3 – 19.05; IQR – 12

d



e Adelaide cooler overall according to both mean and median.
f Adelaide more consistent temperatures, less variation — not as high and not as low; evidenced by narrower range, IQR and standard deviation

2 a 0.5 kg **b** 27 kg **c** 24 kg **d** 83·85%

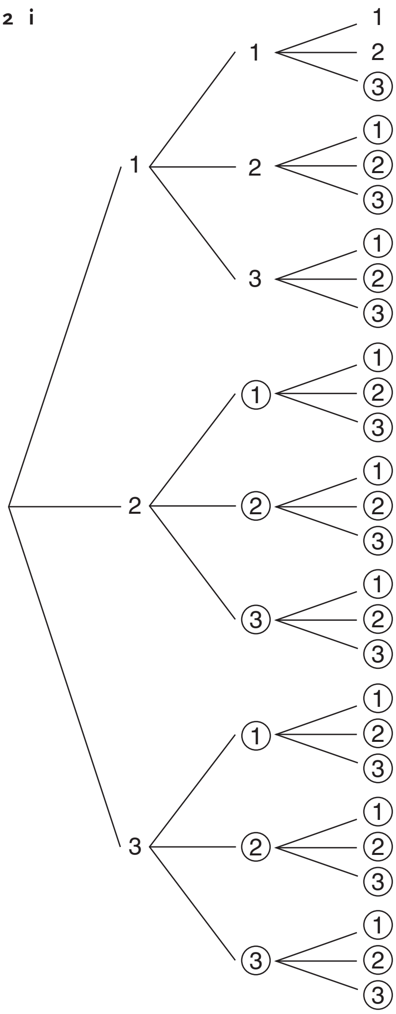
3 a $r = 0{\cdot}681$ **b** Moderately strong, positive correlation. **c** $y = 0{\cdot}070x - 3{\cdot}22$
d $y = 8{\cdot}12$, therefore shoe size 8 or 9. **e** $x = 231{\cdot}7$ cm
f Because size 13 is outside the range of values given (extrapolation).

Chapter 16: Probability

Exercise 16:01

1 i a $\frac{3}{13}$ **b** $\frac{4}{13}$ **c** $\frac{6}{13}$ **ii** $\frac{5}{12}$

2 i



ii a $\frac{1}{9}$ **b** $\frac{2}{9}$ **c** $\frac{1}{3}$ **d** $\frac{2}{3}$ **e** $\frac{2}{3}$

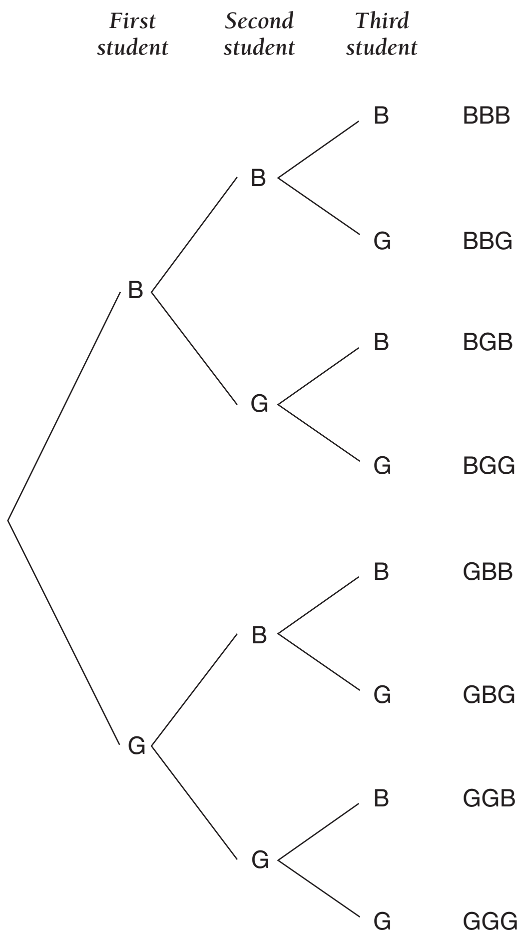
3 a

Die

Coin	1	2	3	4	5	6
H	•	•	•	•	•	•
T	•	•	•	•	•	•

12 **b** $\frac{1}{6}$

4 a

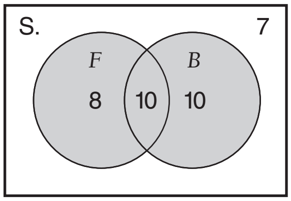


b $\frac{3}{8}$ **c** $\frac{1}{8}$ **d** $\frac{1}{2}$ **e** $\frac{7}{8}$

5 **a** $\frac{3}{20}$ **b** **i** 15 **ii** 9 **c** $P(\text{red}) = \frac{6}{23}$, $P(\text{white}) = \frac{13}{23}$, $P(\text{blue}) = \frac{4}{23}$

6 **a** $\frac{1}{5}$ **b** $\frac{1}{6}$

7 **a**



b **i** The probability they have played football, $\frac{18}{35}$

ii The probability they have played football but not basketball, $\frac{8}{35}$

iii The probability they have played football or basketball, $\frac{4}{5}$

iv The probability they have not played football or basketball, $\frac{1}{5}$

v The probability they have not played basketball, $\frac{3}{7}$

Exercise 16:02A

1 **a** $\frac{5}{12}$ **b** $\frac{7}{12}$ **c** $\frac{1}{4}$ **d** $\frac{1}{3}$ **e** $\frac{1}{6}$ **f** $\frac{2}{3}$

2 **a** $\frac{1}{2}$ **b** $\frac{5}{16}$ **c** $\frac{3}{16}$ **d** $\frac{5}{8}$ **e** $\frac{3}{8}$

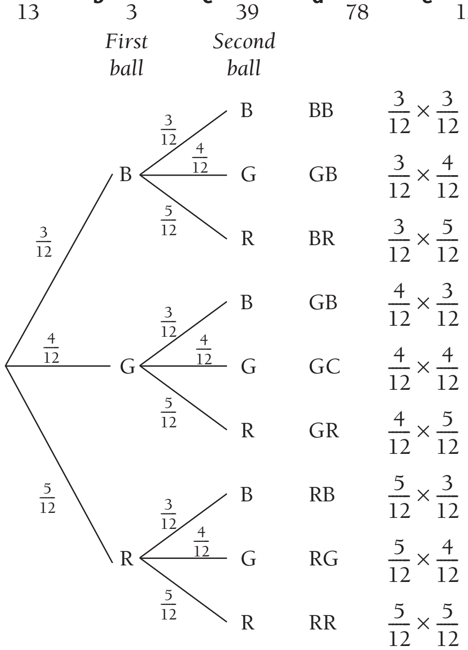
3 **a** 0 **b** $\frac{1}{8}$

4 **i** **a** $\frac{11}{12}$ **b** $\frac{3}{4}$ **c** $\frac{7}{12}$ **d** $\frac{1}{12}$ **ii** 6 **5** $\frac{3}{4}$ **6** $\frac{7}{12}$

Exercise 16:02B

1 **a** $\frac{5}{13}$ **b** $\frac{1}{3}$ **c** $\frac{5}{39}$ **d** $\frac{1}{78}$ **e** $\frac{1}{13}$

2 **a**



b **i** $\frac{25}{144}$ **ii** $\frac{25}{72}$ **iii** $\frac{47}{72}$ **iv** $\frac{4}{9}$ **v** $\frac{5}{9}$

3 **a** $\frac{5}{12}$ **b** $\frac{1}{12}$ **c** $\frac{1}{2}$ **d** $\frac{11}{12}$

4 **a** $\frac{2}{15}$ **b** $\frac{13}{15}$

5 $\frac{1}{100}$

6 **a** $\frac{15}{104}$ **b** $\frac{87}{416}$ **c** $\frac{329}{416}$

7 **a** $\frac{2}{15}$ **b** $\frac{2}{3}$

Exercise 16:03

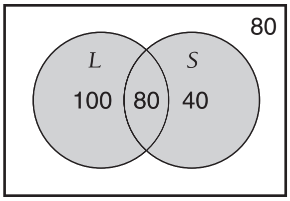
1 0·75 **2 a** 20% **b** 25% **3** $\frac{2}{5}$

4 $\frac{2}{3} \approx 66{\cdot}7\%$ (correct to 3 significant figures) **5** $\frac{8}{25}$ **6** $\frac{1}{8} = 12\frac{1}{2}\,\%$

Diagnostic Test 16 Probability

1 a i $\frac{3}{10}$ **ii** $\frac{2}{5}$ **iii** $\frac{1}{10}$ **b i** $\frac{2}{45}$ **ii** $\frac{1}{9}$ **iii** 0

2 a



They must be connecting with another flight in Singapore. **c** $\frac{2}{15}$

3 $\frac{2}{15}$

4 a They are mutually exclusive events. **b i** 0 **ii** $\frac{15}{23}$

5 a $\frac{132}{529}$ **b** $\frac{110}{529}$ **c** $\frac{419}{529}$ **6** $\frac{16}{19}$ **7** 14%

16A Revision Assignment

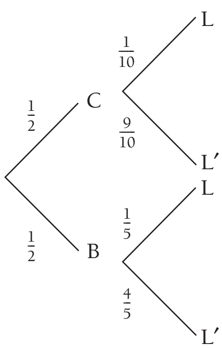
1 a $\frac{4}{9}$ **b** $\frac{2}{9}$

2 a 0·615 $\left(\text{accept } 61{\cdot}5\% \text{ or } \frac{123}{200}\right)$ **b** 0·045 $\left(\text{accept } 4{\cdot}5\% \text{ or } \frac{9}{200}\right)$

3 a 12 **b** $\frac{3}{12} = \frac{1}{4}$ or 25% **c** $\frac{2}{12} = \frac{1}{6}$ or 16·7% **d** $\frac{4}{12} = \frac{1}{3}$ or 33·3%

4 a $\frac{3}{50}$ or 6% or 0·06 **b** $\frac{45}{50} = \frac{9}{10}$ or 90% or 0·9 **c** $\frac{16}{18} = \frac{8}{9}$ or 0·889

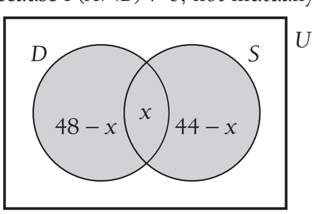
5 a



b i $\frac{1}{20}$(0·05, 5%) **ii** $= \frac{3}{20}$(0·15, 15%) **c** $= \frac{2}{3}$, (0·667)

6 a $P(A \cap B) = 0{\cdot}2$
b Because $0{\cdot}4 \times 0{\cdot}65 \neq 0{\cdot}2$, not independent
c Because $P(A \cap B) \neq 0$, not mutually exclusive

7 a i



ii $x = 32$
iii The set of members who **did not** attend for **both** Drama and Sports (or equivalent)
iv $\frac{28}{60}$ *or* $\frac{7}{15}$
b i $\frac{1}{3}$ **ii** $\frac{2}{5}$

Chapter 17: An Introduction to Algorithms and Number Theory

Exercise 17:01

1 **a** Add a zero to the number (this is not the only possible answer).
b Add a zero to the number, multiply the number by 5 and add the two answers (this is not the only possible answer).
c, d and **e** to be marked by the teacher as a class exercise

2 **a** Exit hotel. Turn left. Walk one block.
Turn right. Walk one block. Turn left.
Enter Bugis MRT station.
Buy a ticket to Clementi. Board west bound train.
Get off the train at Bugis. Exit station.
Turn right. Walk straight ahead. Turn left at Clementi Avenue 2.
Walk straight ahead to block 352. Turn right. Enter block 352.
Catch lift to 23rd floor. Turn right. Enter apartment 2305.
b As can be seen from the mps, there would be other algorithms but these may or may not be more efficient.

3 **Departing**
1. Go to check in. Present ticket and passport
2. Check in luggage if any and collect passport ticket and boarding pass.
3. Walk to immigration and present passport and boarding pass.
4. Collect passport and boarding pass.
5. Go through security check.
6. Go to gate and present boarding pass.
7. Board plane.

At some airports steps 5 and 6 are reversed. At others, step 5 is repeated after step 6.

Arriving
1. Fill in any forms necessary for arrival at your city.
2. Leave the plane after it has come to a complete stop.
3. Go to immigration, present your passport and any necessary forms
4. Go to collect you luggage.
5. If you have goods to declare, go through the red channel at customs and present any necessary forms.
6. If you have nothing to declare, go through the green channel at customs and present any necessary forms.
7. Leave the airport

There may be subtle differences at airports but all answers should be similar in each class.

4
1. If the numerator is bigger than the denominator, divide by the denominator.
2. Write the answer as a whole number.
3. Write the remainder over the denominator as a fraction after the whole number from the last step.
4. If the new numerator and the denominator have a common factor, divide each by that factor.
5. Write the new fraction after the whole number from step 2.
6. Repeat steps 4 and 5 until there are no more common factors.

There may be more than one possible answer.

5
1. If a – b is positive, go to step 5.
2. If b – c is positive, go to step 5.
3. Numbers are in ascending order.
4. Stop.
5. Numbers are not in ascending order.
6. Stop.

Exercise 17:02

1 **a** 6 **b** 7 **c** 3 **d** 3 **e** 6
2 **a** $90 = 6 \times 15 + 5$ **b** $-25 = 4 \times -7 + 3$ **c** $50 = -8 \times 7 + 6$
3 45
4 **a** 12 **b** 15 **c** 248
5 See worked example 3.

6 Let the integers be n, n + 1 and $n + 2$
From worked example 3 we know that any number can be written in the form
$n = 3k$, $n = 3k + 1$ or $n = 3k + 2$.
If $n = 3k$, then n is divisible by 3
If $n = 3k + 1$, then $n + 2 = 3k + 1 + 2$
$= 3k + 3$
$= 3(k+1)$ so $n + 2$ is divisible by 3
If $n = 3k + 2$, then $n + 1 = 3k + 2 + 1$
$= 3k + 3$
$= 3(k + 1)$ so $n + 1$ is divisible by 3
so for any n, at least one of n, $n + 1$ or $n + 2$ is divisible by 3

Exercise 17:03

1 **a** 4 **b** 6 **c** 36 **d** 17 **e** 32
2 **a** 30 **b** 75 **c** 245 **d** 150 **e** 126
3 **a** 14 **b** 18 **c** 24 **d** 98 **e** 150

Exercise 17:04

1 **a** $x = -2, y = 1$ **b** $x = 3, y = -1$ **c** $x = 3, y = -2$ **d** $x = 7, y = -6$ **e** $x = 13, y = -6$
2 **b** $x = -4, y = 3$
3 $x = 15, y = -4$
4 **a** 1 **b** they are relatively prime

Diagnostic Test 17 An Introduction to Algorithms and Number Theory

1 Insert card. Type PIN. Select withdraw. Input amount. Press Finish. Take card. Take cash and receipt.
There may be some differences depending on which country you live in. For example, in China you take the cash before the card.
Ask your teacher if your answer is a correct one.

2
1. Write the numbers as shown (example 13 and 248)
$$13\overline{)248}$$
2. If the two-digit number does not divide the first two digits of the three-digit number, skip to step 8.
3. Divide the first two digits of the three-digit number, writing how many times it divides under the second digit.
$$\begin{array}{r} 1 \\ 13\overline{)248} \end{array}$$
4. Multiply the divisor by the number above the second digit and write below the first 2 digits. Subtract and bring the 3rd digit down.
$$\begin{array}{r} 1 \\ 13\overline{)248} \\ \underline{13} \\ 118 \end{array}$$
5. Divide the two-digit number into the new number. Write how many times it divides above the last digit of the dividend.
$$\begin{array}{r} 19 \\ 13\overline{)248} \\ \underline{13} \\ 118 \end{array}$$
6. Multiply back as before and subtract.
$$\begin{array}{r} 19 \\ 13\overline{)248} \\ \underline{13} \\ 118 \\ \underline{117} \\ 1 \end{array}$$
7. This last number is the remainder and is written beside the quotient with an r in front. You are now finished.
$$\begin{array}{r} 19\ r\ 1 \\ 13\overline{)248} \\ \underline{13} \\ 118 \\ \underline{117} \\ 1 \end{array}$$
8. Divide the two-digit number into the three-digit number. Write how many times it divides above the last digit of the dividend. Go to step 6.

3 **a** $108 = 6 \times 18$ **b** $-259 = 8 \times -33 + 5$ **c** $1267 = -10 \times -126 + 7$
4 26
5 42
6 8
7 **a** 184 **b** $x = 9, y = -4$

17A Revision Assignment

2 **a** $|x| = 1$ **b** $|x| = 5$
3 37
4 8
5 32
6 **a** 92 **b** $x = -3, y = 2$

Answers to ID Cards

ID Card 1 (Metric Units) page xiv

1 *metres* 2 *decimetres* 3 *centimetres* 4 *millimetres*
5 *kilometres* 6 *square metres* 7 *square centimetres* 8 *square kilometres*
9 *hectares* 10 *cubic metres* 11 *cubic centimetres* 12 *seconds*
13 *minutes* 14 *hours* 15 *metres per second* 16 *kilometres per hour*
17 *grams* 18 *milligrams* 19 *kilograms* 20 *tonnes*
21 *litres* 22 *millilitres* 23 *kilolitres* 24 *degrees Celsius*

ID Card 2 (Symbols) page xiv

1 *is equal to* 2 *is approximately equal to* 3 *is not equal to* 4 *is less than*
5 *is less than or equal to* 6 *is not less than* 7 *is greater than* 8 *is greater than or equal to*
9 *4 squared* 10 *4 cubed* 11 *the square root of 2* 12 *the cube root of 2*
13 *is perpendicular to* 14 *is parallel to* 15 *is congruent to* 16 *is similar to*
17 *per cent* 18 *therefore* 19 *for example* 20 *that is*
21 *pi* 22 *the sum of* 23 *the mean* 24 *probability of event E*

ID Card 3 (Language) page xv

1 $6 - 2 = 4$ 2 $6 + 2 = 8$ 3 $6 \div 2 = 3$ 4 $6 - 2 = 4$ 5 $6 \div 2 = 3$
6 2 7 6 8 $6 \times 2 = 12$ 9 $6 - 2 = 4$ 10 $6 \times 2 = 12$
11 $2 + 6 = 8$ 12 $6 - 2 = 4$ 13 $6^2 = 36$ 14 $\sqrt{36} = 6$ 15 $6 - 2 = 4$
16 $6 \times 2 = 12$ 17 $(6 + 2) \div 2 = 4$ 18 $6 + 2 = 8$ 19 $6^2 = 36$ 20 $6 - 2 = 4$
21 $6 - 2 = 4$ 22 $6 + 2 = 8$ 23 $6 \div 2 = 3$ 24 $6 + 2 = 8$

ID Card 4 (Language) page xvi

1 *square* 2 *rectangle* 3 *parallelogram* 4 *rhombus*
5 *trapezium* 6 *regular pentagon* 7 *regular hexagon* 8 *regular octagon*
9 *kite* 10 *scalene triangle* 11 *isosceles triangle* 12 *equilateral triangle*
13 *circle* 14 *oval (or ellipse)* 15 *cube* 16 *rectangular prism*
17 *triangular prism* 18 *square pyramid* 19 *rectangular pyramid* 20 *triangular pyramid*
21 *cylinder* 22 *cone* 23 *sphere* 24 *hemisphere*

ID Card 5 (Language) page xvii

1 *point A* 2 *interval AB* 3 *line AB* 4 *ray AB*
5 *collinear* points 6 *midpoint* 7 *number line* 8 *diagonals*
9 *acute-angled triangle* 10 *right-angled triangle* 11 *obtuse-angled triangle* 12 *vertices*
13 ΔABC 14 *hypotenuse* 15 $180°$ 16 $a° + b°$
17 $360°$ 18 [b] $a° = b°$ 19 $a° = 60°$ 20 $3 \times 180° = 540°$
21 *AB is a diameter: OC is a radius.* 22 *circumference* 23 *semicircle*
24 *AB* is a *tangent*. *CD* is an *arc*. *EF* is a *chord*.

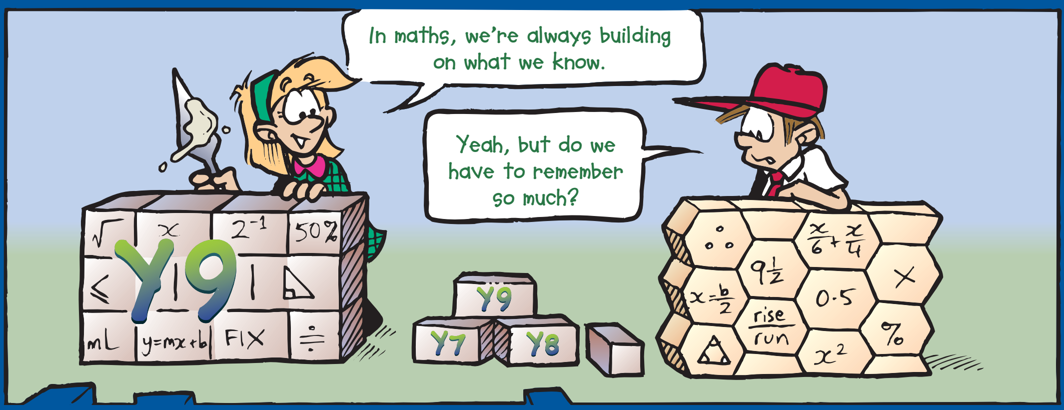
ID Card 6 (Language) page xviii

1 *parallel* lines 2 *perpendicular* lines 3 *vertical, horizontal* 4 *concurrent* lines
5 angle *ABC* or *CBA* 6 *acute* angle 7 *right* angle 8 *obtuse* angle
9 *straight* angle 10 *reflex* angle 11 *revolution* 12 *adjacent* angles
13 *complementary* angles 14 *supplementary* angles 15 *vertically opposite* angles 16 $360°$
17 *transversal* 18 *corresponding* angles 19 *alternate* angles 20 *co-interior* angles
21 *bisecting* an interval 22 *bisecting* an angle 23 $\angle CAB = 60°$ 24 *CD* is *perpendicular* to *AB*

ID Card 7 (Language) page xix

1 *anno Domini* 2 *before Christ* 3 *ante meridiem* 4 *post meridiem*
5 *hectare* 6 *regular* shapes 7 *net* of a cube 8 *cross-section*
9 *face* 10 *vertex* 11 *edge* 12 *axes of symmetry*
13 *reflection (or flip)* 14 *translation (or slide)* 15 *rotation (or turn)* 16 *tessellation*
17 *coordinates* 18 *tally* 19 *picture* graph 20 *column* graph
21 *line* graph 22 *sector (or pie)* graph 23 *bar* graph 24 *scatter diagram*

Appendixes


In maths, we're always building on what we know.
Yeah, but do we have to remember so much?

APPENDIX A

- Much of the mathematics met in Grade 8 will form a foundation for the work to be encountered in Grade 9.
- Revision exercises based on this review can be found in Chapter 1 of the text.

A:01 | Basic Number Skills

A:01A Order of operations

1 **Do operations within grouping symbols.**
2 **Do multiplications and divisions as they appear (from left to right).**
3 **Do additions and subtractions as they appear (from left to right).**

Summary
1 ( )
2 × and ÷
3 + and −

worked examples

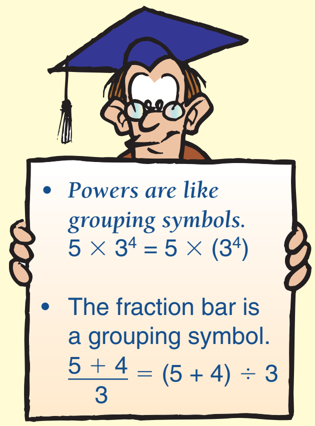
1 $-10 + 20 \div 5$
(÷ before +)
$= -10 + 4$
$= -6$

2 $12 \times 4 - 3 \times 6$
(× before −)
$= 48 - 18$
$= 30$

3 2×5^2
$= 2 \times (5^2)$
$= 2 \times 25$
$= 50$

4 $5 \times (7 - 3) + 1$
(grouping symbols first)
$= 5 \times 4 + 1$
$= 20 + 1$
$= 21$

5 $\dfrac{16 + 14}{10 - 4}$
$= (16 + 14) \div (10 - 4)$
(Grouping symbols first)
$= 30 \div 6$
$= 5$



A:01B Fractions

1 **The fraction bar is like a division sign.**

worked examples

1 $\frac{9}{4} = 9 \div 4$
$= 2\frac{1}{4}$

2 $\frac{365}{7} = 365 \div 7$
$= 52\frac{1}{7}$

3 $\frac{3}{8} = 3 \div 8$
$= 0{\cdot}375$

2 **To write a mixed numeral as an improper fraction, express the whole number with the same denominator as the fraction part.**

worked examples

1 $3\frac{1}{2} = \frac{3}{1}\begin{matrix}\times 2\\ \times 2\end{matrix} + \frac{1}{2}$
$= \frac{6}{2} + \frac{1}{2}$
$= \frac{7}{2}$
or $3\frac{1}{2} = \dfrac{(3 \times 2) + 1}{2}$

2 $6\frac{7}{10} = \frac{6}{1}\begin{matrix}\times 10\\ \times 10\end{matrix} + \frac{7}{10}$
$= \frac{60}{10} + \frac{7}{10}$
$= \frac{67}{10}$
or $6\frac{7}{10} = \dfrac{(6 \times 10) + 7}{10}$

3 $8\frac{3}{7} = \frac{8}{1}\begin{matrix}\times 7\\ \times 7\end{matrix} + \frac{3}{7}$
$= \frac{56}{7} + \frac{3}{7}$
$= \frac{59}{7}$
or $8\frac{3}{7} = \dfrac{(8 \times 7) + 3}{7}$

3 **A fraction may be simplified by casting out equal factors from both numerator and denominator. This process is called cancelling.**

worked examples

1 $\frac{56}{80} = \frac{56 \div 8}{80 \div 8}$

$= \frac{\cancel{56}^{\,7}}{\cancel{80}_{\,10}}$

$= \frac{7}{10}$

2 $\frac{90}{160} = \frac{90 \div 10}{160 \div 10}$

$= \frac{\cancel{90}^{\,9}}{\cancel{160}_{\,16}}$

$= \frac{9}{16}$

3 $\frac{240}{3600} = \frac{24\cancel{0}}{360\cancel{0}}$

$= \frac{24 \div 12}{360 \div 12}$

$= \frac{2 \div 2}{30 \div 2}$

$= \frac{1}{15}$

4 **The size of a fraction is unchanged if both numerator and denominator are multiplied by the same number.**

worked examples

Complete the following equivalent fractions.

1 $\frac{3}{8} = \frac{\square}{80}$

Multiply top and bottom by 10.

$\frac{3 \times 10}{8 \times 10} = \frac{30}{80}$

2 $\frac{17}{20} = \frac{\square}{100}$

Multiply top and bottom by 5.

$\frac{17 \times 5}{20 \times 5} = \frac{85}{100}$

3 $\frac{7}{8} = \frac{\square}{1000}$

1000 ÷ 8 is 125, so multiply top and bottom by 125.

$\frac{7 \times 125}{8 \times 125} = \frac{875}{1000}$

5 **To add or subtract fractions that have equal denominators, add or subtract the numerators. If the denominators are not the same, first express each fraction with a common denominator.**

worked examples

1 $\frac{5}{8} + \frac{7}{8}$

$= \frac{5+7}{8}$

$= \frac{12}{8}$

$= 1\frac{4}{8}$ or $1\frac{1}{2}$

2 $\frac{7}{8} - \frac{3}{4}$

$= \frac{7}{8} - \frac{3 \times 2}{4 \times 2}$

$= \frac{7}{8} - \frac{6}{8}$

$= \frac{1}{8}$

3 $\frac{3}{10} + \frac{1}{4}$

$= \frac{3 \times 2}{10 \times 2} + \frac{1 \times 5}{4 \times 5}$

$= \frac{6}{20} + \frac{5}{20}$

$= \frac{11}{20}$

6 **To add mixed numerals, find the sum of the whole numbers and the sum of the fractions; then add the two answers together. Subtraction can be performed in much the same way.**

worked examples

1 $3\frac{1}{2} + 4\frac{3}{5}$
$= (3 + 4) + (\frac{1}{2} + \frac{3}{5})$
$= 7 + (\frac{5}{10} + \frac{6}{10})$
$= 7 + \frac{11}{10}$
$= 7 + 1\frac{1}{10}$
$= 8\frac{1}{10}$

2 $\frac{3}{4} + 6\frac{5}{8}$
$= 6 + (\frac{3}{4} + \frac{5}{8})$
$= 6 + (\frac{6}{8} + \frac{5}{8})$
$= 6 + \frac{11}{8}$
$= 6 + 1\frac{3}{8}$
$= 7\frac{3}{8}$

3 $5\frac{3}{8} + 9\frac{2}{7}$
$= (5 + 9) + (\frac{3}{8} + \frac{2}{7})$
$= 14 + (\frac{21}{56} + \frac{16}{56})$
$= 14 + \frac{37}{56}$
$= 14\frac{37}{56}$

4 $4\frac{1}{2} - 2\frac{3}{10}$
$= (4 - 2) + (\frac{1}{2} - \frac{3}{10})$
$= 2 + (\frac{5}{10} - \frac{3}{10})$
$= 2 + \frac{2}{10}$
$= 2\frac{1}{5}$

5 $8\frac{3}{4} - 3\frac{2}{5}$
$= (8 - 3) + (\frac{3}{4} - \frac{2}{5})$
$= 5 + (\frac{15}{20} - \frac{8}{20})$
$= 5 + \frac{7}{20}$
$= 5\frac{7}{20}$

6 $5\frac{3}{8} - 1\frac{9}{10}$
$= (5 - 1) + (\frac{3}{8} - \frac{9}{10})$
$= 4 + (\frac{15}{40} - \frac{36}{40})$
$= 4 + (-\frac{21}{40})$
$= 4 - \frac{21}{40}$
$= 3\frac{19}{40}$

7 **When multiplying fractions:**
1 Cast out (cancel) any common factor which appears in a numerator and a denominator.
2 Multiply the numerators to obtain the numerator of your answer.
3 Multiply the denominators to obtain the denominator of your answer.

worked examples

1 $\frac{7}{8} \times \frac{3}{14}$
$= \frac{\cancel{7}^{1}}{8} \times \frac{3}{\cancel{14}_{2}}$
$= \frac{1 \times 3}{8 \times 2}$
$= \frac{3}{16}$

2 $\frac{15}{38} \times \frac{19}{20}$
$= \frac{{}^{3}\cancel{15}}{{}_{2}\cancel{38}} \times \frac{\cancel{19}^{1}}{\cancel{20}_{4}}$
$= \frac{3 \times 1}{2 \times 4}$
$= \frac{3}{8}$

3 $\frac{9}{10}$ of $\frac{8}{15}$
$= \frac{{}^{3}\cancel{9}}{{}_{5}\cancel{10}} \times \frac{\cancel{8}^{4}}{\cancel{15}_{5}}$
$= \frac{3 \times 4}{5 \times 5}$
$= \frac{12}{25}$

8 **When multiplying with mixed numerals, first write each mixed numeral as an improper fraction, then multiply as in rule 7 above.**

worked examples

1 $3\frac{3}{4} \times 1\frac{1}{2}$
$= \frac{15}{4} \times \frac{3}{2}$
$= \frac{45}{8}$
$= 5\frac{5}{8}$

2 $6 \times \frac{5}{8}$
$= \frac{{}^{3}\cancel{6}}{1} \times \frac{5}{\cancel{8}_{4}}$
$= \frac{15}{4}$
$= 3\frac{3}{4}$

3 $5\frac{7}{10} \times 15$
$= \frac{57}{{}_{2}\cancel{10}} \times \frac{\cancel{15}^{3}}{1}$
$= \frac{171}{2}$
$= 85\frac{1}{2}$

9 **To divide by a fraction, invert the fraction and multiply. (Multiply by its reciprocal.) When mixed numerals occur, express as improper fractions.**

worked examples

1 $\frac{5}{8} \div \frac{3}{10}$

$= \frac{5}{{}_4\cancel{8}} \times \frac{\cancel{10}^5}{3}$

$= \frac{25}{12}$

$= 2\frac{1}{12}$

2 $\frac{3}{5} \div 6$

$= \frac{3}{5} \div \frac{6}{1}$

$= \frac{^1\cancel{3}}{5} \times \frac{1}{\cancel{6}_2}$

$= \frac{1}{10}$

3 $3\frac{3}{4} \div 1\frac{1}{5}$

$= \frac{15}{4} \div \frac{6}{5}$

$= \frac{^5\cancel{15}}{4} \times \frac{5}{\cancel{6}_2}$

$= \frac{25}{8}$

$= 3\frac{1}{8}$

A:01C Decimals

1 **To compare decimals, rewrite the decimals so that each has the same number of decimal places. This can be done by placing zeros at the end where necessary.**

worked examples

Put in order, smallest to largest.

1 {0·505, 0·5, 0·55, 0·055}
Write in zeros.
= {0·505, 0·500, 0·550, 0·055}
Compare 505, 500, 550 and 55.
∴ smallest to largest, we have
{0·055, 0·5, 0·505, 0·55}

2 {1·87, 0·187, 1·087}
Write in zeros.
= {1·870, 0·187, 1·087}
Compare 1870, 187 and 1087.
∴ smallest to largest, we have
{0·187, 1·087, 1·87}

2 **To add or subtract decimals use the PUP rule:
keep *p*oints *u*nder *p*oints. An empty space may be filled by a zero.**

worked examples

1 3·6 + 0·68 + 12

```
=   3·60
    0·68
   12·00
   -----
   16·28
```

2 7·683 − 1·3

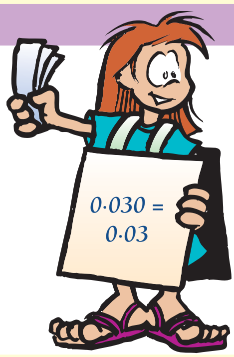
```
=  7·683 −
   1·300
   -----
   6·383
```

3 8 − 2·043

```
=  8·000 −
   2·043
   -----
   5·957
```

3 **For multiplication of decimals, the number of figures after the decimal point in the answer must be the same as the total number of figures that come after the decimal points in the question.**

worked examples



1 0·006 × 5
= 0·030

2 0·02 × 0·4
= 0·008

3
$$\begin{array}{r} 0{\cdot}23\ \times \\ 1{\cdot}7 \\ \hline 16\ 1 \\ 23\ 0 \\ \hline 0{\cdot}391 \\ \hline \end{array}$$

4 $0{\cdot}03^2$
= 0·03 × 0·03
= 0·0009

4 **To multiply by 10, 100, 1000, etc, move the decimal point 1, 2, 3, etc, places to the right, placing zeros at the end if necessary.**

worked examples

1 3·142 × 10
= 3·142 × 10
= 31·42

2 0·04 × 1000
= 0·040 × 1000
= 40

3 0·0025 × 100
= 0·0025 × 100
= 0·25

4 $2{\cdot}1 \times 10^4$
= $2{\cdot}1000 \times 10^4$
= 21 000

5 **When dividing a decimal by a whole number, keep points under points, placing additional zeros after the decimal point if required.**

worked examples

1 6·354 ÷ 6

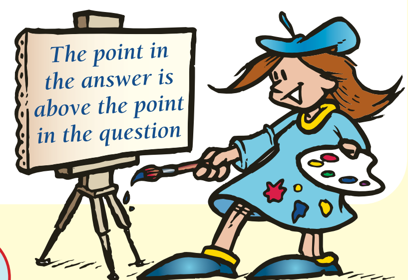
$$= 6\overline{)6{\cdot}3_{5}54}\quad\text{answer } 1{\cdot}059$$

2 19·3 ÷ 5

$$= 5\overline{)19{\cdot}_{4}3_{3}0}\quad\text{answer } 3{\cdot}86$$

3 0·17 ÷ 4

$$= 4\overline{)0{\cdot}1_{1}7_{1}0_{2}0}\quad\text{answer } 0{\cdot}0425$$



6 **To show that a set of digits is repeating, place a dot over the first and last of the repeating digits.**

worked examples

1 8·9 ÷ 6

$$= 6\overline{)8{\cdot}_{2}9_{5}0_{2}0_{2}0_{2}}\ \ldots\quad(1{\cdot}\ 4\ 8\ 3\ 3)$$

$= 1{\cdot}48\dot{3}$

2 5·32 ÷ 11

$$= 11\overline{)5{\cdot}3_{9}2_{4}0_{7}0_{4}0_{7}0_{4}}\ \ldots\quad(0{\cdot}4\ 8\ 3\ 6\ 3\ 6)$$

$= 0{\cdot}48\dot{3}\dot{6}$

3 6 ÷ 7

$$= 7\overline{)6{\cdot}_{6}0_{4}0_{5}0_{1}0_{3}0_{2}0_{6}0_{4}0_{5}0_{1}0_{3}0_{2}0_{6}}\ \ldots\quad(0{\cdot}\ 8\ 5\ 7\ 1\ 4\ 2\ 8\ 5\ 7\ 1\ 4\ 2\ \ldots)$$

$= 0{\cdot}\dot{8}5714\dot{2}$

7 **To divide by 10, 100, 1000, etc, move the decimal point 1, 2, 3, etc, places to the left, writing in zeros where necessary.**

worked examples

1 $36{\cdot}14 \div 10$
$= 36{\cdot}14 \div 10$
$= 3{\cdot}614$

2 $1{\cdot}3 \div 100$
$= 01{\cdot}3 \div 100$
$= 0{\cdot}013$

3 $0{\cdot}7 \div 1000$
$= 000{\cdot}7 \div 1000$
$= 0{\cdot}0007$

4 $45 \div 10^4$
$= 0045 \div 10^4$
$= 0{\cdot}0045$

8 **To change division by a decimal into division by a whole number, multiply both dividend and divisor by a suitable power of ten. This moves the decimal point to the right in both numbers. Move the decimal points to the right as many places as are required to make the divisor a whole number.**

worked examples

1 $6{\cdot}4 \div 0{\cdot}4$
$= 6{\cdot}4 \div 0{\cdot}4$
$= 64 \div 4$
$= 4\overline{)6_24}$ = 1 6

2 $0{\cdot}828 \div 0{\cdot}08$
$= 0{\cdot}828 \div 0{\cdot}08$
$= 82{\cdot}8 \div 8$
$= 8\overline{)82{\cdot}_28_40}$ = 10·3 5

3 $9{\cdot}5 \div 0{\cdot}005$
$= 9{\cdot}500 \div 0{\cdot}005$
$= 9500 \div 5$
$= 5\overline{)9_4500}$ = 1 900

9 **A decimal can be written as a fraction or mixed numeral by placing the part after the decimal point over 10, 100, 1000, etc, depending on the place value of the last digit.**

worked examples

1 $2{\cdot}107$
$= 2\frac{107}{1000}$

2 $0{\cdot}07$
$= \frac{7}{100}$

3 $0{\cdot}74$
$= \frac{\not{74}^{37}}{\not{100}_{50}}$
$= \frac{37}{50}$

1	·	$\frac{1}{10}$	$\frac{1}{100}$	$\frac{1}{1000}$
9	·	7	6	3

$9{\cdot}763 = 9\frac{763}{1000}$

10 **A fraction can be written as a decimal by first expressing the denominator as a power of ten or by dividing the numerator by the denominator. Write in zeros after a decimal if required.**

worked examples

1 $\frac{3}{200} = \frac{3}{200}\begin{smallmatrix}\times 5\\ \times 5\end{smallmatrix}$
$= \frac{15}{1000}$
$= 0{\cdot}015$

2 $\frac{3}{8} = 8\overline{)3_30_60_40}$ = 0·3 7 5
$= 0{\cdot}375$

3 $\frac{9}{7} = 7\overline{)9_20_60_40_50_10_30_20_60_40_50_10_30_2}\ldots$ = 1·2 8 5 7 1 4 2 8 5 7 1 4 …
$= 1{\cdot}\dot{2}85\,71\dot{4}$

11 **To rewrite a terminating decimal as a fraction, put the numbers in the decimal over the correct power of 10, ie 10, 100, 1000, etc, and then simplify.**

For example: $0{\cdot}375 = \frac{375}{1000}\begin{smallmatrix}(\div 125)\\ (\div 125)\end{smallmatrix}$
$= \frac{3}{8}$

Recurring decimals are sometimes called *repeating decimals.*

To rewrite a recurring decimal as a fraction is more difficult. Two examples are shown below.

worked examples

Example 1

When each number in the decimal is repeated.

Write 0·636363 . . . as a fraction

Let $x = 0{\cdot}6363\ldots$

Multiply by 100 because two digits are repeated.

Then $\quad 100x = 63{\cdot}6363\ldots$

Subtract the two lines.

So $\quad 100x - x = 63{\cdot}6363\ldots - 0{\cdot}6363\ldots$

ie $\quad 99x = 63$

$$\therefore x = \frac{63}{99}$$

Simplifying this fraction.

$$\therefore x = \frac{7}{11}$$

Example 2

When only some digits are repeated.

Write 0·617777 . . . as a fraction

Let $x = 0{\cdot}61777\ldots$

Multiply by 100 to move the non-repeating digits to the left of the decimal point.

Then $\quad 100x = 61{\cdot}777\ldots$

Multiply by 1000 to move one set of the repeating digits to the left of the decimal point.

And $\quad 1000x = 617{\cdot}777$

Subtract the previous two lines.

So $\quad 1000x - 100x = 617{\cdot}777 - 61{\cdot}777$

ie $\quad 900x = 556$

$$\therefore x = \frac{556}{900}$$

Simplifying this fraction using your calculator.

$$\therefore x = \frac{139}{225}$$

This answer can be checked by performing 139 ÷ 225 using your calculator.

A:01D Percentages

1 **To express a percentage as a fraction, write it as a fraction with a denominator of 100, and simplify.**

worked examples

1 32%

$= \frac{\cancel{32}^{\,8}}{\cancel{100}_{\,25}}$

$= \frac{8}{25}$

2 307%

$= \frac{307}{100}$

$= 3\frac{7}{100}$

3 $13\frac{1}{4}\%$

$= \frac{13\frac{1}{4} \times 4}{100 \times 4}$

$= \frac{53}{400}$

4 10·7%

$= \frac{10{\cdot}7 \times 10}{100 \times 10}$

$= \frac{107}{1000}$

2 **To express a fraction as a percentage, multiply by $\frac{100}{1}\%$**

worked examples

1 $\frac{13}{20}$

$= \frac{13}{{}_{1}\cancel{20}} \times \frac{\cancel{100}^{\,5}}{1}\%$

$= \frac{65}{1}\%$

$= 65\%$

2 $\frac{5}{9}$

$= \frac{5}{9} \times \frac{100}{1}\%$

$= \frac{500}{9}\%$

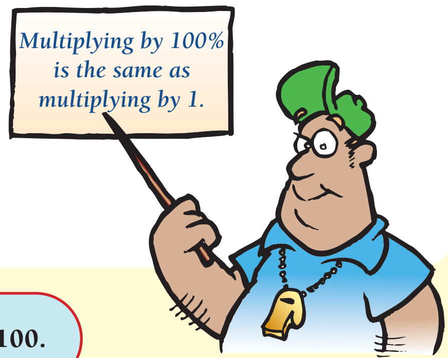
$= 55\frac{5}{9}\%$

3 $2\frac{3}{5}$

$= \frac{13}{{}_{1}\cancel{5}} \times \frac{\cancel{100}^{\,20}}{1}\%$

$= \frac{260}{1}\%$

$= 260\%$



3 **To express 97·2% as a decimal, divide 97·2 by 100.**

worked examples

1 15%

= 15· ÷ 100

= 0·15

2 9·2%

= 09·2 ÷ 100

= 0·092

3 8%

$= \frac{8}{100}$

= 8 ÷ 100

= 0·08

4 $11\frac{1}{4}\%$

$= \frac{11\frac{1}{4}}{100}$

= 11·25 ÷ 100

= 0·1125

4 **To express a decimal as a percentage, multiply by 100%.**

worked examples

1 0·35

= 0·35 × 100%

= 35%

2 0·07

= 0·07 × 100%

= 7%

3 0·125

= 0·125 × 100%

= 12·5%

4 1·8

= 1·80 × 100%

= 180%

5 **To find a percentage of a quantity, change the percentage to a fraction or decimal, and multiply the quantity by this fraction or decimal.**

worked examples

1 38% of 600 m

$= \frac{38}{\not{100}_1} \times \frac{\not{600}^6}{1}$

= 228 m

2 7% of 96 g

= 0·07 × 96

= 6·72 g

3 $6\frac{1}{4}$% of $44

= (6·25 ÷ 100) × 44

= 0·0625 × 44

= $2.75

6 **If a percentage of a quantity is known, to find the quantity, first find 1% (or 10%), then multiply to find 100% (the whole quantity). This is a form of the unitary method.**

worked examples

1 7% of my money was spent on a watch band which cost $1.12. How much money did I have?

7% of my money = $1.12

∴ 1% of my money = $0.16

∴ 100% of my money = $16.00

∴ I had $16

2 30% of my weight is 18 kg. How much do I weigh?

30% of my weight = 18 kg

10% of my weight = 6 kg

∴ 100% of my weight = 60 kg

∴ I weigh 60 kg

7 **To express one quantity as a percentage of another, write the first quantity as a fraction of the second and then convert the fraction to a percentage, by multiplying by $\frac{100}{1}$%.**

worked examples

1 Express 75 cents as a percentage of $2.

Percentage $= \frac{75}{200} \times \frac{100}{1}$%

(since $2 = 200 cents)

$= \frac{75}{\not{200}_2} \times \frac{\not{100}^1}{1}$%

$= \frac{75}{2}$%

= 37·5%

∴ 75 cents is 37·5% of $2

2 5 kg of sugar, 8 kg of salt and 7 kg of flour were mixed. What is the percentage (by weight) of salt in the mixture?

Percentage of salt $= \frac{8}{20} \times \frac{100}{1}$%

(since 5 kg + 8 kg + 7 kg = 20 kg)

$= \frac{8}{\not{20}_1} \times \frac{\not{100}^5}{1}$%

= 40%

∴ salt is 40% of the mixture

A:01E Ratio

1 **A ratio is a comparison of like quantities. Comparing 3 km to 5 km, we write 3 to 5 or 3 : 5 or $\frac{3}{5}$. As with fractions, we can simplify a ratio by multiplying or dividing both terms by the same number.**

worked examples

1 Jan's height is 1 metre while Diane's is 150 cm. Find the ratio of their heights.

Jan's height to Diane's
= 1 m to 150 cm
= 100 cm to 150 cm
= 100 : 150
Divide both terms by 50
= 2 : 3

2 $\frac{3}{5}$ of the class walk to school, while $\frac{1}{4}$ ride bicycles. Find the ratio of those who walk to those who ride bicycles.

Those walking to those cycling
$= \frac{3}{5} : \frac{1}{4}$
Multiply both terms by 20 (the L.C.D.)
$= \frac{3}{\cancel{5}_1} \times \frac{\cancel{20}^4}{1} : \frac{1}{\cancel{4}_1} \times \frac{\cancel{20}^5}{1}$
= 12 : 5

2 **If $x : 15 = 10 : 3$, then $\frac{x}{15} = \frac{10}{3}$**

worked examples

1 The ratio of John's weight to Andrew's is 5 : 4. If Andrew's weight is 56 kg, how much does John weigh?

Let John's weight be x kg
$\frac{\text{John's weight}}{\text{Andrew's weight}} = \frac{5}{4}$
$\therefore \frac{x}{56} \times 56 = \frac{5}{4} \times 56$
Multiply both sides by 56
$x = \frac{5}{\cancel{4}_1} \times \frac{\cancel{56}^{14}}{1}$
$\therefore x = 70$
∴ John's weight is 70 kg

2 The ratio of profit to cost price on sales at a store was 2 : 3. Find the cost price for an item if the store's profit is $600.

Let the cost price be $$x$
Profit : cost = 2 : 3
∴ 600 : x = 2 : 3
∴ x : 600 = 3 : 2
$\frac{x}{600} \times 600 = \frac{3}{2} \times 600$
Multiply both sides by 600
$x = \frac{3}{\cancel{2}_1} \times \frac{\cancel{600}^{300}}{1}$
= 900
∴ the cost price is $900

3 **If a quantity is divided between Alana and Rachel in the ratio 5 : 4, then**
1 the quantity is seen to have 9 parts (ie 5 + 4),
2 Alana receives $\frac{5}{9}$ of the quantity,
3 Rachel receives $\frac{4}{9}$ of the quantity.

worked examples

1 \$44 000 is divided between Tom and Peter in the ratio 9 : 2. How much does Tom receive?

There are 11 parts. Tom receives 9 of them

$\therefore$ Tom's share $= \dfrac{9}{{}_1\cancel{11}} \times \dfrac{\$\cancel{44\,000}^{4000}}{1}$

$= \$36\,000$

$\therefore$ Tom receives \$36 000

2 The sizes of the angles of a triangle are in the ratio 2 : 3 : 5. Find the size of the smallest angle.

There are 10 parts. The smallest angle has 2 of them. The angle sum is 180°.

$\therefore$ smallest angle $= \dfrac{2}{{}_1\cancel{10}} \times \dfrac{\cancel{180°}^{18}}{1}$

$= 36°$

$\therefore$ the smallest angle is 36°

A:01F Rates

A rate is a comparison of unlike quantities. If I travel 180 km in 3 hours, my average rate of speed is 180 km/3 h or 60 km/h. As with ratio, to simplify a rate, both terms may be multiplied or divided by the same number, or units may be converted.

worked examples

1 16 kg per \$10

Cost = \$10 per 16 kg

= 1000 cents per 16 kg

Divide each term by 16

= 62·5 cents per 1 kg

= 62·5 cents/kg

2 72 L/h = … cm^3/s

$\dfrac{72\text{ L}}{1\text{ h}} = \dfrac{72 \times 1000\text{ mL}}{1 \times 60 \times 60\text{ s}}$

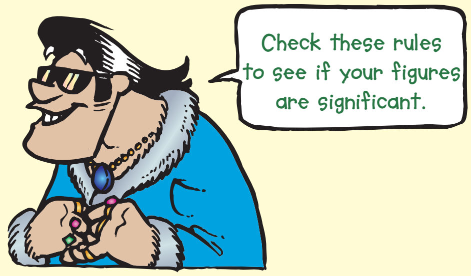
= 20 mL per s

Now 1 mL = 1 cm^3

$\therefore$ 72 L/h = 20 cm^3/s

A:01G Significant figures

You will remember that the accuracy of a measurement is determined by the number of significant figures it contains.



Rules for determining significant figures

1. **Coming from the left, the first non-zero digit is the first significant figure.**
2. **All figures following the first significant figure are also significant unless the number is a *whole* number ending in zeros.**
3. **Final zeros in a whole number may or may not be significant, eg 8800 may have 2, 3 or 4 significant figures, depending on whether it's measured to the nearest 100, 10 or 1.**

worked examples

State the number of significant figures in each of the following.

1 213	2 306	3 0·0006	4 3·00
5 23 000	6 402·6	7 306·0	8 0·030 20

9 16 000 (to the nearest thousand)

Solutions

1 3 sig. figs. (all digits sig., rule 2)
2 3 sig. figs. (all digits sig., rule 2)
3 1 sig. fig. (only the 6 is sig., rule 1)
4 3 sig. figs. (all digits sig., rule 2)
5 Ambiguous. Zeros may or may not be significant. In the absence of further information we would say that only the 2 and 3 are significant. Hence 2 sig. figs., rules 2 and 3.
6 4 sig. figs. (all digits sig., rule 2)
7 4 sig. figs. (all digits sig., rule 2)
8 4 sig. figs. (rules 1 and 2)
9 2 sig. figs. (since it is measured to the nearest thousand, rules 2 and 3)

A:01H Approximations

Often answers to calculations need to be approximated or shortened. There are two main types of approximations.

Type 1: Correcting to a given number of significant figures

To approximate correct to a certain number of significant figures, write down the number that contains only the required number of significant figures and is closest in value to the given number.

Type 2: Correcting to a given number of decimal places

To approximate correct to a certain number of decimal places, write down the number that contains only the required number of decimal places and is closest in value to the given number.

worked examples

1 Write 3·46 correct to: **a** 1 sig. fig. **b** 2 sig. figs.
2 Write 12·682 correct to: **a** 1 dec. pl. **b** 2 dec. pl.

Solutions

1 **a** 3·46 is between 3 and 4 and is closer to 3.
$\therefore$ 3·46 correct to 1 sig. fig. is 3.
b 3·46 is between 3·4 and 3·5 and is closer to 3·5
$\therefore$ 3·46 correct to 2 sig. figs. is 3·5.

2 **a** 12·682 is between 12·6 and 12·7 and is closer to 12·7
$\therefore$ 12·682 correct to 1 dec. pl. is 12·7.
b 12·682 is between 12·68 and 12·69 and is closer to 12·68
$\therefore$ 12·682 correct to 2 dec. pl. is 12·68.

To round off (or approximate) a number correct to a given place, round up if the next figure is 5 or more, and round down if the next figure is less than 5.

A:01I Estimation

An estimate is a valuable means of checking whether your calculator work gives a sensible answer. If your estimate and the actual answer are not similar, then it tells you that a mistake has been made either in your estimate or your calculation.

The following examples will show you how to estimate the size of an answer.

worked examples

Give estimates for: **1** $\dfrac{5\cdot6}{1\cdot7}+\dfrac{14\cdot92}{1\cdot85}$ **2** $\sqrt{\dfrac{9\cdot87\times16\cdot25}{4\cdot52\times1\cdot77}}$ **3** $\dfrac{0\cdot125\times3\cdot92}{0\cdot0836+0\cdot76}$

≈ or ≑ means 'is approximately equal to'.

Solutions

1 $\dfrac{5\cdot6}{1\cdot7}+\dfrac{14\cdot92}{1\cdot85}$

$\doteqdot \dfrac{6}{2}+\dfrac{15}{2}$

$\doteqdot 3+7\cdot5$

$\doteqdot 10\cdot5$

2 $\sqrt{\dfrac{9\cdot87\times16\cdot25}{4\cdot52\times1\cdot77}}$

$\doteqdot \dfrac{\sqrt{10\times16}}{5\times2}$

$\doteqdot \dfrac{\sqrt{160}}{10}$

$\doteqdot \dfrac{13}{10}$

$\doteqdot 1\cdot3$

3 $\dfrac{0\cdot125\times3\cdot92}{0\cdot0836+0\cdot76}$

$\doteqdot \dfrac{0\cdot1\times4}{0\cdot08+0\cdot8}$

$\doteqdot \dfrac{0\cdot4}{0\cdot9}$

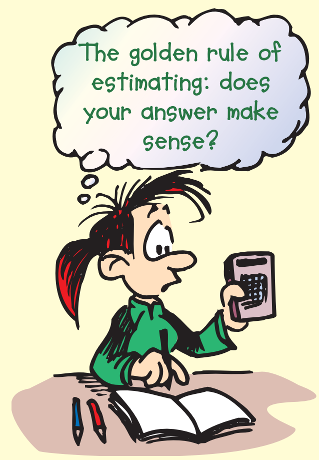
$\doteqdot \dfrac{4}{9}$

$\doteqdot 0\cdot5$

- These hints may be useful.
  - When estimating, look for numbers that are easy to work with, eg 1, 10, 100.
  - Remember it's an estimate. When you approximate a number you don't have to take the nearest whole number.
  - Try thinking of decimals as fractions. It often helps.
    eg $7\cdot6\times0\cdot518\approx8\times\frac{1}{2}$ or 4
  - When dealing with estimates involving fraction bars, look for numbers that nearly cancel out.

    eg $\dfrac{{}^{2}\cancel{17\cdot68}\times5\cdot8}{\cancel{8\cdot9}_{1}}\approx\dfrac{2\times6}{1}=12$

Check that the answer makes sense.



A:02 | Algebraic Expressions

A:02A Generalised arithmetic

The aim of 'generalised arithmetic' is to write an algebraic expression that shows the steps to be taken no matter which numbers are involved.

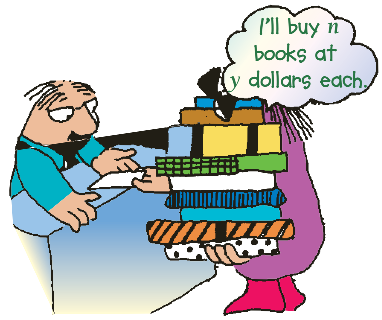
worked examples

1 The sum of y and z is $y + z$.

2 The average of m and n is $\frac{m+n}{2}$.

3 The change from \$100 after buying n books at \$8 each is $100 - 8n$ dollars.

4 Five less than three times a certain number could be written as $3x - 5$.



A:02B Substitution

Algebra involves the use of pronumerals as well as numbers. A pronumeral is usually a letter, such as x, that takes the place of a number in an expression like $3x + 7$.

If a number is substituted for each pronumeral, a value for the expression can then be obtained.

worked examples

Find the value of the following expressions, given that $a = 10$, $b = 4$, $x = 5$ and $y = -3$.

1 $3a + 2b$
$= 3 \times 10 + 2 \times 4$
$= 30 + 8$
$= 38$

2 $x^2 + y^2$
$= 5^2 + (-3)^2$
$= 25 + 9$
$= 34$

3 $\frac{1}{2}ab^2$
$= \frac{1}{2} \times 10 \times 4^2$
$= \frac{1}{2} \times 160$
$= 80$

4 $\frac{1}{x} + \frac{1}{y}$
$= \frac{1}{5} + \frac{1}{(-3)}$
$= \frac{1}{5} - \frac{1}{3}$
$= -\frac{2}{15}$

A:02C Simplifying expressions

- Only like terms may be added or subtracted. Like terms have pronumeral parts that are identical.

worked examples

1 $5p^2 + 2p - 3p^2$
$= 5p^2 - 3p^2 + 2p$
$= 2p^2 + 2p$

2 $6ab - 4ba + b$
$= 6ab - 4ab + b$
$= 2ab + b$

3 $6a - 2y + 5 + y - 2a - 7$
$= 6a - 2a - 2y + y + 5 - 7$
$= 4a - y - 2$

- When multiplying algebraic terms, multiply numbers first and then pronumerals. When dividing algebraic terms, express the division as a fraction and reduce it to its lowest terms.

worked examples

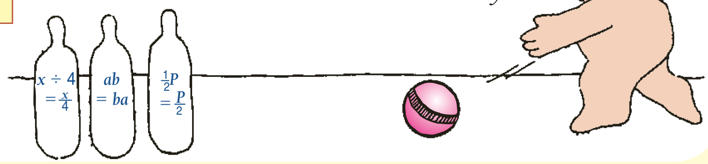
1 $-4n \times 2y$
$= -4 \times 2 \times n \times y$
$= -8ny$

2 $3pa \times 4ar$
$= 12paar$
$= 12pa^2r$

3 $12ac \div 8ab$
$= \dfrac{{}^{3}\cancel{12}^{1}\cancel{a}c}{{}_{2}\cancel{8}\cancel{a}_{1}b}$
$= \dfrac{3c}{2b}$

4 $-6x \div 18xy$
$= \dfrac{-\cancel{6}^{1}\cancel{x}^{1}}{{}_{3}\cancel{18}\cancel{x}_{1}y}$
$= -\dfrac{1}{3y}$

$12ac \div 8ab$ is assumed to mean $(12 \times a \times c) \div (8 \times a \times b)$.



A:02D Fractions

The rules for fractions still hold if pronumerals are present.

worked examples

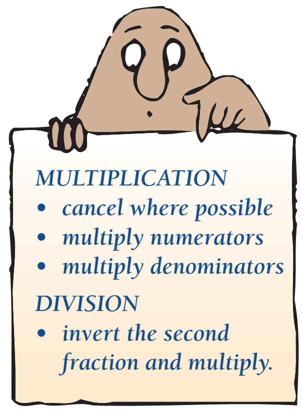
1 $\dfrac{4a}{5} - \dfrac{a}{3}$
$= \dfrac{4a \times 3}{5 \times 3} - \dfrac{a \times 5}{3 \times 5}$
$= \dfrac{12a}{15} - \dfrac{5a}{15}$
$= \dfrac{7a}{15}$

2 $\dfrac{5a}{2x} + \dfrac{2a}{3x}$
$= \dfrac{5a \times 3}{2x \times 3} + \dfrac{2a \times 2}{3x \times 2}$
$= \dfrac{15a}{6x} + \dfrac{4a}{6x}$
$= \dfrac{19a}{6x}$

Addition and subtraction:
Rewrite the fractions as equivalent fractions with a common denominator, then add or subtract the numerators.

3 $\dfrac{3ab}{2} \times \dfrac{4}{5b}$
$= \dfrac{3a\cancel{b}^{1}}{{}_{1}\cancel{2}} \times \dfrac{\cancel{4}^{2}}{5\cancel{b}_{1}}$
$= \dfrac{3a \times 2}{1 \times 5}$
$= \dfrac{6a}{5}$

4 $\dfrac{my}{2} \div \dfrac{y}{5}$
$= \dfrac{m\cancel{y}^{1}}{2} \times \dfrac{5}{\cancel{y}_{1}}$
$= \dfrac{m \times 5}{2 \times 1}$
$= \dfrac{5m}{2}$



A:02E Products

$a(b + c) = ab + ac$

$a(b - c) = ab - ac$

worked examples

1 $p(p + 3)$
$= p \times p + p \times 3$
$= p^2 + 3p$

2 $7(5 - 2a)$
$= 7 \times 5 - 7 \times 2a$
$= 35 - 14a$

3 $-5(3x - 7)$
$= (-5) \times 3x - (-5) \times 7$
$= -15x + 35$

4 $-(3 + 7m)$
$= (-1) \times 3 + (-1) \times 7m$
$= -3 - 7m$

5 $5(3x - 8) + 2(9x + 1)$
$= 15x - 40 + 18x + 2$
$= 33x - 38$

6 $2a(a + b) - a(3a - 4b)$
$= 2a^2 + 2ab - 3a^2 + 4ab$
$= 6ab - a^2$

A:02F Binomial products

$$(a + b)(c + d) = a(c + d) + b(c + d)$$
$$= ac + ad + bc + bd$$

worked examples

1 $(a + 2)(b + 4) = a(b + 4) + 2(b + 4)$
$= ab + 4a + 2b + 8$

2 $(a - 2)(a + 7) = a(a + 7) - 2(a + 7)$
$= a^2 + 7a - 2a - 14$
$= a^2 + 5a - 14$

3 $(x + 2y)(2x + y) = x(2x + y) + 2y(2x + y)$
$= 2x^2 + xy + 4xy + 2y^2$
$= 2x^2 + 5xy + 2y^2$

4 $(1 - x)(x - 3) = 1(x - 3) - x(x - 3)$
$= x - 3 - x^2 + 3x$
$= 4x - 3 - x^2$

Special products

These results should be committed to memory.

Perfect squares:
$(x + y)^2 = x^2 + 2xy + y^2$
$(x - y)^2 = x^2 - 2xy + y^2$

The difference of two squares:
$(x + y)(x - y) = x^2 - y^2$

worked examples

1 $(2y + 7)^2 = (2y)^2 + 2(2y \times 7) + (7)^2$
$= 4y^2 + 28y + 49$

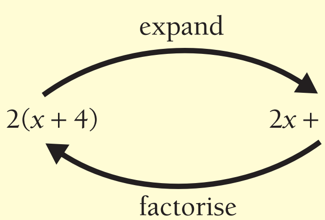
2 $(8m - 5)^2 = (8m)^2 - 2(8m \times 5) + (5)^2$
$= 64m^2 - 80m + 25$

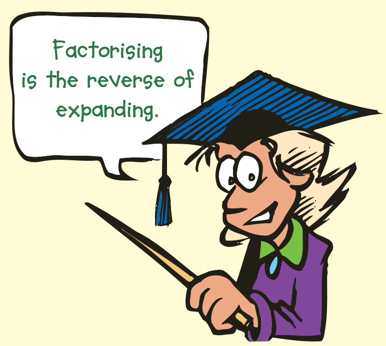
3 $(2a + 3b)(2a - 3b) = (2a)^2 - (3b)^2$
$= 4a^2 - 9b^2$

4 $(9t - 3)(9t + 3) = (9t)^2 - (3)^2$
$= 81t^2 - 9$

A:02G Factorisation

Our aim when factorising is to write the expression as a product.





Taking out common factors

$$ab + ac = a(b + c)$$

worked examples

1 $-6a - 9$
$= -3(2a + 3)$

2 $3x^3 + 6x^2$
$= 3x^2\ (x + 2)$

3 $2(x + 3) + a(x + 3)$
$= (x + 3)(2 + a)$

Factorising by grouping in pairs

$$\begin{aligned} ab + ac + bd + cd &= a(b + c) + d(b + c) \\ &= (b + c)(a + d) \end{aligned}$$

worked examples

1 $ax - bx + am - bm$
$= x(a - b) + m(a - b)$
$= (a - b)(x + m)$

2 $a^2 + 3a + ax + 3x$
$= a(a + 3) + x(a + 3)$
$= (a + 3)(a + x)$

3 $ab + b^2 - a - b$
$= b(a + b) - 1(a + b)$
$= (a + b)(b - 1)$

Factorising the difference of the two squares

$$x^2 - y^2 = (x + y)(x - y)$$

worked examples

1 $4a^2 - b^2$
$= (2a)^2 - (b)^2$
$= (2a - b)(2a + b)$

2 $36m^2 - 49n^2$
$= (6m)^2 - (7n)^2$
$= (6m + 7n)(6m - 7n)$

3 $a^4 - 64$
$= (a^2)^2 - (8)^2$
$= (a^2 + 8)(a^2 - 8)$

Factorising quadratic trinomials

$$x^2 + (a + b)x + ab = (x + a)(x + b)$$

worked examples

1 $x^2 + 7x + 10 = (x + 2)(x + 5)$
2 and 5 multiply to give 10 and add to give 7.

2 $y^2 + y - 12 = (y + 4)(y - 3)$
4 and −3 multiply to give −12 and add to give 1.

3 $x^2 - 9x - 36 = (x - 12)(x + 3)$
−12 and 3 multiply to give −36 and add to give −9.

Further quadratics: The cross method

We use the cross method to factorise trinomials like $ax^2 + bx + c$.

worked examples

1 Factorise $2x^2 + 25x + 12$.

$2x$ ╲╱ ~~3~~, ~~4~~, ~~2~~, ~~6~~, ~~12~~, 1

x ╱╲ ~~4~~, ~~3~~, ~~6~~, ~~2~~, ~~1~~, 12

$2x^2 + 25x + 12 = (2x + 1)(x + 12)$

2 Factorise $6x^2 - x - 15$.

~~6x~~, $3x$ ╲╱ −5 ~~3~~ −~~15~~ ~~1~~

~~x~~, $2x$ ╱╲ 3 −~~5~~ ~~1~~ −~~15~~

$6x^2 - x - 15 = (3x - 5)(2x + 3)$

To discover which pair of factors is correct, multiply across and add the two products until you get the middle term. If you get the opposite sign to the one you want, change the sign of both numbers to the right of the cross.

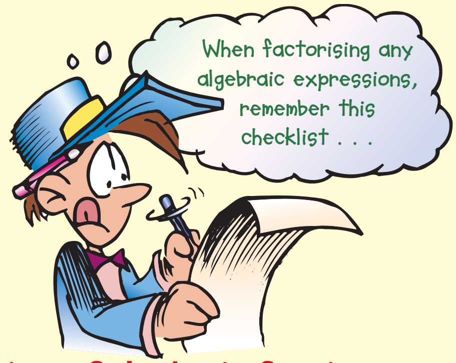
An approach to factorising

First:

- Always take out any common factor.

Then:

- If there are two terms, is it a difference of two squares, $a^2 - b^2$?
- If there are three terms, is it a quadratic trinomial, $ax^2 + bx + c$?
- If there are four terms, can it be factorised by grouping the terms into pairs?



A:02H Factorising in multiplication and division of algebraic fractions

To simplify algebraic fractions, factorise both numerator and denominator, where possible, and then cancel.

worked examples

1
$$\frac{x^2 - 9}{x^2 + 5x + 6} \times \frac{3x + 6}{x^2 - 2x - 3}$$
$$= \frac{(x+3)(x-3)}{(x+3)(x+2)} \times \frac{3(x+2)}{(x-3)(x+2)}$$
$$= \frac{3}{x+1}$$

2
$$\frac{6x - 14}{3x - 9} \div \frac{3x - 7}{5x + 15}$$
$$= \frac{2(3x-7)}{3(x-3)} \times \frac{5(x-3)}{3x-7}$$
$$= \frac{10}{3}$$

A:02I Factorising in the addition and subtraction of algebraic fractions

When adding or subtracting fractions:

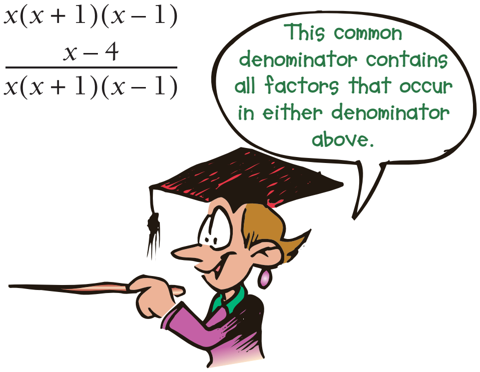
1. factorise the denominator of each fraction
2. find the lowest common denominator
3. rewrite each fraction with this common denominator and simplify.

worked examples

1
$$\frac{1}{x^2 + 5x + 6} + \frac{2}{x + 3}$$
$$= \frac{1}{(x+2)(x+3)} + \frac{2}{x+3}$$
$$= \frac{1 + 2(x+2)}{(x+2)(x+3)}$$
$$= \frac{1 + 2x + 4}{(x+2)(x+3)}$$
$$= \frac{2x + 5}{(x+2)(x+3)}$$

2
$$\frac{4}{x^2 + x} - \frac{3}{x^2 - 1}$$
$$= \frac{4}{x(x+1)} - \frac{3}{(x-1)(x+1)}$$
$$= \frac{4(x-1) - 3x}{x(x+1)(x-1)}$$
$$= \frac{4x - 4 - 3x}{x(x+1)(x-1)}$$
$$= \frac{x - 4}{x(x+1)(x-1)}$$

3
$$\frac{x + 3}{x^2 + 2x + 1} - \frac{x - 1}{x^2 - x - 2}$$
$$= \frac{x+3}{(x+1)(x+1)} - \frac{x-1}{(x+1)(x-2)}$$
$$= \frac{(x+3)(x-2) - (x-1)(x+1)}{(x+1)(x+1)(x-2)}$$
$$= \frac{x^2 + x - 6 - (x^2 - 1)}{(x+1)(x+1)(x-2)}$$
$$= \frac{x - 5}{(x+1)^2(x-2)}$$



A:03 | Probability

A:03A Experimental probability

One way of determining the chance of something happening is by observing what occurs in a sample 'experiment'.

If simple equipment such as coins, dice, spinners, cards or random numbers are used to represent real events, then the 'experiment' is called a **simulation**.

Experimental probability formula:

The experimental probability of an event $= \dfrac{\textbf{number of times this event occurred}}{\textbf{total number in sample}}$

Experimental probabilities are usually based on an examination of a **sample** or **trial run** of the activity under examination.

worked example

The contents of 20 matchboxes were examined and the results recorded.

Number of matches	48	49	50	51	52	53
Number of boxes	1	5	8	3	2	1

■ The probability of an event occurring in an experiment is the same as its 'relative frequency'.

If the contents of a similar box of matches were counted, what would be the experimental probability that it would contain 50 matches or more?

Solution

In the sample, 14 of the 20 boxes had more than 50 matches.

$$\text{Experimental probability} = \frac{\text{number of times this event occurred}}{\text{total number in sample}}$$
$$= \frac{14}{20}$$

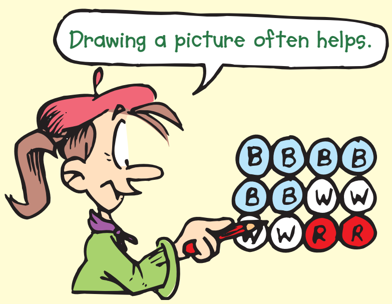
In boxes like these, we would expect the chance of choosing one with 50 or more matches to be $\frac{14}{20}$ or $\frac{7}{10}$.

A:03B Theoretical probability

In many cases, we can work out the expected or theoretical probability of an event by considering the possible outcomes. For example, when tossing a coin there are two possible outcomes, a head or a tail.

Since there is only one head, the probability of throwing a head would be 1 out of 2, ie $\frac{1}{2}$.

■ When calculating the probability of an event, we shall assume that each possible outcome is equally likely, ie no two-headed coins or loaded dice.



worked examples

1 If a dice is rolled, what is the probability of getting:

a a six? **b** an odd number? **c** a number less than seven?

2 In a bag there are six blue marbles, four white marbles and two red marbles. What is the probability of choosing at random:

a a blue marble? **b** a blue or white marble? **c** a pink marble?

Solutions

1 The possible outcomes when rolling a dice are 1, 2, 3, 4, 5, 6. So the number of possible outcomes is 6.

a The number of sixes on a dice is 1. So the probability of throwing a six is 1 out of 6, or $\frac{1}{6}$. This can be written as:

$P(6) = \frac{1}{6}$

b The number of odd numbers on a dice is 3. So the probability of throwing an odd number is 3 out of 6.

$$\begin{aligned} P(\text{odd no.}) &= \frac{3}{6} \\ &= \frac{1}{2} \end{aligned}$$

c Since all six numbers on a dice are less than seven, the probability of throwing a number less than seven is:

$$\begin{aligned} P(\text{no.} < 7) &= \frac{6}{6} \\ &= 1 \end{aligned}$$

■ The probability of an event *certain* to happen is 1.
P(sure thing) = 1

2 The total number of marbles in the bag is twelve. So the number of possible outcomes is 12.

a Number of blue marbles is six.

$$\begin{aligned} \therefore P(\text{blue marble}) &= \frac{6}{12} \\ &= \frac{1}{2} \end{aligned}$$

b Number of blue or white marbles is ten.

$$\begin{aligned} \therefore P(\text{blue or white}) &= \frac{10}{12} \\ &= \frac{5}{6} \end{aligned}$$

c Number of pink marbles is zero.

$$\begin{aligned} \therefore P(\text{pink}) &= \frac{0}{12} \\ &= 0 \end{aligned}$$

■ The probability of an event that *cannot happen* is 0.
P(impossibility) = 0

If each possible outcome is equally likely, then:

$$\textbf{probability of an event, } P(E) = \frac{n(E)}{n(S)}$$

where $n(E)$ = number of ways the event can occur
$n(S)$ = number of ways all events can occur
(S is used to represent the sample space, which is the set of possible outcomes.)
The probability of any event occurring must lie in the range $0 \leqslant P(E) \leqslant 1$.

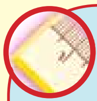
It must be pointed out that the probabilities of each possible event must add up to 1. As a consequence of this, if the probability of an event occurring is P(E), then the probability of E *not* occurring is 1 − P(E).

■ $P(E') = 1 - P(E)$ where $P(E')$ is the probability of E not occurring.

E' is set notation for the 'complement' of E, ie those outcomes outside of E. For example:

- The complementary event for *getting an even number* after rolling a dice is getting an odd number.
- The complementary event for *drawing a red card* from a deck of cards is drawing a black card.

A:03C Addition principle for mutually exclusive events



Mutually exclusive events are events that cannot happen at the same time.

Examples:

1 Selecting a boy and selecting a girl.

2 Throwing an even number on a dice and throwing a three.

If two events, A and B are mutually exclusive, then:

P(either A or B) = P(A) + P(B)

(If the events were not mutually exclusive, we would need to subtract the probability that both events would happen at the same time.)

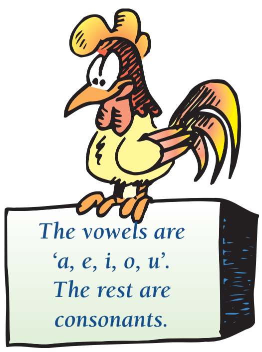
worked example

A different letter of the alphabet was placed on each of 26 cards. One of these cards was then drawn at random. What is the probability that the card drawn is:

1 either a vowel or a consonant?

2 either a letter of the word *final* or a letter of the word *method*?

3 either a letter of the word *thick* or a letter of the word *pick*?



Solutions

1 There are 5 vowels out of 26 letters and there are 21 consonants.

$$\therefore \text{P(either vowel or consonant)} = \frac{5}{26} + \frac{21}{26} = \frac{26}{26} = 1$$

∴ The card is certain to be either a vowel or a consonant.

2 P(a letter in *final*) = $\frac{5}{26}$, P(a letter in *method*) = $\frac{6}{26}$

All letters in *final* and in *method* are different.

$$\therefore \text{P(either a letter in } \textit{final} \text{ or } \textit{method}) = \frac{5}{26} + \frac{6}{26} = \frac{11}{26}$$

3 P(a letter in *thick*) = $\frac{5}{26}$, P(a letter in *pick*) = $\frac{4}{26}$

Several of the letters in *thick* are also in *pick*.

These events are *not* mutually exclusive.

$$\therefore \text{P(either a letter in } \textit{thick} \text{ or } \textit{pick}) \neq \frac{5}{26} + \frac{4}{26}$$

The number of different letters in these two words is only 6.

$$\therefore \text{P(either a letter in } \textit{thick} \text{ or } \textit{pick}) = \frac{6}{26} = \frac{3}{13}$$

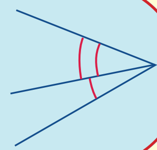
A:04 | Geometry

A:04A Deductive exercises

Following are the basic angle facts met up to Year 9.

Adjacent angles

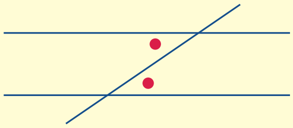
1 **They have a common vertex (or point).**
2 **They have a common arm.**
3 **They lie on opposite sides of this common arm.**



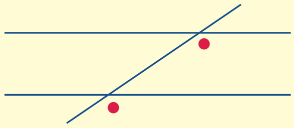
The Greek letters α, β, γ, δ, θ are often used for the size of angles.
eg $\alpha = 75°$

- **If two adjacent angles add up to 180°, then together they form a straight angle.**
- **The sum of the angles at a point is 360° or one revolution.**
- **When two straight lines intersect, the vertically opposite angles are equal.**

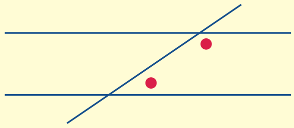
Angles and parallel lines



Alternate angles



Corresponding angles

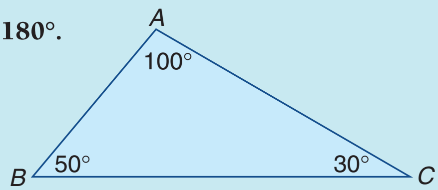


Co-interior angles

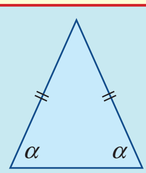
- **Parallel lines are straight lines in the same plane that do not meet.**
- **If a transversal cuts two parallel lines, then:**
  - **a** **alternate angles are equal, and**
  - **b** **corresponding angles are equal, and**
  - **c** **co-interior angles are supplementary.**
- **Two straight lines are parallel if:**
  - **a** **alternate angles are equal,**
  - **b** **corresponding angles are equal,**
  - **c** **co-interior angles are supplementary.**
- **If two lines are parallel to a third then they are parallel to one another.**

A transversal is a line cutting two or more other lines.

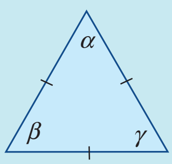
In any triangle, the angles add up to 180°.



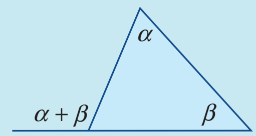
An *isosceles triangle* has:
- **two equal sides**
- **two equal angles**
- **one axis of symmetry**



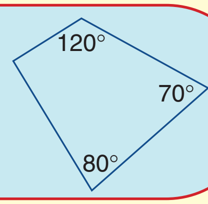
An *equilateral triangle* has:
- **three equal sides**
- **three equal angles**
- **three axes of symmetry**



The exterior angle of a triangle is equal to the sum of the two interior opposite angles.

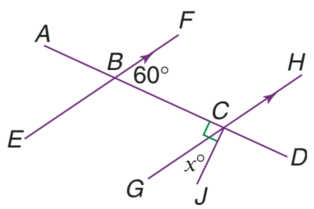


The angle sum of a quadrilateral is 360°.



worked examples

1

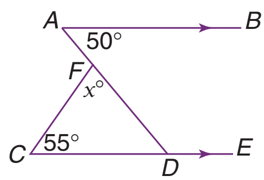


$EF \| GH$. Find the size of x.
Give reasons.

Solution

$\angle BCG = 60°$
(alt. $\angle$s, $EF \| GH$
$x + 60 = 90$ (comp. $\angle$s)
$x = 30$

2

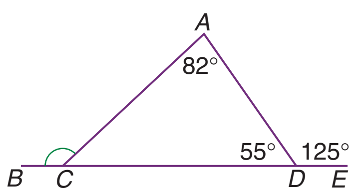


$AB \| CE$. Find the size of x.
Give reasons.

Solution

$\angle FDC = 50°$ (alt. $\angle$s, $AB \| CE$)
$x + 55 + 50 = 180$ ($\angle$sum of Δ)
$\therefore\ x = 75$

3

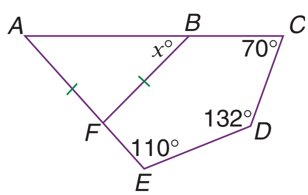


Find the size of $\angle BCA$.
Give reasons.

Solution

$\angle ADC = 55°$ (adj. supp. $\angle$s)
$\angle BCA = 82° + 55°$ (ext. $\angle$ of Δ)
$\therefore\ \angle BCA = 137°$

4



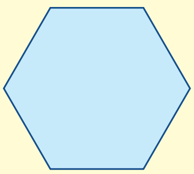
$AF = BF$. Find the size of x.
Give reasons.

Solution

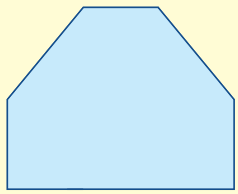
$\angle EAC = 360° - (70° + 132° + 110°)$
($\angle$ sum of quadrilateral)
$= 48°$
$\therefore\ x = 48$ (base $\angle$s of isos. Δ)

A:04B Polygons

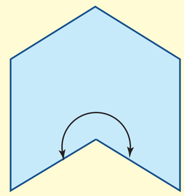
- A polygon is a plane figure with straight sides.
- A polygon is said to be regular if all of its sides and angles are equal. (If they are not, it is said to be irregular.)
- Some polygons are named according to the number of sides they have.



A regular hexagon

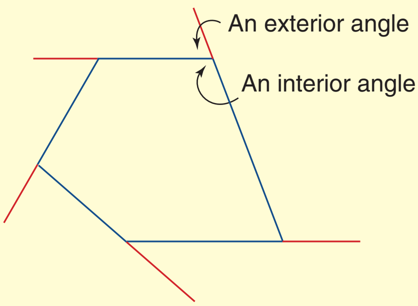


An irregular hexagon



A concave hexagon

- A polygon can be concave or *convex*.
- In a *convex* polygon, all the angles are acute or obtuse. If a polygon has any reflex angles, it is said to be concave.



- The sum of the interior angles of a polygon of n sides is: $(n - 2) \times 180°$.
- The sum of the exterior angles of any convex polygon is 360°.

worked examples

1 Find the sum of the interior angles of an octagon and the size of an interior angle, if the octagon is regular.

2 A regular polygon has an exterior angle of 20°. How many sides does the polygon have?

Solutions

1 Sum of interior angles $= (n - 2) \times 180°$

For an octagon, n is equal to 8.

Sum of interior angles $= (8 - 2) \times 180°$

$= 1080°$

If the octagon is regular, all angles are equal.

$\therefore$ Size of an interior angle $= 1080° \div 8$

$= 135°$

2 Sum of exterior angles = 360°

For a polygon, the number of sides is equal to the number of exterior angles.

$$\text{Number of angles} = \frac{360°}{20°}$$

$= 18$

$\therefore$ Number of sides = 18

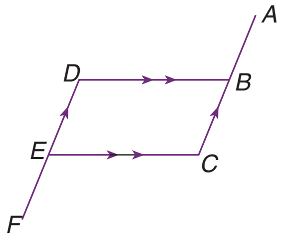
A:04C Deductive reasoning: Non-numerical exercises

Many problems in geometry are non-numerical. In these problems, the reasoning process becomes more involved. As there are no numbers involved, pronumerals are used to represent unknown quantities. With the use of pronumerals, the reasoning will involve algebraic skills learned in other parts of the course.

Because exercises do not involve specific numbers, the results we obtain will be true irrespective of the numbers used. The results obtained are called **generalisations** or, more commonly, **proofs**.

worked examples

1 In the diagram, prove that $\angle ABD = \angle CEF$.



Solution

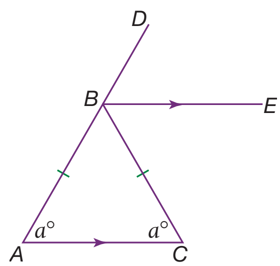
1 $\angle ABD = \angle BDE$ (alt. $\angle$s, $AC \| DF$)

$\angle CEF = \angle BDE$ (corresp. $\angle$s $DB \| EC$)

$\therefore \angle ABD = \angle CEF$ (both equal to $\angle BDE$)

2 ΔABC is isosceles with $AB = BC$. AB is produced to D and BE is drawn through B parallel to the base AC. Prove that BE bisects $\angle CBD$.

Solution



Let $\angle BAC = a°$

$\therefore \angle BCA = a°$ (base $\angle$s of isos. Δ)

Now $\angle EBC = a°$ (alt. $\angle$s, $BE \| AC$)

$\angle DBE = a°$ (corresp. $\angle$s, $BE \| AC$)

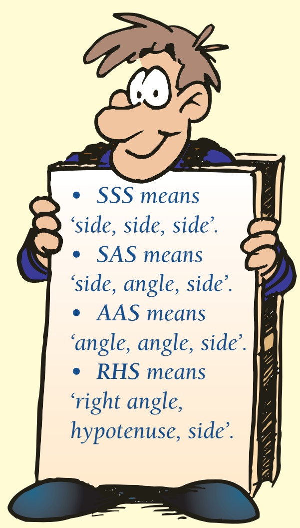
$\therefore \angle DBE = \angle EBC$

$\therefore BE$ bisects $\angle CBD$

A:04D Congruent triangles

Summary

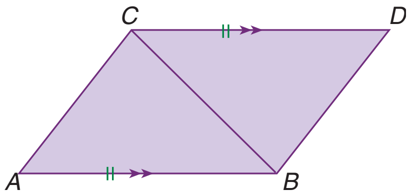
- **Two triangles are congruent if three sides of one triangle are equal to three sides of the other. (SSS)**
- **Two triangles are congruent if two sides and the included angle of one triangle are equal to two sides and the included angle of the other. (SAS)**
- **Two triangles are congruent if two angles and a side of one triangle are equal to two angles and the matching side of the other. (AAS)**
- **Two right-angled triangles are congruent if the hypotenuse and one side of one triangle are equal to the hypotenuse and one side of the other triangle. (RHS)**



The minimum conditions deduced in the last section are used to prove that two triangles are congruent. Special care must be taken in exercises that involve overlapping triangles.

worked examples

1 Show that $\Delta ABC \equiv \Delta DCB$.



Solution

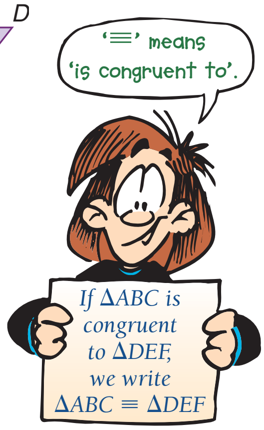
In ΔABC and ΔDCB

1 $\angle ABC = \angle DCB$
(alt. angles, $DC \| BA$)

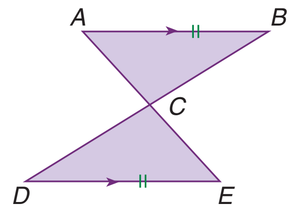
2 $AB = CD$

3 $BC = BC$

$\therefore \Delta ABC \equiv \Delta DCB$ (SAS)



2 Prove that $\Delta ABC \equiv \Delta EDC$.



Solution

In ΔABC and EDC:

1 $\angle ABC = \angle ECD$ (vert. opp. $\angle$s)

2 $\angle CAB = \angle CED$ (alt. $\angle$s $AB \| DE$)

3 $AB = DE$ (given)

$\therefore \Delta ABC \equiv \Delta EDC$ (SAS)

When working with congruent figures, the term 'corresponding' is often used instead of the term 'matching' to refer to angles or sides in the same position.

When writing congruent triangle proofs, write the vertices in matching order as shown in the examples.

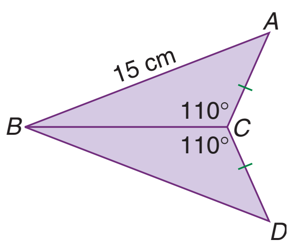
A:04E Using congruent triangles to find sides and angles

If two triangles can be shown to be congruent, then, of course, all matching sides and angles are equal.

Using congruent triangles to find the values of unknown angles and sides or to prove relationships is very important in geometry.

worked examples

1 $AC = CD$, $\angle ACB = \angle BCD$.
Prove that $\Delta ACB \equiv \Delta DCB$, and hence that $BD = 15$ cm.



Solution

In Δs ABC and DBC:

1 $AC = DC$ (given)

2 $\angle ACB = \angle DCB$ (given)

3 BC is common to both Δs.

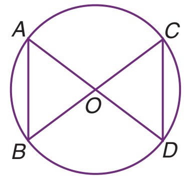
$\therefore \Delta ABC \equiv \Delta DBC$ (SAS)

$\therefore AB = DB$ (corresp. sides of cong't Δs)

$\therefore BD = 15$ cm

2 AD and BC are two chords which cross at O, the centre of the circle. Prove that ΔABO is congruent to ΔCDO and hence that $AB = CD$.

Solution



Draw a diagram if one is not given in the question.

In the Δs ABO and CDO:

1 $AO = OC$ (radii of a circle are equal)

2 $\angle AOB = \angle COD$ (vertically opposite angles)

3 $BO = DO$ (radii of a circle are equal)

$\therefore \Delta ABO \equiv \Delta CDO$ (SAS)

$\therefore AB = CD$ (corresponding sides of congruent Δs)

A:04F Triangles and quadrilaterals

Triangles

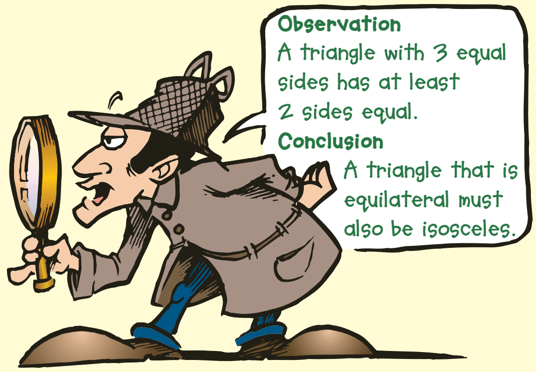
Geometrical figures have many properties and it is not practicable to mention them all when defining the figure.

A definition is the minimum amount of information needed to identify a particular figure.

In deductive geometry, the definitions serve as starting points. The properties of the figures can then be proved using basic geometrical facts.

The proved result is known as a theorem and this can then be used to produce other theorems.

Definitions
- **A scalene triangle is a triangle with no two sides equal in length.**
- **An isosceles triangle is a triangle with at least two sides equal in length.**
- **An equilateral triangle is a triangle with all sides equal in length.**



The definitions imply that an equilateral triangle must also be an isosceles triangle. Hence, any property of an isosceles triangle must also be a property of an equilateral triangle.

Quadrilaterals

As we have seen with triangles, the definitions of the quadrilaterals are minimum definitions.

Definitions
- **A trapezium is a quadrilateral with at least one pair of opposite sides parallel.**
- **A parallelogram is a quadrilateral with both pairs of opposite sides parallel.**
- **A rhombus is a parallelogram with two adjacent sides equal in length.**
- **A rectangle is a parallelogram with one angle a right angle.**
- **A square is a rectangle with two adjacent sides equal**

  OR

  A square is a rhombus with one angle a right angle.

Tests for parallelograms

A quadrilateral is a parallelogram if any one of the following is true.

1. **Both pairs of opposite sides are equal.**
2. **Both pairs of opposite angles are equal.**
3. **One pair of sides is both equal and parallel.**
4. **The diagonals bisect each other.**

Tests for a rhombus

1. **All sides are equal.** ***OR***
2. **Diagonals bisect each other at right angles.**

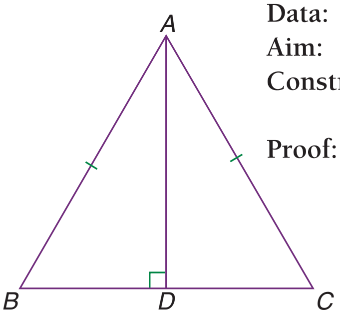
Tests for a rectangle

1. **All angles are equal.** ***OR***
2. **Diagonals are equal and bisect each other.**

worked examples

1 Use congruent triangles to prove that the angles opposite equal sides in an isosceles triangle are equal.

Solution



Data: ΔABC is isosceles with $AB = AC$.

Aim: To prove that $\angle ABC = \angle ACB$.

Construction: Draw AD perpendicular to BC, meeting BC in D.

Proof: In Δs ABD and ACD:

1 $AB = AC$ (data)

2 AD is common.

3 $\angle ABD = \angle ADC$ ($AD \perp BC$)

$\therefore \Delta ABD \equiv \Delta ACD$ (RHS)

$\therefore \angle ABD = \angle ACD$ (corresponding $\angle$s of congruent Δs)

$\therefore \angle ABC = \angle ACB$

2 Prove that a quadrilateral is a parallelogram if its opposite angles are equal.

Solution

Data: $ABCD$ is a quadrilateral with $\angle A = \angle C$ and $\angle B = \angle D$.

Aim: To prove that $AB \| DC$ and $AD \| BC$.

Proof: Let $\angle A = \angle C = b°$ and $\angle B = \angle D = a°$.

$2(a + b) = 360$ ($\angle$ sum of quad.)

$\therefore a + b = 180$

$\therefore \angle ADC + \angle DAB = 180°$

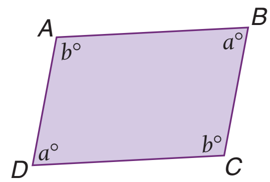
$\therefore AB \| DC$ (co-int. $\angle$s are supp.)

Also, $\angle ADC + \angle DCB = 180°$

$\therefore AD \| BC$ (co-int. $\angle$s are supp.)

$\therefore ABCD$ has opposite sides parallel.

$\therefore ABCD$ is a parallelogram.

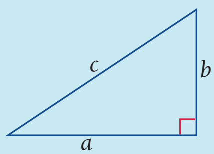


A:04G Pythagoras' theorem and its converse

The theorem states that:

If a triangle is right-angled then the square on the longest side is equal to the sum of the squares on the two smaller sides.

For the triangle shown, this means that $c^2 = a^2 + b^2$.



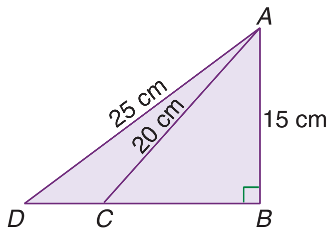
• Pythagoras' theorem is still used to check that buildings are square.

Furthermore, the converse states that:

If the square on the longest side is equal to the sum of the squares on the two smaller sides then the triangle is right-angled.

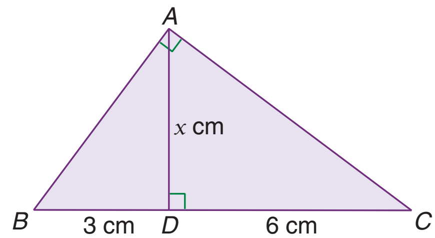
worked examples

1



Find the value of DC.

2



Find the value of x.

Solutions

1

In ΔABD,

$$25^2 = 15^2 + BD^2$$
$$625 = 225 + BD^2$$
$$BD^2 = 400$$
$$BD = 20$$

In ΔABC,

$$20^2 = 15^2 + BC^2$$
$$400 = 225 + BC^2$$
$$BC^2 = 175$$
$$BC = \sqrt{175}$$

Now $DC = BD - BC$

$$= 20 - \sqrt{175}$$
$$= (20 - 5\sqrt{7})\text{ cm}$$

2

In ΔABD,

$$AB^2 = x^2 + 9$$

In ΔADC,

$$AC^2 = x^2 + 36$$

In ΔABC,

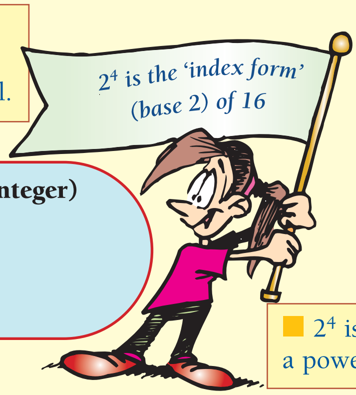
$$BC^2 = AB^2 + AC^2$$
$$9^2 = (x^2 + 9) + (x^2 + 36)$$
$$81 = 2x^2 + 45$$
$$2x^2 = 36$$
$$x^2 = 18$$
$$x = \sqrt{18}$$
$$= 3\sqrt{2}$$

A:05 | Indices

A:05A Index form

$$2 \times 2 \times 2 \times 2 = 16$$
$$2^4 = 16$$

- 2 is called the base.
- 4 is called the index.
- 16 is called the basic numeral.



$x^n = \underbrace{x \times x \times x \times \ldots \times x \times x}_{n \text{ factors}}$ **(where n is a positive integer)**

For: x^n — **x is the base, n is the index.**

2^4 is called a power of 2.

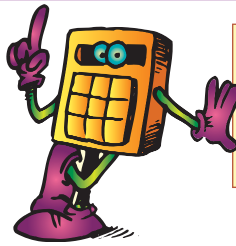
worked examples

1 $a \times a = a^2$

2 $m \times m \times m \times m = m^4$

3 $y = y^1$

4 $2^4 = 2 \times 2 \times 2 \times 2$
$= 16$



Remember the x^y button. Enter the base x first, press x^y then enter the index y.

A:05B Index laws

Law 1 $x^m \times x^n = x^{m+n}$
Law 2 $x^m \div x^n = x^{m-n}$
Law 3 $(x^m)^n = x^{m \times n}$ **or simply** x^{mn}

Law 4 $x^0 = 1$

worked examples

1 $a^3 \times a^5 = a^{3+5} = a^8$

2 $a^{12} \div a^3 = a^{12-3} = a^9$

3 $(a^3)^4 = a^{3 \times 4} = a^{12}$

4 $a^0 \times 4 = 1 \times 4 = 4$

5 $7a^5 \times 3ab^2$
$= 7 \times a^5 \times 3 \times a \times b^2$
$= 7 \times 3 \times a^5 \times a^1 \times b^2$
$= 21a^6b^2$

6 $(2a^2)^4$
$= 2^4 \times (a^2)^4$
$= 16 \times a^8$
$= 16a^8$

7 $24a^5b^2 \div 8a^3b^2$
$= \frac{24}{8} \times \frac{a^5}{a^3} \times \frac{b^2}{b^2}$
$= 3a^2b^0$
$= 3a^2$ since $b^0 = 1$

A:05C Negative indices

In general, the meaning of a negative index can be summarised by the rule: $x^{-m} = \frac{1}{x^m}$, $(x \neq 0)$

x^{-m} **is the reciprocal of** x^m**, since** $x^m \times x^{-m} = 1$

Examples
$x^{-3} = \frac{1}{x^3}$
$x^3 \times x^{-3} = x^0$
$= 1$

worked examples

1 Simplify the following:
a 3^{-2} b 5^{-1} c $x^7 \times x^{-3}$ d $6x^2 \div 3x^4$ e $(\frac{1}{4})^{-2}$ f $(\frac{2}{3})^{-3}$

2 Evaluate, using the calculator:
a 2^{-3} b $(\frac{1}{3})^{-2}$

Solutions

1 a $3^{-2} = \frac{1}{3^2}$
$= \frac{1}{9}$

b $5^{-1} = \frac{1}{5^1}$
$= \frac{1}{5}$

c $x^7 \times x^{-3} = x^{7+(-3)}$
$= x^4$

d $6x^2 \div 3x^4 = 2x^{2-4}$
$= 2x^{-2}$
$= \frac{2}{x^2}$

e $\left(\frac{1}{4}\right)^{-2} = \dfrac{1}{\left(\frac{1}{4}\right)^2}$

$= \dfrac{1}{\left(\frac{1}{16}\right)}$

$= 16$

■ $\dfrac{1}{\left(\frac{1}{16}\right)} = 1 \div \dfrac{1}{16} = 1 \times \dfrac{16}{1} = 16$

f $\left(\frac{2}{3}\right)^{-3} = \dfrac{1}{\left(\frac{2}{3}\right)^3}$

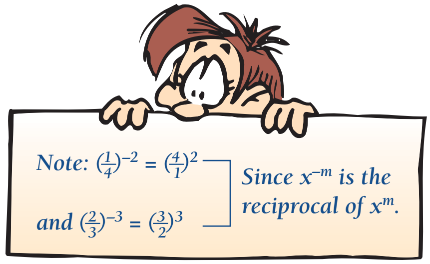
$= \dfrac{1}{\left(\frac{8}{27}\right)}$

$= \dfrac{27}{8}$

$= 3\frac{3}{8}$

2 **a** $2^{-3} = 0{\cdot}125$

b $\left(\frac{1}{3}\right)^{-2} = 9$



A:05D Fractional indices

$x^{\frac{1}{2}} = \sqrt{x}$, $x^{\frac{1}{3}} = \sqrt[3]{x}$, $x^{\frac{1}{n}} = n$th root of x

$x^{\frac{1}{3}}$ is the number that, when used three times in a product, gives x.

worked examples

1 $9^{\frac{1}{2}} \times 9^{\frac{1}{2}} = 9^{\left(\frac{1}{2}+\frac{1}{2}\right)}$

$= 9^1$

$= 9$

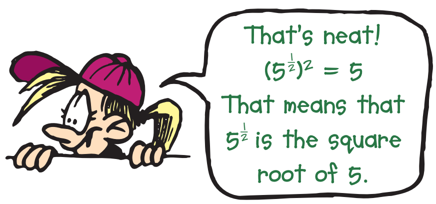
$3 \times 3 = 9$

$\sqrt{9} \times \sqrt{9} = 9$

$9^{\frac{1}{2}}$ multiplied by itself gives 9 and $\sqrt{9}$ multiplied by itself gives 9.

So $9^{\frac{1}{2}}$ is the square root of 9.

$\therefore\ 9^{\frac{1}{2}} = \sqrt{9}$



2 $5^{\frac{1}{2}} \times 5^{\frac{1}{2}} = 5^{\left(\frac{1}{2}+\frac{1}{2}\right)}$

$= 5^1$

$= 5$

Now $\sqrt{5} \times \sqrt{5} = 5$

So $5^{\frac{1}{2}} = \sqrt{5}$

The number that multiplies itself to give 5 (ie $5^{\frac{1}{2}}$) is the square root of 5.

3 Similarly:

$8^{\frac{1}{3}} \times 8^{\frac{1}{3}} \times 8^{\frac{1}{3}} = 8^{\left(\frac{1}{3}+\frac{1}{3}+\frac{1}{3}\right)}$

$= 8^1$

$= 8$

$2 \times 2 \times 2 = 8$

$\sqrt[3]{8} \times \sqrt[3]{8} \times \sqrt[3]{8} = 8$

Two is the cube root of 8.

So $8^{\frac{1}{3}} = \sqrt[3]{8}$, (the cube root of 8)

$\therefore\ 8^{\frac{1}{3}} = 2$

■ Since $(\sqrt[3]{x})^3 = x$, $\sqrt[3]{x} = x^{\frac{1}{3}}$

A:05E Scientific notation (or standard notation)

When expressing numbers in scientific (or standard) notation, each number is written as the product of a number between 1 and 10, and a power of 10.

$6{\cdot}1 \times 10^5$

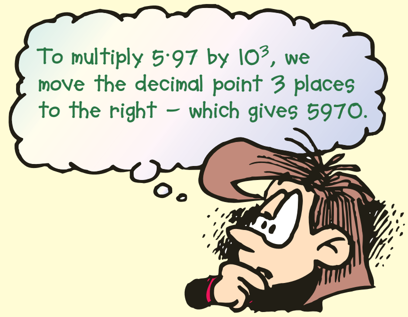
- This number is written in scientific notation (or standard form).
- The first part is between 1 and 10.
- The second part is a power of 10.

Scientific notation is useful when writing very large or very small numbers.

Numbers greater than 1

$5970\cdot = 5{\cdot}97 \times 10^3$

To write 5970 in standard form:
- put a decimal point after the first digit
- count the number of places you have to move the decimal point to the left from its original position. This will be the power needed.



worked examples

1 Express the following in scientific notation.
  a 243 b 60 000 c 93 800 000

2 Write the following as a basic numeral.
  a $1{\cdot}3 \times 10^2$ b $2{\cdot}431 \times 10^2$ c $4{\cdot}63 \times 10^7$

■ 'Scientific notation' is sometimes called 'standard notation'.

Solutions

1 a $243 = 2{\cdot}43 \times 100$
  $= 2{\cdot}43 \times 10^2$

  b $60\,000 = 6 \times 10\,000$
  $= 6 \times 10^4$

  c $93\,800\,000 = 9{\cdot}38 \times 10\,000\,000$
  $= 9{\cdot}38 \times 10^7$

If end zeros are significant, write them in your answer.
eg 60 000 (to nearest 100) = $6{\cdot}00 \times 10^4$

■ We have moved the decimal point 7 places from its original position.

2 a $1{\cdot}3 \times 10^2 = 1{\cdot}30 \times 100$
  $= 130$

  b $2{\cdot}431 \times 10^2 = 2{\cdot}431 \times 100$
  $= 243{\cdot}1$

  c $4{\cdot}63 \times 10^7 = 4{\cdot}6300000 \times 10\,000\,000$
  $= 46\,300\,000$

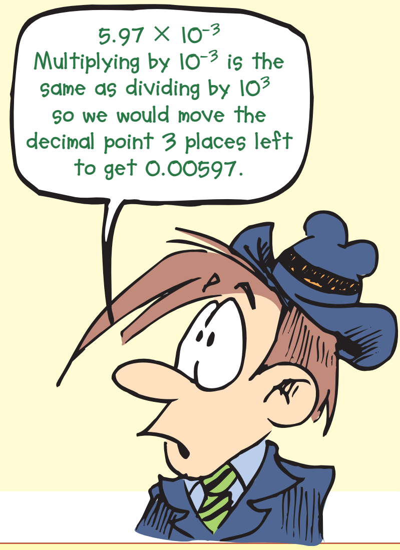
■ To multiply by 10^7, move the decimal point 7 places right.

Numbers less than 1

$0{\cdot}005\,97 = 5{\cdot}97 \times 10^{-3}$

To write 0·005 97 in scientific notation:

- put a decimal point after the first non-zero digit
- count the number of places you have moved the decimal point to the right from its original position. This will show the negative number needed as the power of 10.



$5{\cdot}97 \times 10^{-3}$ is the same as $5{\cdot}97 \div 10^3$.

worked examples

1 Express each number in scientific notation.
  a 0·043 b 0·000 059 7
  c 0·004

2 Write the basic numeral for:
  a $2{\cdot}9 \times 10^{-2}$ b $9{\cdot}38 \times 10^{-5}$
  c $1{\cdot}004 \times 10^{-3}$

Short-cut method:
0·043
- How many places must we move the decimal point for scientific notation? Answer = 2
- Is 0·043 bigger or smaller than 1? Answer = smaller
- So the power of 10 is '−2'.
  $\therefore\ 0{\cdot}043 = 4{\cdot}3 \times 10^{-2}$

Solutions

1 a $0{\cdot}043 = 4{\cdot}3 \div 100 = 4{\cdot}3 \times 10^{-2}$

  b $0{\cdot}000\,059\,7 = 5{\cdot}97 \div 100\,000 = 5{\cdot}97 \times 10^{-5}$

  c $0{\cdot}004 = 4 \div 1000 = 4 \times 10^{-3}$

2 a $2{\cdot}9 \times 10^{-2} = 002{\cdot}9 \div 100 = 0{\cdot}029$

  b $9{\cdot}38 \times 10^{-5} = 000009{\cdot}38 \div 100\,000 = 0{\cdot}000\,093\,8$

  c $1{\cdot}004 \times 10^{-3} = 0001{\cdot}004 \div 1000 = 0{\cdot}001\,004$

A:06 | Surds

A:06A The real number system

The real number system is made up of two groups of numbers: rational and irrational numbers.

Rational numbers

Any number that can be written as a fraction, $\frac{a}{b}$ where a and b are whole numbers and $b \neq 0$, is a rational number. These include integers, fractions, mixed numbers, terminating decimals and recurring decimals.

eg $\frac{7}{8}$, $6\frac{3}{5}$, 1·25, 0·07, $0{\cdot}\dot{4}$, $\sqrt{81}$

These examples can all be written as fractions.

$\frac{7}{8}$, $\frac{33}{5}$, $\frac{5}{4}$, $\frac{7}{100}$, $\frac{4}{9}$, $\frac{9}{1}$

Note: An integer is a rational number whose denominator is 1.

Irrational numbers

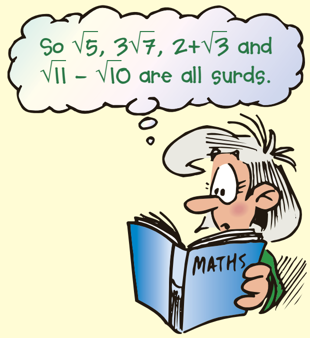
It follows that irrational numbers cannot be written as a fraction, $\frac{a}{b}$ where *a* and *b* are whole numbers. We have met a few numbers like this in our study of the circle and Pythagoras' theorem.

eg π, $\sqrt{2}$, $\sqrt[3]{4}$, $\sqrt{3} + 2$

The calculator can only give approximations for these numbers. The decimals continue without terminating or repeating.

3·141 592 65 . . ., 1·414 213 56 . . ., 1·587 401 05 . . ., 3·732 050 80 . . .

A:06B Surds



***Surds* are numerical expressions that involve irrational roots. They are irrational numbers.**

Surds obey the following rules.

Rule 1 $\sqrt{xy} = \sqrt{x} \times \sqrt{y}$

worked examples

1 $\sqrt{100} = \sqrt{4} \times \sqrt{25}$
$= 2 \times 5$
$= 10$ (which is true)

2 $\sqrt{27} = \sqrt{9} \times \sqrt{3}$
$= 3 \times \sqrt{3}$
$= 3\sqrt{3}$

3 $\sqrt{5} \times \sqrt{7} = \sqrt{5 \times 7}$
$= \sqrt{35}$

Rule 2 $\sqrt{\frac{x}{y}} = \frac{\sqrt{x}}{\sqrt{y}}$

Note: $\sqrt{x}$ means *the positive square root of x* when $x > 0$.
$\sqrt{x} = 0$ when $x = 0$.

worked examples

1 $\sqrt{\frac{16}{4}} = \frac{\sqrt{16}}{\sqrt{4}}$
ie $\sqrt{4} = \frac{4}{2}$
$= 2$ (which is true)

2 $\sqrt{125} \div \sqrt{5} = \sqrt{125 \div 5}$
$= \sqrt{25}$
$= 5$

3 $\sqrt{30} \div \sqrt{5} = \sqrt{30 \div 5}$
$= \sqrt{6}$

Rule 3 $(\sqrt{x})^2 = x$

Note: For $\sqrt{x}$ to exist, x cannot be negative.

worked examples

1. $(\sqrt{25})^2 = (5)^2$
$= 25$

2. $(\sqrt{7})^2 = 7$

3. $(3\sqrt{2})^2 = 3^2 \times (\sqrt{2})^2$
$= 9 \times 2$
$= 18$

Simplifying surds

A surd is in its simplest form when the number under the square root sign is as small as possible. To simplify a surd we make use of Rule 1 by expressing the square root as the product of two smaller square roots, one being the root of a square number. Examine the examples below.

worked examples

Simplify the following surds.

1. $\sqrt{18} = \sqrt{9} \times \sqrt{2}$
$= 3 \times \sqrt{2}$
$= 3\sqrt{2}$

2. $\sqrt{75} = \sqrt{25} \times \sqrt{3}$
$= 5 \times \sqrt{3}$
$= 5\sqrt{3}$

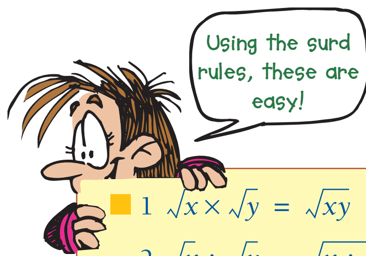
3. $5\sqrt{48} = 5 \times \sqrt{16} \times \sqrt{3}$
$= 5 \times 4 \times \sqrt{3}$
$= 20\sqrt{3}$

A:06C Multiplication and division of surds

worked examples

Simplify the following.

1. $\sqrt{7} \times \sqrt{3}$
2. $3\sqrt{2} \times 5\sqrt{2}$
3. $5\sqrt{8} \times 3\sqrt{6}$
4. $\sqrt{3}(2\sqrt{3} - \sqrt{5})$



1. $\sqrt{x} \times \sqrt{y} = \sqrt{xy}$
2. $\sqrt{x} \div \sqrt{y} = \sqrt{x \div y}$

Solutions

1. $\sqrt{7} \times \sqrt{3} = \sqrt{7 \times 3}$
$= \sqrt{21}$

2. $3\sqrt{2} \times 5\sqrt{2} = 3 \times 5 \times \sqrt{2} \times \sqrt{2}$, $(\sqrt{2} \times \sqrt{2} = 2)$
$= 15 \times 2$
$= 30$

3. $5\sqrt{8} \times 3\sqrt{6}$
$= 5 \times 3 \times \sqrt{8} \times \sqrt{6}$
$= 15 \times \sqrt{48}$
$= 15 \times 4\sqrt{3}$
$= 60\sqrt{3}$

4. $\sqrt{3}(2\sqrt{3} - \sqrt{5}) = \sqrt{3} \times 2\sqrt{3} - \sqrt{3} \times \sqrt{5}$
$= 2 \times 3 - \sqrt{3 \times 5}$
$= 6 - \sqrt{15}$

A:06D Binomial products

worked examples

Expand and simplify:

1 **a** $(\sqrt{2}+3)(\sqrt{2}-5)$ **b** $(3\sqrt{2}-\sqrt{5})(2\sqrt{2}+3\sqrt{5})$

2 **a** $(2\sqrt{3}+5)^2$ **b** $(\sqrt{7}-\sqrt{3})^2$

3 **a** $(\sqrt{5}-\sqrt{2})(\sqrt{5}+\sqrt{2})$ **b** $(5\sqrt{2}-\sqrt{7})(5\sqrt{2}+\sqrt{7})$

Solutions

1 **a** $(\sqrt{2}+3)(\sqrt{2}-5)$

$= \sqrt{2}(\sqrt{2}-5)+3(\sqrt{2}-5)$

$= (\sqrt{2})^2-5\sqrt{2}+3\sqrt{2}-15$

$= 2-2\sqrt{2}-15$

$= -13-2\sqrt{2}$

b $(3\sqrt{2}-\sqrt{5})(2\sqrt{2}+3\sqrt{5})$

$= 3\sqrt{2}(2\sqrt{2}+3\sqrt{5})-\sqrt{5}(2\sqrt{2}+3\sqrt{5})$

$= 12+9\sqrt{10}-2\sqrt{10}-15$

$= 7\sqrt{10}-3$

2 **a** $(2\sqrt{3}+5)^2$

$= (2\sqrt{3})^2+2\times 2\sqrt{3}\times 5+(5)^2$

$= 12+20\sqrt{3}+25$

$= 37+20\sqrt{3}$

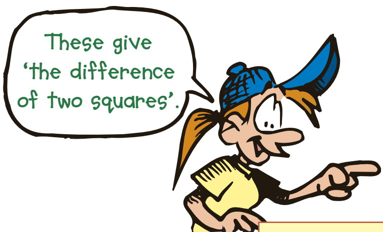
b $(\sqrt{7}-\sqrt{3})^2$

$= (\sqrt{7})^2-2\times\sqrt{7}\times\sqrt{3}+(\sqrt{3})^2$

$= 7-2\sqrt{21}+3$

$= 10-2\sqrt{21}$

Remember!
$(a+b)^2 = a^2+2ab+b^2$
$(a-b)^2 = a^2-2ab+b^2$



Remember!
$(a+b)(a-b) = a^2-b^2$

3 **a** $(\sqrt{5}-\sqrt{2})(\sqrt{5}+\sqrt{2}) = (\sqrt{5})^2-(\sqrt{2})^2$

$= 5-2$

$= 3$

b $(5\sqrt{2}-\sqrt{7})(5\sqrt{2}+\sqrt{7}) = (5\sqrt{2})^2-(\sqrt{7})^2$

$= 50-7$

$= 43$

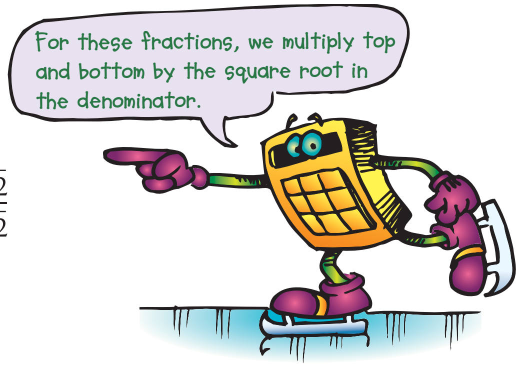
A:06E Rationalising the denominator

If a fraction has a surd (ie an irrational number) in its denominator, we generally rewrite the fraction with a 'rational' denominator by using the method shown below.

worked examples

Rewrite with rational denominators:

1 $\dfrac{3}{\sqrt{3}}$ **2** $\dfrac{1}{5\sqrt{2}}$ **3** $\dfrac{\sqrt{5}}{\sqrt{12}}$ **4** $\dfrac{2+\sqrt{3}}{2\sqrt{3}}$



Solutions

1
$$\frac{3}{\sqrt{3}} = \frac{3}{\sqrt{3}} \times \frac{\sqrt{3}}{\sqrt{3}} = \frac{\not{3}\sqrt{3}}{\not{3}} = \sqrt{3}$$

2
$$\frac{1}{5\sqrt{2}} = \frac{1}{5\sqrt{2}} \times \frac{\sqrt{2}}{\sqrt{2}} = \frac{\sqrt{2}}{5 \times 2} = \frac{\sqrt{2}}{10}$$

3
$$\frac{\sqrt{5}}{\sqrt{12}} = \frac{\sqrt{5}}{2\sqrt{3}} \times \frac{\sqrt{3}}{\sqrt{3}} = \frac{\sqrt{15}}{2 \times 3} = \frac{\sqrt{15}}{6}$$

> *Note:*
> Multiplying by $\dfrac{\sqrt{3}}{\sqrt{3}}$ is the same as multiplying by 1.

4
$$\frac{2+\sqrt{3}}{2\sqrt{3}} = \frac{2+\sqrt{3}}{2\sqrt{3}} \times \frac{\sqrt{3}}{\sqrt{3}} = \frac{\sqrt{3}(2+\sqrt{3})}{2 \times 3} = \frac{2\sqrt{3}+3}{6}$$

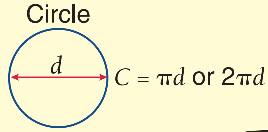
A:07 | Measurement

A:07A Perimeter

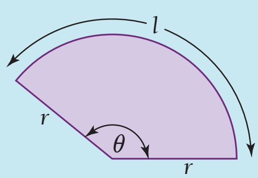
The perimeter of a plane figure is the length of its boundary.

To calculate the perimeter:
- find the lengths of all the sides
- add the lengths together.
- To find the arc length of a sector, l, first find what fraction the sector is of the circle by dividing the sector angle θ by 360°. Then find this fraction of the circumference.

Circle



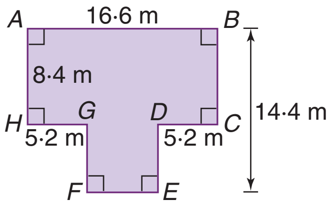
$$l = \frac{\theta}{360°} \times 2\pi r$$



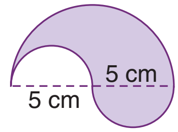
- Composite figures are formed by putting simple figures together or by removing parts of a figure. The calculation of the perimeter of composite figures is shown in the examples below.

worked examples

1 Find the perimeter of this figure.

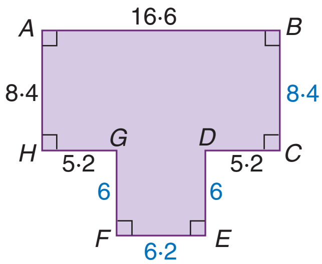


2 Find the perimeter of the figure formed from three semicircles.



Solutions

1



$5{\cdot}2 + FE + 5{\cdot}2 = 16{\cdot}6$
$FE = 6{\cdot}2$

$8{\cdot}4 + GF = 14{\cdot}4$
$\therefore GF = 6$
$BC = 8{\cdot}4$
$DE = 6$

Perimeter = sum of horizontal sides + sum of vertical sides

Sum of horizontal sides $= 16{\cdot}6 + 5{\cdot}2 + 5{\cdot}2 + 6{\cdot}2$
$= 33{\cdot}2$

Sum of vertical sides $= 8{\cdot}4 + 6 + 8{\cdot}4 + 6$
$= 28{\cdot}8$

$\therefore$ Perimeter $= 33{\cdot}2 + 28{\cdot}8$
$= 62$ m

2 Arc length of large semicircle $= \dfrac{\pi D}{2}$
$= \dfrac{\pi \times 10}{2}$
$= 5\pi$

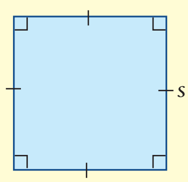
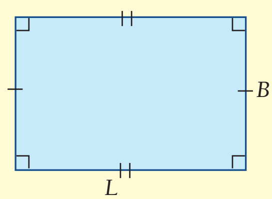
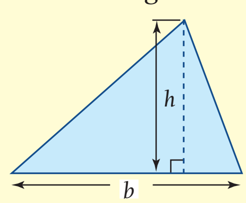
Arc length of small semicircle $= \dfrac{\pi \times 5}{2}$
$= 2{\cdot}5\pi$

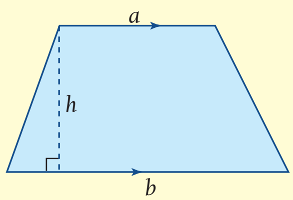
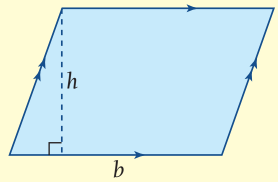
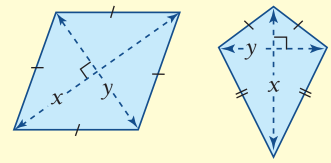
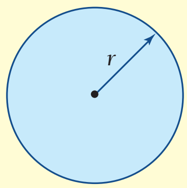
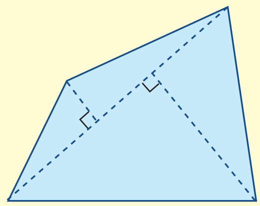
$\therefore$ Perimeter $= 5\pi + 2 \times 2{\cdot}5\pi$
$= 10\pi$
$= 31{\cdot}4$ cm (correct to one decimal place)

A:07B Area

- The area of a plane figure is the amount of space it occupies.

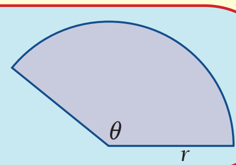
Area formulae

Square	Rectangle	Triangle
		
$A = s^2$	$A = LB$	$A = \frac{1}{2}bh$ or $A = \dfrac{bh}{2}$

Trapezium	Parallelogram	Rhombus and kite
 $A = \frac{1}{2}h(a + b)$	 $A = bh$	 $A = \frac{1}{2}xy$
Circle	**Quadrilateral**	
 $A = \pi r^2$		There is no formula. The area is found by joining opposite corners to form two triangles. The area of each triangle is calculated and the two areas added to give the area of the quadrilateral.

- To find the area of a sector, first find what fraction the sector is of the circle by dividing the sector angle θ by 360°. Then find this fraction of the area of the circle.

Area $= \dfrac{\theta}{360} \times \pi r^2$



- The area of composite figures can be calculated by either of the two methods.

Method I (by addition of parts)

We imagine that smaller figures have been joined to form the figure, as in Figures 1 and 2.

1 Copy the figure.
2 Divide the figure up into simpler parts. Each part is a shape whose area can be calculated directly, eg square or rectangle.
3 Calculate the area of the parts separately.
4 Add the area of the parts to give the area of the figure.

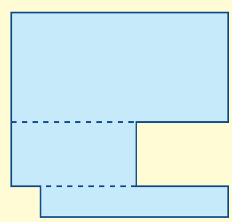


Figure 1

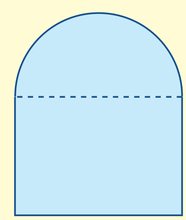
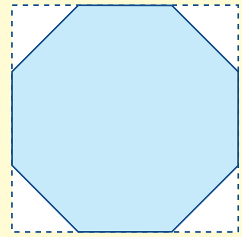


Figure 2

Method II (by subtraction)

We imagine the figure is formed by cutting away simple shapes from a larger complete figure, as shown.



1 Copy the figure and mark in the original larger figure from which it has been cut.
2 Calculate the area of the larger original figure.
3 Calculate the area of the parts that have been removed.
4 Area of figure = (area of original figure) − (area of parts that have been removed).

A:07C Surface area of prisms

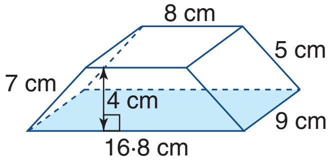
The surface area of a solid is the sum of the areas of its faces.

To calculate the surface area, you must know the number of faces and the shapes of the faces of the solid.

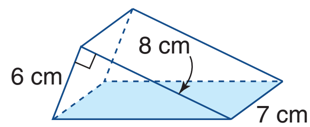
worked examples

Find the surface area of each of the following solids.

1

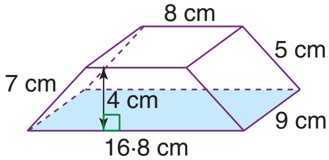


2



Solutions

1



Area of trapezoidal faces

$= 2 \times \frac{1}{2} h(a + b)$

$= 2 \times \frac{1}{2} \times 4 \times (16{\cdot}8 + 8)$

$= 99{\cdot}2\ \text{cm}^2$

Area of rectangular faces

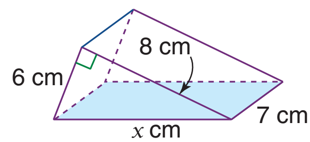
$= (7 + 8 + 5 + 16{\cdot}8) \times 9$

$= 331{\cdot}2\ \text{cm}^2$

$\therefore$ Surface area $= 331{\cdot}2 + 99{\cdot}2\ \text{cm}^2$

$= 430{\cdot}4\ \text{cm}^2$

2



First calculate x.

Now $x^2 = 6^2 + 8^2$ (Pythagoras' theorem)

$= 100$

$\therefore x = 10$

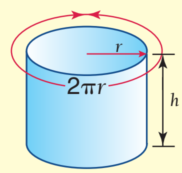
Surface area = area of triangular faces + area of rectangular faces

$= 2 \times \frac{1}{2} \times 6 \times 8 + (6 + 8 + 10) \times 7$

$= 216\ \text{cm}^2$

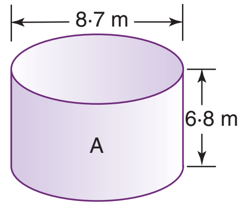
A:07D Surface area of cylinders

Surface area = curved surface area + area of circles
$= 2\pi rh + 2\pi r^2$



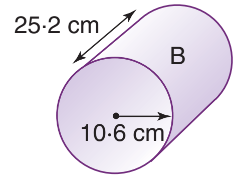
worked examples

1 Find the surface area of a cylinder that has a radius of 8 cm and a height of 9·5 cm. Give your answer correct to two decimal places.



2 For cylinder A, find:
   a the curved surface area
   b the area of the circular ends
   c the surface area
   Give the answers correct to three significant figures.

3 Find the curved surface area of cylinder B, correct to one decimal place.



Solutions

1 Surface area $= 2\pi r^2 + 2\pi rh$
$= 2 \times \pi \times 8^2 + 2 \times \pi \times 8 \times 9{\cdot}5$
$= 879{\cdot}65\text{ cm}^2$ (correct to two decimal places)

2 a Curved surface area
$= 2\pi rh$
$= 2\pi \times 4{\cdot}35 \times 6{\cdot}8$
$= 59{\cdot}16\pi$
$= 186\text{ m}^2$ (correct to 3 sig. figs.)

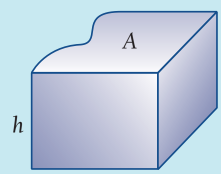
b Area of circular ends
$= 2\pi r^2$
$= 2\pi \times (4{\cdot}35)^2$
$= 37{\cdot}845\pi$
$= 119\text{ m}^2$ (correct to 3 sig. figs.)

c Surface area
$= 59{\cdot}16\pi + 37{\cdot}845\pi$
$= 305\text{ m}^2$ (correct to 3 sig. figs.)

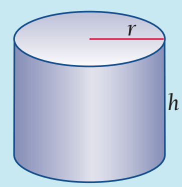
3 Curved surface area
$= 2\pi rh$
$= 2 \times \pi \times 10{\cdot}6 \times 25{\cdot}2$
$= 1678{\cdot}4\text{ cm}^2$ (correct to 1 dec. pl.)

A:07E Volume of prisms and cylinders

The volume of all prisms, cylinders and prism-like solids is given by the formula
$$V = Ah$$
where:
V = **volume**
A = **cross-sectional area**
h = **height of the prism.**



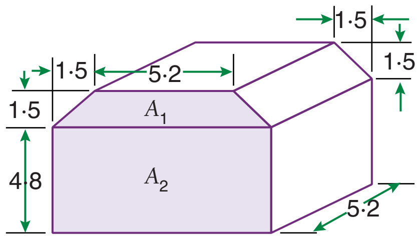
For a cylinder, the cross-section is a circle and $A = \pi r^2$. The formula is then rewritten as
$$V = \pi r^2 h$$



worked examples

Find the volume of these solids.

1

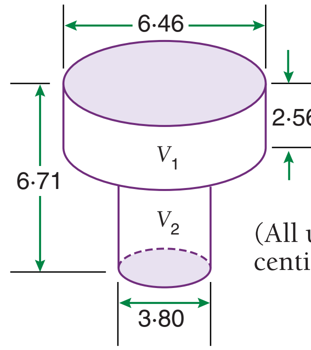


(All units are centimetres.)

Volume = *Ah*

Area of cross-section
$= A_1 + A_2$
$= \frac{1}{2}(1{\cdot}5)(5{\cdot}2 + 8{\cdot}2) + (8{\cdot}2 \times 4{\cdot}8)$
$= 49{\cdot}41\ \text{cm}^2$
$\therefore\ V = Ah$
$= 49{\cdot}41 \times 5{\cdot}2$
$= 256{\cdot}932\ \text{cm}^3$

2



(All units are centimetres.)

Volume = $V_1 + V_2$

Volume of cylinder V_1
$= \pi r^2 h$
$= \pi \times 3{\cdot}23^2 \times 2{\cdot}56$
$\doteqdot 83{\cdot}906\ \text{cm}^3$
Volume of cylinder V_2
$= \pi r^2 h$
$= \pi \times 1{\cdot}90^2 \times (6.71 - 2{\cdot}56)$
$\doteqdot 47{\cdot}066\ \text{cm}^3$
$\therefore$ Total volume
$\doteqdot 83{\cdot}906 + 47{\cdot}066$
$\doteqdot 130{\cdot}972\ \text{cm}^3$

A:08 | Equations, Inequations and Formulae

A:08A Equivalent equations

If one equation can be changed into another by performing the same operation on both sides, then the equations are said to be equivalent.

We solve equations by making a series of equivalent equations, each one in the series being simpler than the one before it. In this way we reduce a complicated equation to a simple one. We must remember to perform the same operation on both sides of the equation.

worked examples

1 $8a + 6 = 15$ **2** $1 - 3b = 7$ **3** $5a - 7 = a + 2$ **4** $7 - y = 5 - 2y$

Solutions

1 $8a + 6 = 15$

$-6 \quad -6$ Subtract 6 from both sides.

$8a = 9$

$\div 8 \quad \div 8$ Divide both sides by 8.

$\frac{8a}{8} = \frac{9}{8}$

$\therefore a = 1\frac{1}{8}$

2 $1 - 3b = 7$

$-1 \quad -1$ Subtract 1 from both sides.

$-3b = 6$

$\div -3 \quad \div -3$ Divide both sides by −3.

$\frac{-3b}{-3} = \frac{6}{-3}$

$\therefore b = -2$

3 $5a - 7 = a + 2$

$-a = -a$ Subtract *a* from both sides.

$4a - 7 = 2$

$+7 \quad +7$ Add 7 to both sides.

$4a = 9$

$\div 4 \quad \div 4$ Divide both sides by 4.

$\frac{4a}{4} = \frac{9}{4}$

$a = 2\frac{1}{4}$

4 $7 - y = 5 - 2y$

$+2y \quad +2y$ Add 2*y* to both sides.

$7 + y = 5$

$-7 \quad -7$ Subtract 7 from both sides.

$y = 5 - 7$

$\therefore y = -2$

A:08B Equations with grouping symbols

If you remember how to 'expand' grouping symbols, these equations are no harder than the ones you have already revised. Look at these worked examples.

worked examples

1 Expand the grouping symbols and then solve the equation.

a $2(x + 3) = 8$

$2x + 6 = 8$

$-6 \quad -6$

$2x = 2$

$\div 2 \quad \div 2$

$\therefore x = 1$

b $5(a - 3) = 3$

$5a - 15 = 3$

$+15 \quad +15$

$5a = 18$

$\div 5 \quad \div 5$

$\therefore a = \frac{18}{5}$ or $3\frac{3}{5}$

c $3(2m - 4) = 4m - 6$

$6m - 12 = 4m - 6$

$-4m \quad -4m$

$2m - 12 = -6$

$+12 \quad +12$

$2m = 6$

$\div 2 \quad \div 2$

$\therefore m = 3$

2 Expand each set of grouping symbols and then solve the equations.

a $3(a + 7) = 4(a - 2)$

$3a + 21 = 4a - 8$

$-3a \quad -3a$

$21 = a - 8$

$+8 \quad +8$

$29 = a$

$\therefore a = 29$

b $3(x + 4) + 2(x + 5) = 4$

$3x + 12 + 2x + 10 = 4$

Collect like terms.

$5x + 22 = 4$

$-22 \quad -22$

$5x = -18$

$\div 5 \quad \div 5$

$\therefore x = -3\frac{3}{5}$

A:08C Equations with fractions

Single denominator

worked examples

Find the value of the pronumeral in each of the following equations.

1 $\frac{y}{6} - 1 = 3$ **2** $\frac{x+7}{4} = 8$ **3** $\frac{3x-1}{5} = 7$

Solutions

1

$$\frac{y}{6} - 1 = 3$$
$$\times 6 \quad \times 6$$
$$6\left(\frac{y}{6} - 1\right) = 6 \times 3$$
$$\frac{6y}{6} - 6 = 18$$
$$y - 6 = 18$$
$$\therefore y = 24$$

2

$$\frac{x+7}{4} = 8$$
$$\times 4 \quad \times 4$$
$$\frac{\cancel{4}(x+7)}{\cancel{4}} = 8 \times 4$$
$$x + 7 = 32$$
$$\therefore x = 25$$

3

$$\frac{3x-1}{5} = 7$$
$$\times 5 \quad \times 5$$
$$\frac{\cancel{5}(3x-1)}{\cancel{5}} = 7 \times 5$$
$$3x - 1 = 35$$
$$3x = 36$$
$$\therefore x = 12$$

More than one denominator

To simplify these equations, we must multiply by the lowest common multiple of all the denominators. (Or, in other words, we must multiply by some number that will cancel out every denominator.)

worked examples

Solve:

1 $\frac{3x}{5} - \frac{x}{4} = 1$ **2** $\frac{a}{5} - 1 = \frac{3a-1}{2} + 4$ **3** $\frac{m+2}{3} - \frac{m-5}{4} = 6$

Solutions

1

$$\frac{3x}{5} - \frac{x}{4} = 1$$

Multiply both sides by 20.

$$\therefore 20\left(\frac{3x}{5} - \frac{x}{4}\right) = 1 \times 20$$
$$\therefore \frac{60x}{5} - \frac{20x}{4} = 20$$
$$\therefore 12x - 5x = 20$$
$$\therefore 7x = 20$$
$$\therefore x = \frac{20}{7}$$
$$\therefore x = 2\tfrac{6}{7}$$

2

$$\frac{a}{5} - 1 = \frac{3a-1}{2} + 4$$
$$\therefore \frac{a}{5} = \frac{3a-1}{2} + 5$$

Multiply both sides by 10.

$$\therefore \frac{{}^{2}\cancel{10}a}{\cancel{5}_1} = \frac{\cancel{10}^{5}(3a-1)}{\cancel{2}_1} + 50$$
$$\therefore 2a = 5(3a-1) + 50$$
$$\therefore 2a = 15a - 5 + 50$$
$$\therefore -13a = 45$$
$$\therefore a = \frac{45}{-13}$$
$$\therefore a = -3\tfrac{6}{13}$$

3 $\dfrac{m+2}{3} - \dfrac{m-5}{4} = 6$

Multiply both sides by 12.

$$\therefore\ 12\left(\frac{m+2}{3} - \frac{m-5}{4}\right) = 6 \times 12$$

$$\therefore\ 4(m+2) - 3(m-5) = 72$$

$$\therefore\ 4m + 8 - 3m + 15 = 72$$

$$\therefore\ m + 23 = 72$$

$$\therefore\ m = 49$$

A:08D Solving problems using equations

worked examples

Translate the following into number sentences.
In all cases use the x to represent the unknown number.

1 I multiply a number by 2 and the result is 50.
2 If I add 6 to a number the answer is 11.
3 I subtract a number from 6 and the answer is 2.
4 A certain number is multiplied by 3 then 6 is added and the result is 17.

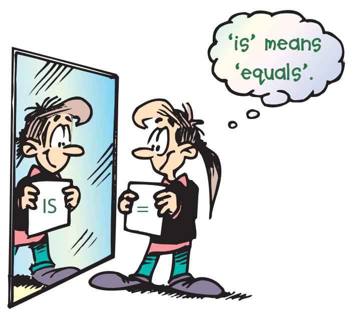
We often use x to represent an unknown number.

Solutions

1 I multiply a number by 2 and the result is 50.

$\underbrace{\text{I multiply a number by 2}}_{2 \times x}$ and the result $\underbrace{\text{is}}_{=}$ $\underbrace{\text{50}}_{50}$ → $2 \times x$ $= 50$

The equation is $2x = 50$.



2 If I add 6 to a number the answer is 11.

$6 + x$ $= 11$

The equation is $6 + x = 11$.

3 I subtract a number from 6 and the answer is 2.

$6 - x$ $= 2$

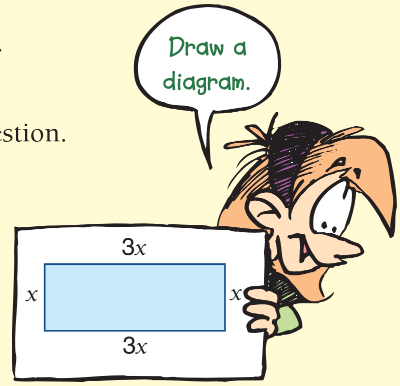
The equation is $6 - x = 2$.

4 A certain number is multiplied by 3, then 6 is added and the result is 17.

$x \times 3$ $+ 6$ $= 17$

The equation is $3x + 6 = 17$.

To use equations to solve problems we must be able to analyse a written problem, translate it into an equation and then solve it.



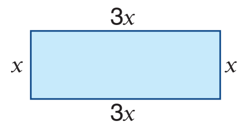
Approach

- Read the problem carefully, examining the wording of the question.
- Establish what is to be found and what information is given.
- Ask yourself whether any other information can be assumed, eg that a pack of cards mentioned is a standard pack.
- Try to connect the given information to form an equation. This will often require a knowledge of a formula or the meaning of mathematical terms.

worked examples

Example 1

A rectangle is three times longer than it is wide. If it has a perimeter of 192 m, what are its dimensions?



Solution 1

Let the width be x metres.

$\therefore$ The length $= 3 \times x$ metres

$= 3x$ metres

Now perimeter means the sum of the lengths of the sides (or the distance around the outside of the figure).

$\therefore\ 3x + x + 3x + x = 192$

$\therefore\ 8x = 192$

$\therefore\ x = 24$

$\therefore$ The width $= 24$ m

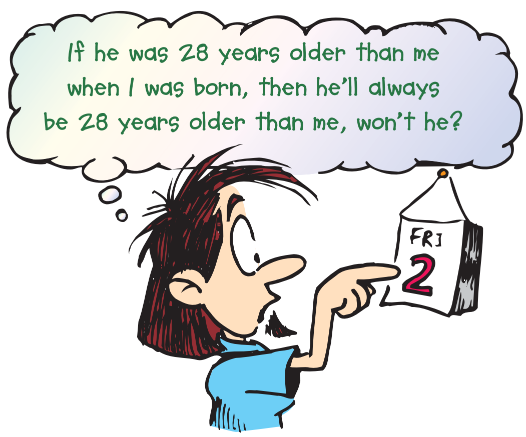
and the length $= 3 \times 24$ m

$= 72$ m

In the first line of each solution, indicate what the pronumeral represents.

Example 2

My father was 28 years old when I was born. If he is now three times as old as I am, what are our present ages?



Solution 2

Let my present age be x years.

$\therefore$ My father's present age is $3 \times x$ years.

When I was born my father was 28.

$\therefore$ The difference in our ages is 28 years.

$\therefore$ Father's age − my age always equals 28 years.

$\therefore\ 3x - x = 28$

$2x = 28$

$x = 14$

$\therefore$ I am 14 years old and my father is 42 years old (ie 3×14 years).

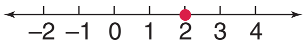
A:08E Inequations

An inequation is a number sentence where the 'equals' sign has been replaced by an inequality sign. The most common inequality signs are:

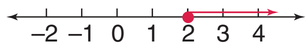
$>$	$<$	$\geqslant$	$\leqslant$
'is greater than'	'is less than'	'is greater than or equal to'	'is less than or equal to'

The solutions of inequations are often graphed on a number line.

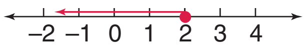
worked examples

1


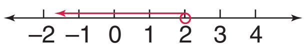
This shows the solution $x = 2$.

2


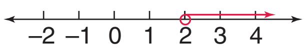
This shows the solution $x \geqslant 2$.

3


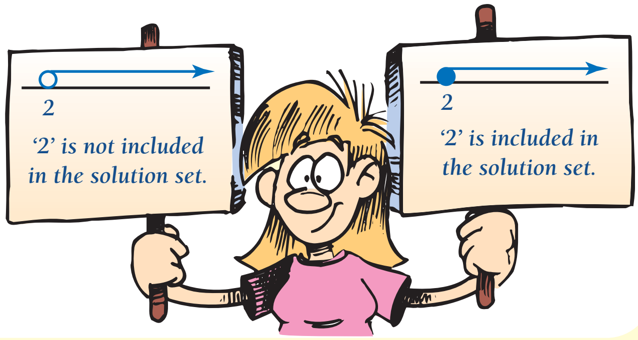
This shows the solution $x \leqslant 2$.

4


This shows the solution $x < 2$.

5


This shows the solution $x > 2$.



Inequations, unlike equations, usually have more than one solution. For instance:

- the equation $x + 6 = 10$ has one solution, namely $x = 4$.
- the inequation $x + 6 > 10$ has an infinite number of solutions. The numbers $4\frac{1}{2}$, 8, 9·5, 30 are some solutions. The full set of solutions is written as $x > 4$.

When multiplying or dividing an inequation by a negative numeral, the inequality sign must be reversed to obtain an equivalent inequality.

■ $-x \leqslant 1$ is the same as $x \geqslant -1$.

worked examples

Solve the following inequations.

1 $2x + 3 < 6$

2 $\dfrac{x}{2} - 3 \leqslant 7$

3 $5 - 3x > 6$

4 $-\frac{1}{3}x < 5$

5 $2(1 - 2x) \leqslant 6$

Solutions

1

$$
\begin{aligned}
2x + 3 &< 6 \\
-3 \quad & \quad -3 \\
\therefore 2x &< 3 \\
\div 2 \quad & \quad \div 2 \\
\therefore x &< \tfrac{3}{2} \\
\therefore x &< 1\tfrac{1}{2}
\end{aligned}
$$

2

$$
\begin{aligned}
\frac{x}{2} - 3 &\leqslant 7 \\
+3 \quad & \quad +3 \\
\therefore \frac{x}{2} &\leqslant 10 \\
\times 2 \quad & \quad \times 2 \\
\therefore 2 \times \frac{x}{2} &\leqslant 10 \times 2 \\
\therefore x &\leqslant 20
\end{aligned}
$$

3

$$
\begin{aligned}
5 - 3x &> 6 \\
-5 \quad & \quad -5 \\
\therefore -3x &> 1 \\
\div(-3) \quad & \quad \div(-3) \\
&\text{(reverse sign)} \\
\therefore x &< -\tfrac{1}{3}
\end{aligned}
$$

4

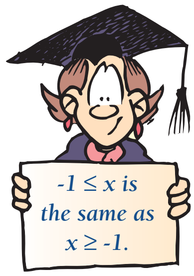
$$
\begin{aligned}
-\tfrac{1}{3}x &< 5 \\
\times(-3) \quad & \quad \times(-3) \\
&\text{(reverse sign)} \\
\therefore -\tfrac{1}{3}x \times (-3) &> 5 \times (-3) \\
\therefore x &> -15
\end{aligned}
$$

5

$$
\begin{aligned}
2(1 - 2x) &\leqslant 6 \\
\therefore 2 - 4x &\leqslant 6 \\
-2 \quad & \quad -2 \\
\therefore -4x &\leqslant 4 \\
\div -4 \quad & \quad \div -4 \\
\therefore x &\geqslant -1
\end{aligned}
$$

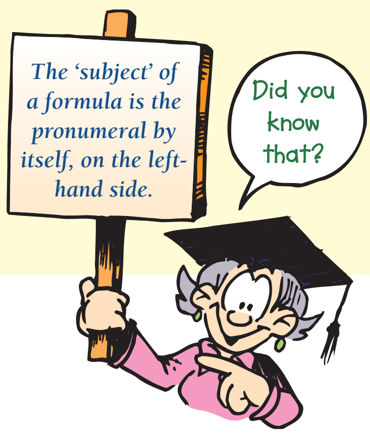
OR

$$
\begin{aligned}
2(1 - 2x) &\leqslant 6 \\
\therefore 2 - 4x &\leqslant 6 \\
+4x \quad & \quad +4x \\
\therefore 2 &\leqslant 6 + 4x \\
-6 \quad & \quad -6 \\
\therefore -4 &\leqslant 4x \\
\div 4 \quad & \quad \div 4 \\
\therefore -1 &\leqslant x \\
\therefore x &\geqslant -1
\end{aligned}
$$



A:08F Formulae

A formula is different from an equation in that it will always have more than one pronumeral. However, to find the value of a pronumeral in a formula we must be told the values of every other pronumeral in the formula.



Evaluating the subject

worked examples

1. Given that $I = \dfrac{Prn}{100}$, find I when $P = 500$, $r = 12$ and $n = 4$.
2. If $V = \frac{1}{3}Ah$, find V when $A = 15$ and $h = 4$.
3. Given that $a = 4$ and $b = 3$, find c when $c = \sqrt{a^2 + b^2}$.

Solutions

1 $P = 500$, $r = 12$ and $n = 4$

$$
\begin{aligned}
I &= \frac{Prn}{100} \\
&= \frac{{}^{5}\cancel{500} \times 12 \times 4}{\cancel{100}_1} \\
&= 5 \times 12 \times 4 \\
\therefore I &= 240
\end{aligned}
$$

2 $A = 15$ and $h = 4$

$$
\begin{aligned}
V &= \frac{1}{3}Ah \\
&= \frac{1}{3} \times 15 \times 4 \\
&= \frac{1}{\cancel{3}_1} \times \cancel{60}^{20} \\
V &= 20
\end{aligned}
$$

3 $a = 4$ and $b = 3$

$$
\begin{aligned}
c &= \sqrt{a^2 + b^2} \\
&= \sqrt{4^2 + 3^2} \\
&= \sqrt{16 + 9} \\
&= \sqrt{25} \\
c &= 5
\end{aligned}
$$

Equations arising from substitution

- We often know the value of the subject and are asked to find the value of one of the other pronumerals.
- To find the value of this pronumeral we will need to solve an equation.

worked examples

1 Given that $V = \dfrac{AH}{3}$, find H when $V = 12$ and $A = 5$.

2 $A = \frac{1}{2}h(x + y)$. Find the value of x correct to one decimal place if $A = 11$, $h = 3{\cdot}6$ and $y = 4{\cdot}5$.

3 If $S = \dfrac{a}{1 - r}$, find r when $s = 10$ and $a = 1{\cdot}5$.

Solutions

1 $V = 12$ and $A = 5$

$$V = \frac{AH}{3}$$
$$\therefore\ 12 = \frac{5H}{3}$$
$$36 = 5H$$
$$\therefore\ H = \frac{36}{5}$$
$$= 7\tfrac{1}{5}$$

2 $A = 11$, $h = 3{\cdot}6$ and $y = 4{\cdot}5$

$$A = \tfrac{1}{2}h(x + y)$$
$$\therefore\ 11 = \frac{1}{\cancel{2}_1} \times \cancel{3{\cdot}6}^{1{\cdot}8}(x + 4{\cdot}5)$$
$$11 = 1{\cdot}8(x + 4{\cdot}5)$$
$$= 1{\cdot}8x + 1{\cdot}8 \times 4{\cdot}5$$
$$11 = 1{\cdot}8x + 8{\cdot}1$$
$$11 - 8{\cdot}1 = 1{\cdot}8x$$
$$2{\cdot}9 = 1{\cdot}8x$$
$$\frac{2{\cdot}9}{1{\cdot}8} = x$$
$$\therefore\ x = 1{\cdot}6 \text{ (correct to 1 dec. pl.)}$$

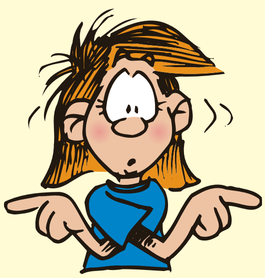
3 $S = 10$ and $a = 1{\cdot}5$

$$S = \frac{a}{1 - r}$$
$$\therefore\ 10 = \frac{1{\cdot}5}{1 - r}$$
$$10(1 - r) = 1{\cdot}5$$
$$10 - 10r = 1{\cdot}5$$
$$-10r = -8{\cdot}5$$
$$\therefore\ r = 0{\cdot}85$$

A:08G Literal equations

A formula such as $A = lb$ is written with A as its subject. This means that we can quite easily calculate A if we know the values of l and b. Sometimes, however, we need to rearrange the formula so that one of the other pronumerals is the subject. To do this, the same procedures as for solving equations are used.

In the examples, compare the solving of each equation with the changing of the subject of the formula to x, on the right.



Remember:
'+' is the opposite of '−'
'−' is the opposite of '+'
'×' is the opposite of '÷'
'÷' is the opposite of '×'

Another name for a formula is a *literal equation*.

Note:
Another way of saying 'make x the subject of this formula' is: 'solve this literal equation for x'.

worked examples

1 Solve for x.

a
$$3x + 1 = 13 \quad \text{−1 both sides}$$
$$3x = 12 \quad \text{÷3 both sides}$$
$$x = 4$$

b
$$5 - 2x = 1 \quad +2x \text{ both sides}$$
$$5 = 1 + 2x \quad \text{−1 both sides}$$
$$4 = 2x \quad \text{÷2 both sides}$$
$$2 = x$$
ie $x = 2$

c
$$3(x + 2) = 5 \quad \text{Expand}$$
$$3x + 6 = 5 \quad \text{−6 both sides}$$
$$3x = -1 \quad \text{÷3 both sides}$$
$$x = \frac{-1}{3}$$

2 Make x the subject.

a
$$ax + b = c \quad -b \text{ both sides}$$
$$ax = c - b \quad \div a \text{ both sides}$$
$$x = \frac{c - b}{a}$$

b
$$m - nx = p \quad +nx \text{ both sides}$$
$$m = p + nx \quad -p \text{ both sides}$$
$$m - p = nx \quad \div n \text{ both sides}$$
$$\frac{m - p}{n} = x$$
ie $x = \dfrac{m - p}{n}$

c
$$a(x + b) = c \quad \text{Expand}$$
$$ax + ab = c \quad -ab \text{ both sides}$$
$$ax = c - ab \quad \div a \text{ both sides}$$
$$x = \frac{c - ab}{a}$$

3 Each formula below has had its subject changed to the capital letter. The operation done to each side is shown for each step.

a
$$v = u + aT \quad -u \text{ both sides}$$
$$v - u = aT \quad \div a \text{ both sides}$$
$$\frac{v - u}{a} = T$$
$$\therefore T = \frac{v - u}{a}$$

b
$$m = \tfrac{1}{2}(x + Y) \quad \times 2 \text{ both sides}$$
$$2m = x + Y \quad -x \text{ both sides}$$
$$2m - x = Y$$
$$\therefore Y = 2m - x$$

c
$$t = a + (N - 1)d \quad \text{Expand}$$
$$t = a + Nd - d \quad -a \text{ both sides}$$
$$t - a = Nd - d \quad +d \text{ both sides}$$
$$t - a + d = Nd \quad \div d \text{ both sides}$$
$$\frac{t - a + d}{d} = N$$
$$\therefore N = \frac{t - a + d}{d}$$

d
$$a = 2\pi r(r + H) \quad \text{Expand}$$
$$a = 2\pi r^2 + 2\pi rH \quad -2\pi r^2 \text{ both sides}$$
$$a - 2\pi r^2 = 2\pi rH \quad \div 2\pi r \text{ both sides}$$
$$\frac{a - 2\pi r^2}{2\pi r} = H$$
$$\therefore H = \frac{a - 2\pi r^2}{2\pi r}$$

To change the subject of a formula (solve a literal equation):
1 Expand parentheses if applicable.
2 By using inverse operations, isolate the pronumeral required to be the subject.

The formulae may also contain a squared term or a square root sign, or the pronumeral to become the subject may appear more than once.

Remember!
$\sqrt{\ }$ is the opposite of $(\ )^2$.
$(\ )^2$ is the opposite of $\sqrt{\ }$.

worked examples

Change the subject of the formula to the letter indicated in brackets.

1 $E = mc^2$ [c] **2** $v^2 = u^2 - 2as$ [u] **3** $r = \sqrt{\dfrac{A}{\pi}}$ [A]

4 $a = 6 - \dfrac{12}{R}$ [R] **5** $y = \dfrac{A}{A + 2}$ [A]

Solutions

1
$$E = mc^2 \quad \div m \text{ both sides}$$
$$\frac{E}{m} = c^2 \quad \sqrt{\ } \text{ both sides}$$
$$\therefore c = \pm\sqrt{\frac{E}{m}}$$

2
$$v^2 = u^2 - 2as \quad + 2as \text{ both sides}$$
$$v^2 + 2as = u^2 \quad \sqrt{\ } \text{ both sides}$$
$$\pm\sqrt{v^2 + 2as} = u$$
$$\therefore u = \pm\sqrt{v^2 + 2as}$$

3
$$r = \sqrt{\frac{A}{\pi}} \quad \text{Square both sides}$$
$$r^2 = \frac{A}{\pi} \quad \times \pi \text{ both sides}$$
$$\pi r^2 = A$$
$$\therefore A = \pi r^2$$

4
$$a = 6 - \frac{12}{R} \quad \times R \text{ both sides}$$
$$aR = 6R - 12 \quad - 6R \text{ both sides}$$
$$aR - 6R = -12 \quad \text{Factorise L.H.S.}$$
$$R(a - 6) = -12 \quad \div (a - 6) \text{ both sides}$$
$$\therefore R = \frac{-12}{a - 6}$$

5
$$y = \frac{A}{A + 2} \quad \times (A + 2) \text{ both sides}$$
$$y(A + 2) = A \quad \text{Expand L.H.S.}$$
$$Ay + 2y = A \quad -Ay \text{ both sides}$$
$$2y = A - Ay \quad \text{Factorise R.H.S.}$$
$$2y = A(1 - y) \quad \div (1 - y) \text{ both sides}$$
$$\frac{2y}{1 - y} = A$$
$$\therefore A = \frac{2y}{1 - y}$$

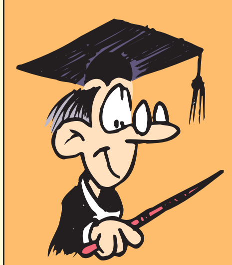
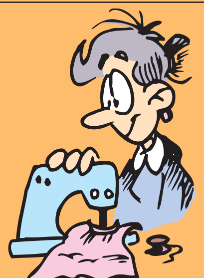
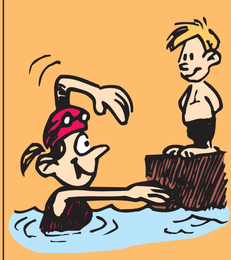
Remember!
Sometimes formulae are called *literal equations*. When literal equations are 'solved' for a certain pronumeral, it is the same as changing the subject of the formula to that pronumeral.

If the pronumeral that is to be the subject appears in more than one term in the formula, gather the terms together and factorise as in examples 4 and 5.

A:09 | Consumer Arithmetic

A:09A Earning an income

Some people work for themselves and charge a fee for their services or sell for a profit, but most people work for others to obtain an income. In the chart below, the main ways of earning an income from an employer are introduced.

Employment				
Salary	*Piece work*	*Casual*	*Commission*	*Wages*
Meaning				
A fixed amount is paid for the year's work even though it may be paid weekly or fortnightly.	The worker is paid a fixed amount for each piece of work completed.	A fixed rate is paid per hour. The person is not permanent, but is employed when needed.	This payment is usually a percentage of the value of goods sold.	Usually paid weekly to a permanent employee and based on an hourly rate, for an agreed number of hours per week.
Advantages				
Permanent employment. Holiday and sick pay. Superannuation. A bonus may be given as an incentive or time off for working outside normal working hours.	The harder you work, the more you earn. You can choose how much work you do and in some cases the work may be done in your own home.	A higher rate of pay is given as compensation for other benefits lost. Part-time work may suit some, or casual work may be a second job. Superannuation may be paid.	The more you sell, the more you are paid. Some firms pay a low wage plus a commission to act as an incentive.	Permanent employment. Holiday and sick pay. Superannuation. If additional hours are worked, additional money is earned, sometimes at a higher hourly rate of pay.
Disadvantages				
During busy periods, additional hours might be worked, without additional pay. Very little flexibility in working times, eg 9 am–5 pm.	No holiday or sick pay. No fringe benefits. No permanency of employment in most piece work.	No holiday or sick pay. No permanency of employment. Few fringe benefits. Less job satisfaction.	There may be no holiday or sick pay. If you sell nothing, you are paid nothing. Your security depends on the popularity of your product.	There is little incentive to work harder, since your pay is fixed to time, not effort. Little flexibility in working times, eg 9 am–5 pm.
Salary	*Piece work*	*Casual*	*Commission*	*Wages*
 teachers	 dressmakers	 swimming instructors	 sales people	 mechanics

Superannuation is an investment fund usually contributed to by both employer and employee on the employee's behalf. It provides benefits for employees upon retirement or for the widow or widower if the member dies.

Overtime is time worked in excess of a standard day or week. Often a rate of $1\frac{1}{2}$ or 2 times the normal rate of pay is paid for overtime.

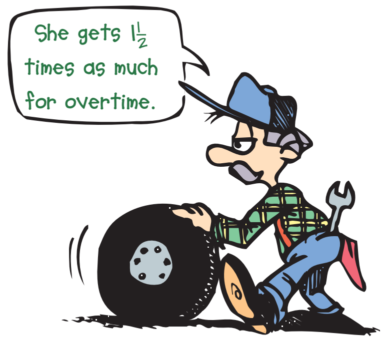
A **bonus** is money or an equivalent given in addition to an employee's usual income.

Holiday loadings are payments made to workers in addition to their normal pay. It is calculated as a set percentage of the normal pay which would be earned in a fixed number of weeks. It is usually paid at the beginning of annual holidays to meet the increased expenses often occurring then.

worked examples

1 June is paid \$10.56 per hour and time-and-a-half for overtime. If a normal day's working time is 7 hours, how much would she be paid for 10 hours' work in one day?

Overtime worked = 3 hours
Overtime pay = (\$10.56 × 3) × 1·5
= \$47.52
Normal pay = \$10.56 × 7
= \$73.92
∴ pay for the 10 h = \$47.52 + \$73.92
= \$121.44
∴ June would be paid \$121.44 for the 10 hours' work.



2 Michael receives a holiday loading of $17\frac{1}{2}$% on 4 weeks' normal pay. If he works 37 hours in a normal week and is paid \$11.25 per hour, how much money does he receive as his holiday loading?

Michael's pay for 4 weeks = (\$11.25 × 37) × 4
= \$1665
$17\frac{1}{2}$% holiday loading = 17·5% of \$1665
= \$291.38 (to the nearest cent)
∴ Michael receives \$291.38 as his holiday loading.

A:09B Income tax

- The annual *Income Tax Return* is a form, filled out each year, to determine the exact amount of tax that has to be paid, for the preceding 12 months. Since most people have been paying tax as they have earned their income, this exercise may mean that a *tax rebate* is given.
- Some expenses, such as those necessary in the earning of our income, are classified as *tax deductions* and the tax we have paid on this money will be returned to us. On the other hand, if we have additional income (such as interest on savings) that has not yet been taxed, additional taxes will have to be paid.

  The tax deductions are subtracted from the total income to provide the *taxable income*.
- The tax to be paid on the *taxable income* can be calculated from the table on the next page (2003 scale).

- If your taxable income is more than $13 800, your Medicare levy is 1·5% of that taxable income. This covers you for basic medical costs.

TABLE 1—Resident for full year (2003)	
Taxable income	*Tax on this income*
$1–$6000	Nil
$6001–$20 000	17 cents for each $1 over $6000
$20 001–$50 000	$2380 + 30 cents for each $1 over $20 000
$50 001–$60 000	$11 380 + 42 cents for each $1 over $50 000
$60 001 and over	$15 580 + 47 cents for each $1 over $60 000

worked example

Alan received a salary of $47 542 and a total from other income (investments) of $496. His total tax deductions were $1150. During the year he had already paid tax instalments amounting to $10 710.75. Find:

1 his total income
2 his taxable income
3 how much Alan must pay as his Medicare levy
4 the tax payable on his taxable income
5 his refund due or balance payable when the Medicare levy is included
6 how much extra Alan would receive each week if he is given a wage rise of $10 per week

Solutions

1 Alan's total income
= $47 542 + $496
= $48 038

2 Alan's taxable income
= total income − tax deductions
= $48 038 − $1150
= $46 888

3 Medicare levy
= 1·5% of the taxable income
= 1·5% of $46 888
= $703.32

4 Taxable income = $46 888 (or $20 000 + $26 888)
Tax on $20 000 = $2380.00 (from the table above) . . . (A)
Tax on $26 888 at 30 cents = $8066.40 (30c/$ for amount over $20 000) . . . (B)
∴ Tax on $46 888 = (A) + (B)
= $2380 + $8066.40
= $10 446.40

5 Tax on $46 888 + Medicare levy
= $10 446.40 + $703.32
= $11 149.72
Tax instalments paid = $10 710.75
∴ Balance payable = $11 149.72 − $10 710.75
= $438.97

6 For salaries over $20 000 and less than $50 001, for each additional $1 earned you pay 30 cents tax and a Medicare levy of 1·5%.
∴ Tax on an extra $10 per week = 10 × $0.30 + 1·5% of $10
= $3.00 + $0.15
∴ Amount left after tax = $10 − $3.15
= $6.85 per week

A:09C Best buy

worked example

'Aussi' coffee costs \$12.40 for 500 g. 'Ringin' coffee costs \$7.80 for 300 g. Which brand is the better value? (Assume quality is similar.)

Cost of 500 g of 'Aussi' coffee = \$12.40
∴ cost of 100 g of 'Aussi' coffee = \$12.40 ÷ 5
= \$2.48

Cost of 300 g of 'Ringin' coffee = \$7.80
∴ cost of 100 g of 'Ringin' coffee = \$7.80 ÷ 3
= \$2.60

Clearly, 'Aussi' coffee is the better value.

A:09D Goods and services tax (GST)

The GST is a broad-based tax of 10% on most goods and services you buy. GST is included in the price you pay. However, because no GST is applied to some items such as basic food goods, a bill or shopping docket may itemise each product showing whether the GST was charged and how much GST was included in the bill.

It is simple, however, to calculate the GST included in a price by dividing by 11, since the base price has been increased by 10% or $\frac{1}{10}$.

■ To calculate the GST to add on to a price, simply find 10% of the price.

■ To find the GST included in a price, divide the price by 11.

worked examples

1 Find the GST that needs to be applied to a price of \$325.
2 What is the retail price of a DVD player worth \$325 after the GST has been applied?
3 How much GST is contained in a price of \$357.50?
4 What was the price of an item retailing at \$357.50 before the GST was applied?

Solutions

1 The GST is 10% of the price.

$\therefore$ GST = \$325 × 10%
= \$32.50

2 The GST is added on to get the retail price.

$\therefore$ Retail price = \$325 + \$32.50
= \$357.50

Note: The retail price can also be calculated by multiplying the original price by 110% (or 1·1) since 10% is added on

ie Retail price = \$325 × 1·1
= \$357.50

3 To find the GST contained in a price, we divide it by 11. (If the original price is increased by $\frac{1}{10}$, then the retail price, including the GST, is $\frac{11}{10}$ of the original price.)

$\therefore$ GST = \$357.50 ÷ 11
= \$32.50

4 To find the original price, simply subtract the GST from the retail price.

$\therefore$ Original price = \$357.50 − \$32.50
= \$325

Note: The original price can also be found by multiplying the retail price by $\frac{10}{11}$.

ie Original price = \$357.50 × $\frac{10}{11}$
= \$325

A:09E Ways of paying: Discounts

When buying the things we need, we can pay cash (or cheque or use electronic fund transfer), buy on terms or use credit cards. The wise buyer will seek discounts wherever possible, comparing prices at different stores.

EFTPOS stands for electronic funds transfer (at) point of sale.

Using money			
Seeking discount	*Buying with credit card*	*Buying on terms*	*Paying cash or transferring funds*
Meaning			
A process of bargaining to seek a reduced price.	A readily acceptable method of making credit purchases. 'Buy now pay later.'	A way of having the item and spreading the payment over a period of time. (Hire-purchase)	An immediate payment with cheque, electronic funds transfer (EFTPOS) card or money.
Advantages			
You pay less because you can challenge one shop to beat the price of another. Taking time allows you to compare the quality of items.	Convenient. Safer than carrying large sums of money. Useful in meeting unexpected costs. If payment is made promptly, the charge is small. Many stores accept credit cards.	You can buy essential items and make use of them as you pay. Buying a house on terms saves rent. The item bought may be used to generate income. Little immediate cost.	Paying cash may help you get a discount. Money is accepted anywhere. You own the item. You keep out of debt. It doesn't encourage impulse buying. With cheque or EFTPOS card you don't have to carry a lot of money.
Disadvantages			
It takes time and energy to compare prices. To get the best price you may have less choice in things like colour, after-sales service and maybe condition of the item. 'Specials' are discounts.	There is a tendency to overspend, and to buy on impulse and not out of need. The interest charged on the debt is high. Hidden costs (stamp duty and charge on stores) generally lift prices.	Relies on a regular income and, if you cannot continue payments, the item can be repossessed, sold and, if its value has depreciated (dropped), you still may owe money. High interest rates. You are in debt.	Carrying large sums of money can be dangerous (risk of loss) and some shops won't accept cheques or EFTPOS cards. You may miss out on a good buy if you don't carry much money with you.

worked examples

1 Brenda bought a car on terms of $100 deposit and 60 monthly repayments of $179.80. The price of the car was $5000.

a How much did she pay for the car?

b How much interest did she pay on the money borrowed?

c How much money had she borrowed?

2 a Greg was given a $12\frac{1}{2}$% discount on a rug with a marked price of \$248. How much did he pay?

b A television marked at \$2240 was eventually sold for \$2128. What was the discount and what was the percentage discount given on the marked price?

c After a discount of 14% was given, I paid \$5848 for my yellow Holden. What was the original marked price?

3 Brenda bought a TV priced at \$1200 after it was discounted by 10%. Brenda received a further 5% discount because she was a member of staff. How much did she pay for the TV?

Solutions

1 a Total payments for car = deposit + payments
= \$100 + 60 × \$179.80
= \$10 888

b Interest = extra money paid
= \$10 888 − \$5000
= \$5888

c Amount borrowed = price of car − deposit
= \$5000 − \$100
= \$4900

2 a Discount on rug = 12·5% of \$248
= 0·125 × \$248
= \$31

Amount paid = \$248 − \$31
= \$217

b Discount on TV = \$2240 − \$2128
= \$112

Percentage discount = (\$112 ÷ \$2240) × 100%
= 5%

c Price paid = (100 − 14)% of marked price
= 86% of marked price

1% of marked price = \$5848 ÷ 86
= \$68

100% of marked price = \$6800

3 Price after original 10% discount = (100 − 10)% of \$1200
= 90% of \$1200
= \$1080

This is an example of successive discounts.

Price after a further 5% discount = (100 − 5)% of \$1080
= 95% of \$1080
= \$1026

(*Note:* This is not the same as a 15% discount off the original price ie 85% of \$1200 = \$1020.)

A:09F Working for a profit

People who work for themselves may charge a fee for their services or sell for a profit. However, they are not the only people concerned with profit and loss. We all, from time to time, will need to consider whether our investment of time, money and effort is justified by the results. This may be in our work for charity, organisations or in our hobbies.

- **When buying and selling:**

$$\textbf{Selling price} = \textbf{Cost price} + \textbf{Profit}$$

or

$$\textbf{Profit} = \textbf{Selling price} - \textbf{Cost price}$$

Note: **If the *profit is negative* we have made a *loss*.**

- **When calculating money made:**

$$\textbf{Profit} = \textbf{Money received} - \textbf{Expenses}$$

worked examples

1 Julia bought a bicycle for \$450 which had an original cost price of \$360. Find the profit as a percentage:

a of the cost price   b of the selling price

2 A motor cycle bought for \$12 000 was sold for \$8800. Find the loss as a percentage of the original cost price.

Solutions

1 Julia's profit = Selling price − Cost price
= \$450 − \$360
= \$90

a Profit as a percentage of cost price $= \frac{\$90}{\$360} \times 100\%$
$= 25\%$

b Profit as a percentage of selling price $= \frac{\$90}{\$450} \times 100\%$
$= 20\%$

2 Profit = Selling price − Cost price
= \$8800 − \$12 000
= −\$3200

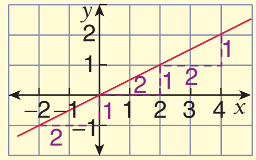
∴ A loss of \$3200 (since profit is negative)

∴ Loss as a percentage of cost price $= \frac{\$3200}{\$12\,000} \times 100\%$
$= 26{\cdot}7\%$ (to 1 dec. pl.)

A:10 | Coordinate Geometry

A:10A Gradient

The **gradient** or **slope** of a line is a measure of *how steep* it is.



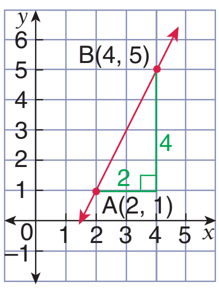
Gradient $= \dfrac{\textbf{rise}}{\textbf{run}} = \dfrac{\textbf{change in } y}{\textbf{change in } x}$

- So a gradient of $\frac{1}{2}$ means that for every run of 2 there is a rise of 1 (or for every 2 that you go across you go up 1).

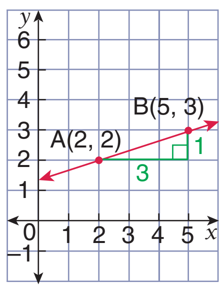
worked examples

Use the points A and B to find the gradient of the line AB in each case.

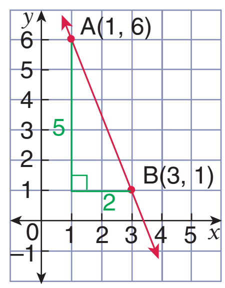
1



2



3



Solutions

1 Gradient

$= \dfrac{\text{change in } y}{\text{change in } x}$

$= \dfrac{\text{up } 4}{\text{across } 2}$

$= \dfrac{4}{2}$

$= 2$

2 $m = \dfrac{\text{change in } y}{\text{change in } x}$

$= \dfrac{\text{up } 1}{\text{across } 3}$

$= \dfrac{1}{3}$

3 $m = \dfrac{\text{change in } y}{\text{change in } x}$

$= \dfrac{\text{down } 5}{\text{across } 2}$

$= \dfrac{-5}{2}$

$= -2\frac{1}{2}$

m is used for 'gradient'

The gradient of the line that passes through the points A(x_1, y_1) and B(x_2, y_2) is given by the formula:

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

worked examples

Find the gradient of the straight line passing through the following points.

1 (1, 3) and (4, 7)

2 (6, −2) and (2, −1)

Solutions

1 Let (x_1, y_1) be (1, 3) and (x_2, y_2) be (4, 7).

$$\text{Gradient} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{7-3}{4-1} = \frac{4}{3}$$

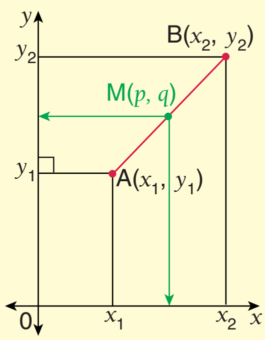
$\therefore$ The gradient is $1\frac{1}{3}$.

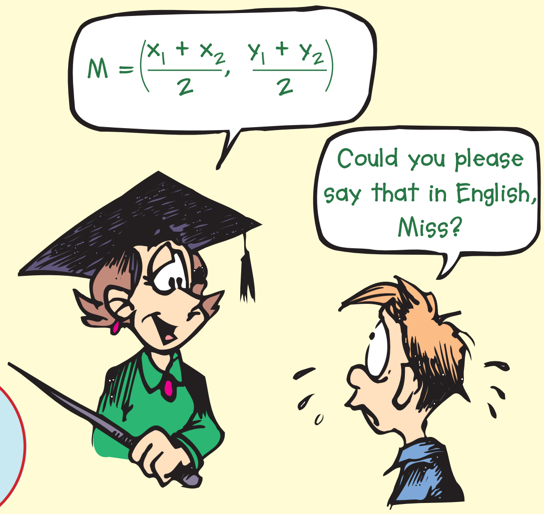
2 Let (x_1, y_1) be (6, −2) and (x_2, y_2) be (2, −1).

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{-1-(-2)}{2-6} = \frac{1}{-4}$$

$\therefore$ The gradient is $-\frac{1}{4}$.

A:10B Midpoint





The midpoint, M, of interval AB, where A is (x_1, y_1) and B is (x_2, y_2), is given by:

$$\mathbf{M} = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right).$$

worked examples

1 Find the midpoint of the interval joining (2, 6) and (8, 10).

2 Find the midpoint of interval AB, if A is the point (−3, 5) and B is (4, −2).

Solutions

1 $$\text{Midpoint} = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right) = \left(\frac{2+8}{2}, \frac{6+10}{2}\right) = (5, 8)$$

2 $$\text{Midpoint} = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right) = \left(\frac{-3+4}{2}, \frac{5+-2}{2}\right) = \left(\tfrac{1}{2}, \tfrac{3}{2}\right) \text{ or } \left(\tfrac{1}{2}, 1\tfrac{1}{2}\right)$$

A:10C Distance

Pythagoras' theorem can be used to find the distance between two points on the number plane.

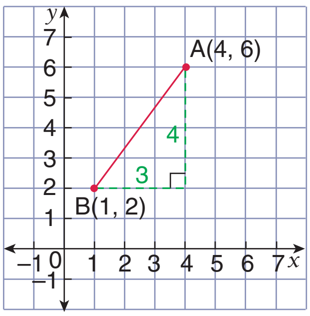
worked examples

1 Find the distance between the points (1, 2) and (4, 6).

2 If A is (−2, 2) and B is (4, 5) find the length of AB.

Solutions

1



$c^2 = a^2 + b^2$

$AB^2 = AC^2 + BC^2$

$= 4^2 + 3^2$

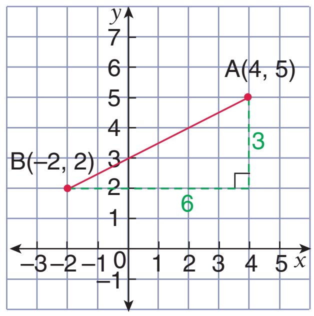
$= 16 + 9$

$= 25$

$\therefore\ AB = \sqrt{25}$

$\therefore$ the length of AB is 5 units.

2



$c^2 = a^2 + b^2$

$AB^2 = AC^2 + BC^2$

$= 3^2 + 6^2$

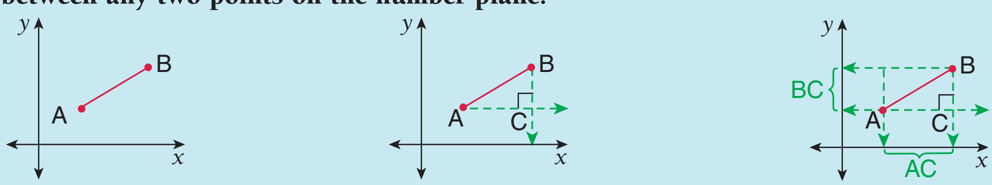
$= 9 + 36$

$= 45$

$\therefore\ AB = \sqrt{45}$

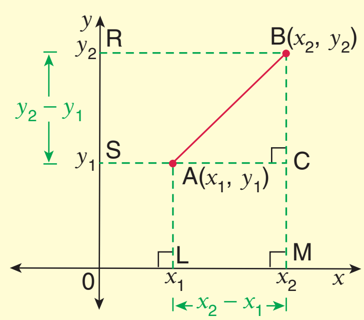
$\therefore$ the length of AB is $\sqrt{45}$ unit.

By drawing a right-angled triangle we can use Pythagoras' theorem to find the distance between any two points on the number plane.



The distance AB between $A(x_1, y_1)$ and $B(x_2, y_2)$ is given by:

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$



worked examples

1 Find the distance between the points (3, 8) and (5, 4).

2 Find the distance between the points (−2, 0) and (8, −5)

Solutions

1 Distance $= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$

$(x_1, y_1) = (3, 8)$ and $(x_2, y_2) = (5, 4)$

$$\therefore\ d = \sqrt{(5-3)^2 + (4-8)^2}$$
$$= \sqrt{(2)^2 + (-4)^2}$$
$$= \sqrt{4 + 16}$$
$$= \sqrt{20}$$

∴ Distance ≑ 4·47 (using a calculator to answer to two decimal places).

2 Distance $= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$

$(x_1, y_1) = (-2, 0)$ and $(x_2, y_2) = (8, -5)$

$$\therefore\ d = \sqrt{(8 - -2)^2 + (-5-0)^2}$$
$$= \sqrt{(10)^2 + (-5)^2}$$
$$= \sqrt{100 + 25}$$
$$= \sqrt{125}$$

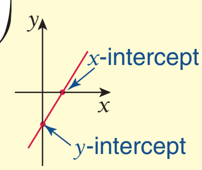
∴ Distance ≑ 11·18 (using a calculator to answer to two decimal places).

- You should check that the formula will still give the same answer if the coordinates are named in the reverse way. Hence, in example **1**, if we call $(x_1, y_1) = (5, 4)$ and $(x_2, y_2) = (3, 8)$, we would produce the same answer.

A:10D Graphing straight lines

To graph a straight line we need:
- an equation to allow us to calculate the *x*- and *y*-coordinates for each point on the line
- a table to store at least two sets of coordinates
- a number plane on which to plot the points.

Two important points on a line are:
- the *x*-intercept (where the line crosses the *x*-axis) This is found by substituting $y = 0$ into the line's equation and then solving for *x*.
- the *y*-intercept (where the line crosses the *y*-axis) This is found by substituting $x = 0$ into the line's equation and then solving for *y*.



worked examples

Draw the graph of each straight line. From the graph write down the line's *x* and *y*-intercepts.

1 $x + y = 5$ **2** $y = 3x - 2$ **3** $4x + y = 2$

Solutions

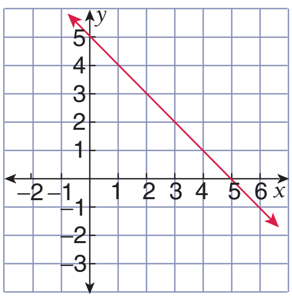
1 $x + y = 5$

x	0	1	2
y	5	4	3

When $x = 0$,
$0 + y = 5$
$\therefore y = 5$

When $x = 1$,
$1 + y = 5$
$\therefore y = 4$

When $x = 2$,
$2 + y = 5$
$\therefore y = 3$



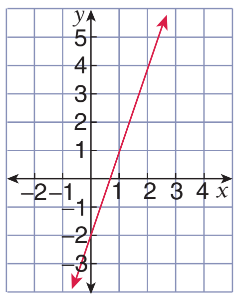
2 $y = 3x - 2$

x	0	1	2
y	−2	1	4

When $x = 0$,
$y = 3 \times 0 - 2$
$= -2$

When $x = 1$,
$y = 3 \times 1 - 2$
$\therefore y = 1$

When $x = 2$,
$y = 3 \times 2 - 2$
$\therefore y = 4$



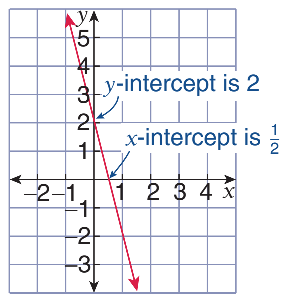
3 $4x + y = 2$

x	0	$\frac{1}{2}$
y	2	0

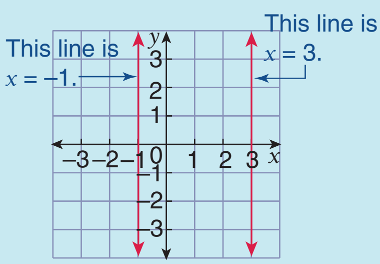
When $x = 0$,
$4 \times 0 + y = 2$
$0 + y = 2$
$\therefore y = 2$

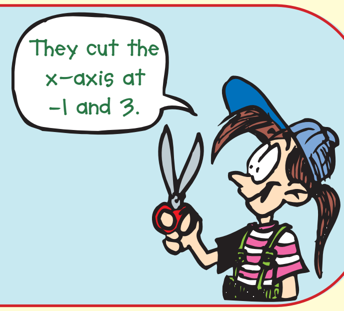
When $y = 0$,
$4x + 0 = 2$
$4x = 2$
$x = \frac{1}{2}$

$\therefore$ x-intercept $= \frac{1}{2}$
$\therefore$ y-intercept $= 2$

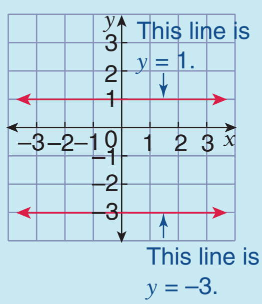


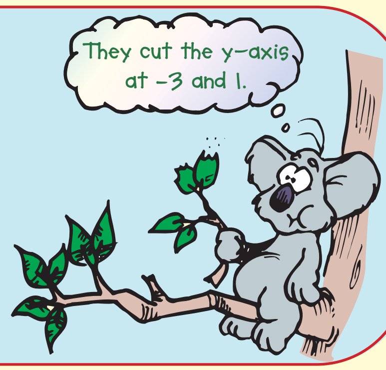
Vertical lines have equations of the form $x = a$ where a is where the line cuts the x-axis.





Horizontal lines have equations of the form $y = b$ where b is where the line cuts the y-axis.





A:10E Gradient–intercept form of a straight line: $y = mx + b$

- **When an equation of a line is written in the form $y = mx + b$,**
  m gives the gradient of the line and
  b gives the y-intercept of the line.
- **Clearly, lines with the same gradient are parallel.**
- **When an equation of a line is written in the form $ax + by + c = 0$, where a, b and c are integers and $a > 0$, it is said to be in general form.**

worked examples

1 Write down the gradient and y-intercept of these lines.

a $y = 3x - 5$

Here $m = 3$, $b = -5$.
The gradient is 3,
the y-intercept is -5.

b $y = -2x$

Here $m = -2$, $b = 0$.
The gradient is -2,
the y-intercept is 0.

c $y = 4 - 3x$

Here $m = -3$, $b = 4$.
The gradient is -3,
the y-intercept is 4.

2 Find the gradient and y-intercept from the graph and write down the equation of the line.

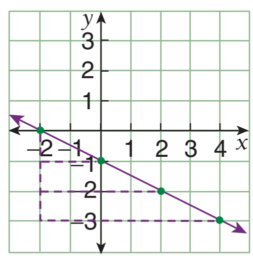
From the graph:

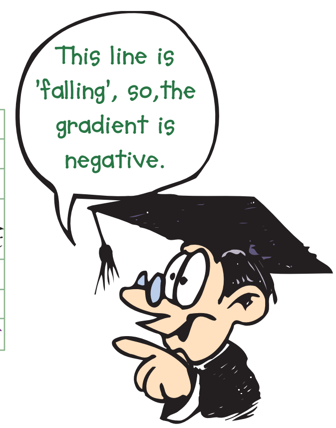
For every run of 2 there is a fall of 1.

So gradient $= -\frac{1}{2}$

y-intercept $= -1$

$\therefore$ Equation of the line is $y = -\frac{1}{2}x - 1$





A:10F Equation of a line given point and gradient

Method 1

To find the equation of a straight line that has a gradient of 2 and passes through (7, 5):
1 Substitute $m = 2$, $x = 7$ and $y = 5$ into the formula $y = mx + b$ to find the value of b.
2 Rewrite $y = mx + b$ replacing m and b with their numerical values.

Method 2

The equation of a line with gradient m, that passes through the point (x_1, y_1) is given by:

$$y - y_1 = m(x - x_1) \text{ or } \frac{y - y_1}{x - x_1} = m.$$

worked examples

1 Find the equation of the line that passes through (1, 4) and has gradient 2.

2 A straight line has gradient $-\frac{1}{2}$ and passes through the point (1, 3). Find the equation of this line.

- $y - y_1 = m(x - x_1)$
or
$y = mx + b$

Solutions

1 Let the equation of the line be:

$y = mx + b$

$\therefore y = 2x + b$ ($m = 2$ is given)

$4 = 2(1) + b$ [(1, 4) lies on the line]

$4 = 2 + b$

$\therefore b = 2$

$\therefore$ The equation is $y = 2x + 2$.

or 1 $y - y_1 = m(x - x_1)$

(x_1, y_1) is (1, 4), $m = 2$

$\therefore y - 4 = 2(x - 1)$

$y - 4 = 2x - 2$

$\therefore y = 2x + 2$ is the equation of the line.

2 Let the equation be:

$y = mx + b$

$\therefore y = -\frac{1}{2}x + b$ ($m = -\frac{1}{2}$ is given)

$3 = -\frac{1}{2}(1) + b$ [(1, 3) is on the line]

$3 = -\frac{1}{2} + b$

$\therefore b = 3\frac{1}{2}$

$\therefore$ The equation is $y = -\frac{1}{2}x + 3\frac{1}{2}$.

or 2 $y - y_1 = m(x - x_1)$

(x_1, y_1) is (1, 3), $m = -\frac{1}{2}$

$\therefore y - 3 = -\frac{1}{2}(x - 1)$

$y - 3 = -\frac{1}{2}x + \frac{1}{2}$

$\therefore y = -\frac{1}{2}x + 3\frac{1}{2}$ is the equation of the line.

A:10G Equation of a line given two points

To find the equation of a straight line that passes through the two points (1, 2) and (3, 6):

1. **Find the value of the gradient m, using the given points.**
2. **For $y = mx + b$, find the value of b by substituting the value of m and the coordinates of one of the given points.**
3. **Rewrite $y = mx + b$ replacing m and b with their numerical values.**

Another method is to use the formula:

$$y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1)$$

where (x_1, y_1) and (x_2, y_2) are points on the line.

worked example

Find the equation of the line that passes through the points (−1, 2) and (2, 8).

Solution

Let the equation of the line be:

$y = mx + b$

Now $m = \dfrac{y_2 - y_1}{x_2 - x_1}$

$= \dfrac{8 - 2}{2 - (-1)}$

$= \dfrac{6}{3}$

$\therefore m = 2$

$\therefore y = 2x + b$ (since $m = 2$)

(2, 8) lies on the line.

$\therefore 8 = 2(2) + b$

$\therefore b = 4$

$\therefore$ The equation is $y = 2x + 4$.

$(x_1, y_1) = (-1, 2)$
$(x_2, y_2) = (2, 8)$

or

$y - y_1 = \dfrac{y_2 - y_1}{x_2 - x_1}(x - x_1)$

(x_1, y_1) is (−1, 2), (x_2, y_2) is (2, 8)

$\therefore y - 2 = \dfrac{8 - 2}{2 - (-1)}[x - (-1)]$

$y - 2 = \dfrac{6}{3}(x + 1)$

$y - 2 = 2(x + 1)$

$y - 2 = 2x + 2$

$\therefore y = 2x + 4$ is the equation of the line.

A:10H Parallel and perpendicular lines

Two lines with gradients of m_1 and m_2 are:

- **parallel if $m_1 = m_2$**
- **perpendicular if $m_1 m_2 = -1$**

(or $m_1 = \dfrac{-1}{m_2}$) where neither m_1 nor m_2 can equal zero.

worked examples

1 Which of the lines $y = 4x$, $y = 3x + 2$ and $y = x$ is perpendicular to $x + 4y + 2 = 0$?

2 Find the equation of the line that passes through the point (2, 4) and is perpendicular to $y = 3x - 2$.

3 Find the equation of the line that passes through the point (1, 4) and is parallel to $y = 3x - 2$.

Solutions

1 Step 1: Find the gradient of $x + 4y + 2 = 0$.

Writing this in gradient form gives:

$y = -\frac{1}{4}x - 2$

$\therefore$ The gradient of this line is $-\frac{1}{4}$.

Step 2: Find the gradients of the other lines.

The gradient of $y = 4x$ is 4.

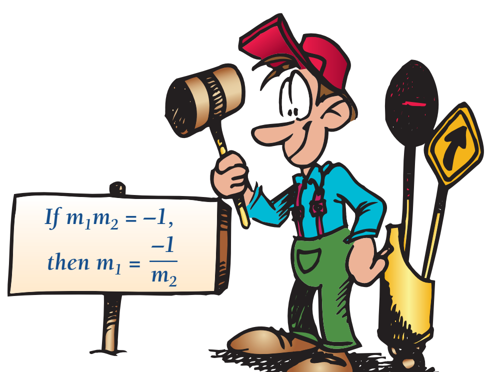
The gradient of $y = 3x + 2$ is 3.

The gradient of $y = x$ is 1.

Step 3: Find which gradient in step **2** will multiply $-\frac{1}{4}$ to give −1.

Conclusion: $-\frac{1}{4} \times 4 = -1$

$\therefore x + 4y + 2 = 0$ is perpendicular to $y = 4x$.



2 Let the equation of the line be $y = mx + b$.
Now the gradient of $y = 3x - 2$ is 3.

$\therefore\ m = -\frac{1}{3}$ (since $-\frac{1}{3} \times 3 = -1$)

$\therefore\ y = -\frac{1}{3}x + b$

$4 = -\frac{1}{3}(2) + b$ [since (2, 4) lies on line]

$4 = -\frac{2}{3} + b$

$\therefore\ b = 4\frac{2}{3}$

$\therefore$ The equation of the line is $y = -\frac{1}{3}x + 4\frac{2}{3}$.

3 Let the equation of the line be $y = mx + b$.
$y = 3x - 2$ has gradient 3

$\therefore\ m = 3$ (Parallel lines have equal gradients.)

$\therefore\ y = 3x + b$

$4 = 3(1) + b$, [(1, 4) lies on the line]

$\therefore\ b = 1$

$\therefore$ The equation of the line is $y = 3x + 1$.

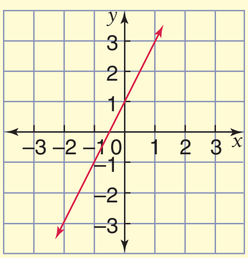
A:10I Graphing inequalities

On the number plane, all points satisfying the equation $y = 2x + 1$ lie on one straight line.

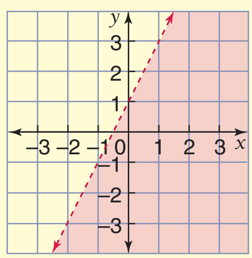
All points satisfying the inequation $y < 2x + 1$ will lie on one side of the line.

All points satisfying the inequation $y > 2x + 1$ will lie on the other side of the line.

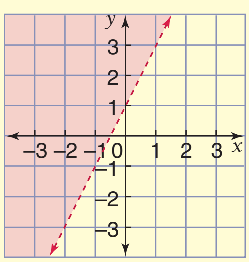
A $y = 2x + 1$



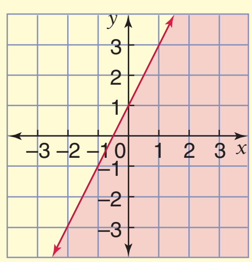
B $y < 2x + 1$



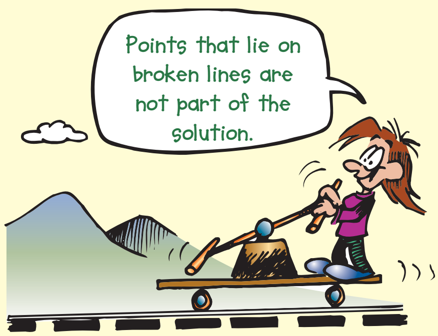
C $y > 2x + 1$



D $y \leqslant 2x + 1$



Note:
- Inequations **B**, **C** and **D** are often called 'half planes'.
- In **D**, the line is part of the solution set. In **B** and **C**, the line acts as a boundary only, and so is shown as a broken line.
- Choose points at random in each of the half planes in **B**, **C** and **D** to confirm that all points in each half plane satisfy the appropriate inequation.



worked examples

1 Graph the region $3x + 2y > 6$ on the number plane.

2 Graph **a** the union and **b** the intersection of the half planes representing the solutions of $x + 2y \geqslant 2$ and $y < 3x - 1$.

Solutions

1 Graph the boundary line $3x + 2y = 6$ as a broken line since it is not part of $3x + 2y > 6$.

$3x + 2y = 6$

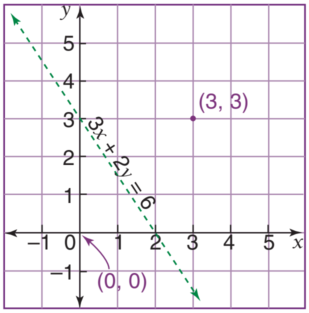
x	0	1	2
y	3	1·5	0

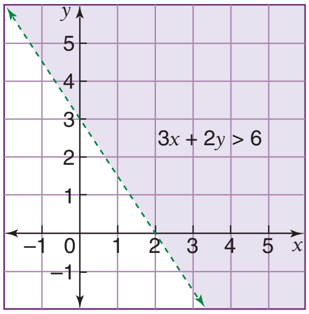
Discover which half plane satisfies the inequation $3x + 2y > 6$ by substituting a point from each side of the boundary into $3x + 2y > 6$.

(0, 0) is obviously to the left of $3x + 2y = 6$.
∴ substitute (0, 0) into $3x + 2y > 6$.
$3(0) + 2(0) > 6$, which is false.
∴ (0, 0) does not lie in the half plane.

(3, 3) is obviously to the right of $3x + 2y = 6$.
∴ substitute (3, 3) into $3x + 2y > 6$.
$3(3) + 2(3) > 6$, which is true.
∴ (3, 3) lies in the half plane $3x + 2y > 6$.
Shade in the half plane on the (3, 3) side.

Points that lie on broken lines are not part of the solution.



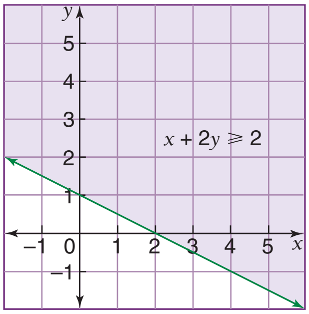


2 Graph the two half planes using the method above.

$x + 2y = 6$

x	0	1	2
y	1	0·5	0

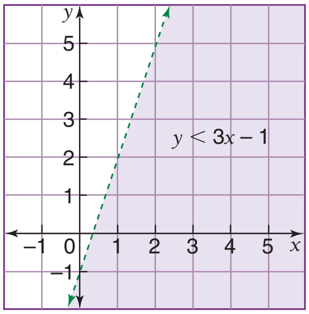
Points above the boundary line satisfy $x + 2y \geqslant 2$.



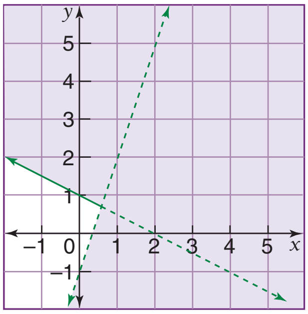
$y = 3x - 1$

x	0	1	2
y	−1	2	5

Points to the right of the boundary satisfy $y < 3x - 1$.

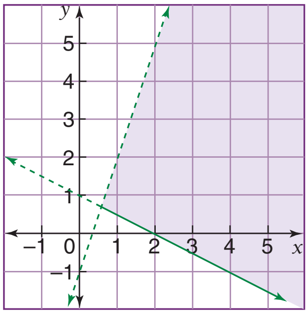


a The union of the two half planes is the region that is part of one or the other or both graphs.



The union is written:
$\{(x, y): x + 2y \geqslant 2 \cup y < 3x - 1\}$

b The intersection is the region that belongs to both half planes. It is the part that the graphs have in common.



The intersection is written:
$\{(x, y): x + 2y \geqslant 2 \cap y < 3x - 1\}$

Note:
- Initially draw the boundary lines as broken lines.
- Part of each region has a part of the boundary broken and a part unbroken.

A:11 | Statistics

A:11A Frequency distribution tables

One method of organising data is to use a frequency distribution table.

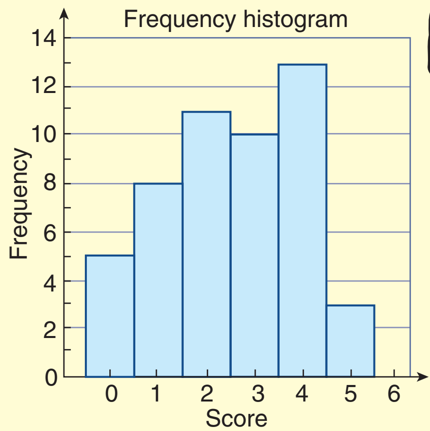
3, 2, 1, 3, 2, 4, 1, 2, 0, 3
1, 4, 2, 4, 4, 5, 3, 4, 1, 0
3, 2, 4, 4, 1, 4, 3, 5, 2, 2
1, 3, 4, 2, 5, 0, 1, 3, 4, 2
0, 4, 1, 3, 4, 2, 3, 4, 2, 0

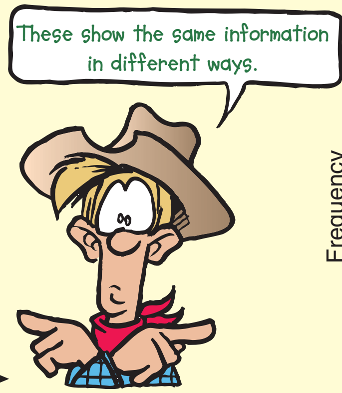
Score	*Tally*	*Frequency*
0	卌	5
1	卌 |||	8
2	卌 卌		11
3	卌 卌	10
4	卌 卌 |||	13
5					3
		$\sum f = 50$

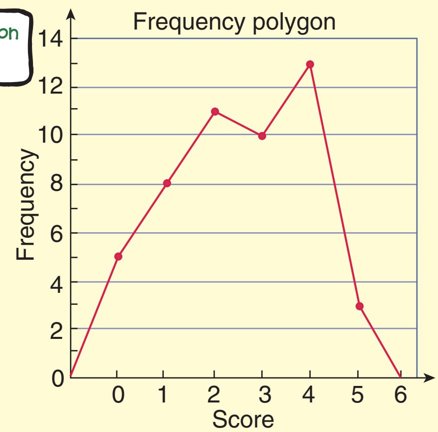
■ The total of the frequencies should equal the number of scores.

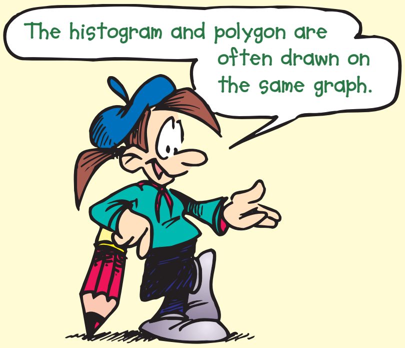
A:11B Frequency graphs

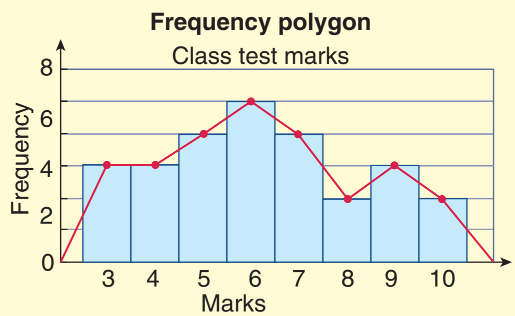
It is often desirable to present information in the form of a diagram.





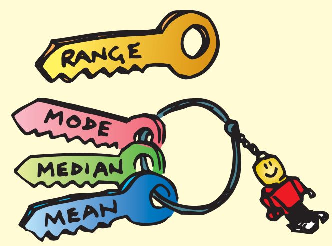






A:11C Analysing data

After data has been collected, certain 'key' numbers can be calculated that give us further information about the data being examined.



- The **range** is a measure of the spread of the scores.
- The **mean**, **median** and **mode** are all measures that try to summarise or average the scores. There are three averages because all have some disadvantage in certain situations.

The *range* = highest score − lowest score.
The *mode* is the outcome that occurs the *most*.
The *median* is the middle score for an odd number of scores.
The *median* is the average of the middle two scores for an even number of scores.
The *mean* is the arithmetic average.

$$\text{mean} = \frac{\text{sum of scores}}{\text{total number of scores}} = \left[\frac{\text{sum of } fx \text{ column}}{\text{sum of } f \text{ column}}\right]$$

worked examples

1 Find the range, mode, median and mean of each set of scores.

a 4 4 4 12
9 6 10

Range = highest score − lowest score
= 12 − 4
= 8

Mode = outcome occurring most
= 4

Median = middle score
= 6

Mean = $\dfrac{\text{sum of scores}}{\text{total number of scores}}$

$= \dfrac{4+4+4+12+9+6+10}{7}$

= 7

b 15 36 40 23 18
46 21 28 32 36

Range = highest score − lowest score
= 46 − 15
= 31

Mode = outcome occurring most
= 36

Median = average of two middle scores

$= \dfrac{28+32}{2}$

= 30

Mean = $\dfrac{295}{10}$

= 29·5

2 Find the mean of the set of scores described by this frequency distribution table.

Score (*x*)	*Frequency* (*f*)	*Score × frequency* (*fx*)
0	5	5 × 0 = 0
1	8	8 × 1 = 8
2	11	11 × 2 = 22
3	10	10 × 3 = 30
4	13	13 × 4 = 52
5	3	3 × 5 = 15
	$\sum f = 50$	$\sum fx = 127$

To find the sum of all the scores, we use the *fx* (or *f* × *x*) column to work out the sum of the 0s, 1s, 2s, 3s, 4s and 5s separately (by multiplying each score by its frequency), and then we add these figures to get the sum of all the scores (ie $\sum fx$).

$\bar{x} = \dfrac{\sum fx}{\sum f}$

$= \dfrac{127}{50}$

= 2·54

∴ the mean of the scores is 2·54.

3 Use the cumulative frequency column in the frequency distribution table to determine the median.

The cumulative frequency of an outcome is the number of scores equal to, or less than, that particular outcome.

a

Outcome *x*	*Frequency* *f*	*Cumulative frequency*
3	5	5
4	3	8
5	7	15
6	8	23
7	4	27
8	2	29

The middle score is the 15th score (14 above it and 14 below it).

The 15th score is a 5.

Hence the median = 5.

b

Outcome (*x*)	*f*	*c.f.*
5	2	2
6	4	6
7	3	9
8	7	16
9	5	21
10	1	22

Here there is an even number of scores, ie 22; so the middle two scores are the 11th and 12th scores.

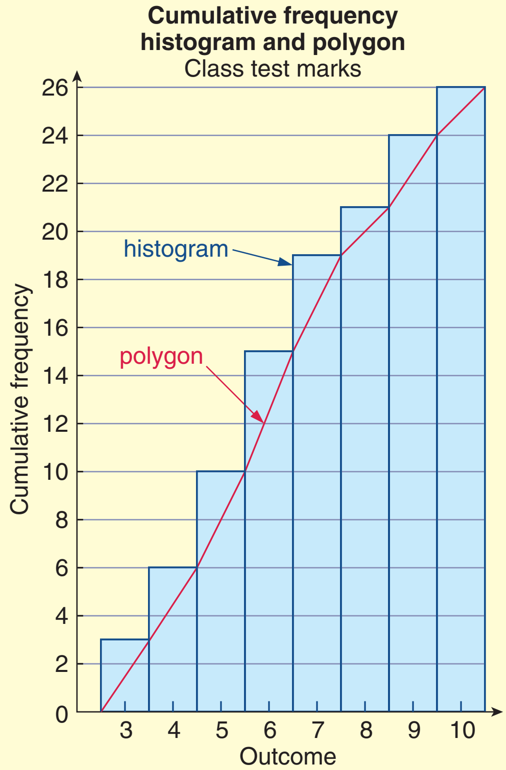
From the *c.f.* column it can be seen that each of these scores is 8.

$\therefore$ median = 8.

A:11D Cumulative frequency graphs

- The histogram progressively steps upwards to the right.
- The polygon is obtained by joining the top right-hand corner of each column. (Why is it drawn this way?)
- Imagine that the column before the '3' column has zero height.

Another name for the cumulative frequency polygon is the 'ogive'.



Finding the median from an ogive

The cumulative frequency polygon or ogive can be used to find the median.

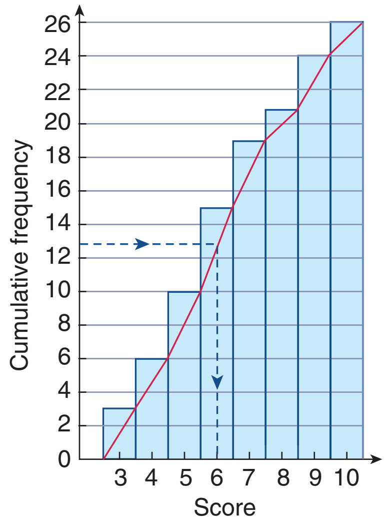
Note that the method used is different from that used for the table.

worked example

To find the median, follow these steps.

- Find the halfway point ($\frac{1}{2} \times 26 = 13$).
- Draw a horizontal line from this point to the ogive.
- Then draw a vertical line to meet the horizontal axis.
- This meets the horizontal axis within the '6' column.

$\therefore$ The median is 6.



A:11E Grouped data

In cases where there are a large number of possible scores, it is usually more convenient to group the scores into *classes*.

For the measures of central tendency:

1 The *mode* becomes the '*modal class*'.
In the example on the next page, the modal class is 65–73.

2 The *mean* is estimated by using the class centre as a representative figure for each class. So the mean is given by:

$$\bar{x} = \frac{\sum(f \times \text{c.c.})}{\sum f}$$

In the example following, $\bar{x} = \frac{3969}{60} = 66{\cdot}15$.

3 The *median* becomes the *median class*.
In the following example there are 60 scores, so the middle score is the average of the 30th and 31st scores. Both lie in the 65–73 class, so the median class is 65–73.

worked examples

The percentage results for 60 students in an examination were:

78	63	89	55	92	74	62	69	43	90
71	83	49	37	58	73	78	65	62	87
95	77	69	82	71	60	61	53	59	42
43	33	98	88	73	82	75	63	67	59
57	48	50	51	66	73	68	46	69	70
91	83	62	47	39	63	67	74	52	78

The frequency distribution table for this set of scores would look like this.

Class	*Class centre (c.c.)*	*Tally*	*Frequency*	$f \times$ *c.c.*	*Cumulative frequency (c.f.)*
29–37	33	||	2	66	2
38–46	42	~~||||~~	5	210	7
47–55	51	~~||||~~ |||	8	408	15
56–64	60	~~||||~~ ~~||||~~ ||	12	720	27
65–73	69	~~||||~~ ~~||||~~ ||||	14	966	41
74–82	78	~~||||~~ ||||	9	702	50
83–91	87	~~||||~~ ||	7	609	57
92–100	96	|||	3	288	60
		Totals:	60	3969	

1 The modal class is the class with the highest frequency.

∴ modal class is 65–73

2 The mean is estimated by using the class centre as a representative figure for each class. So the mean is given by

$$\bar{x} = \frac{\sum(f \times \text{c.c.})}{\sum f}$$

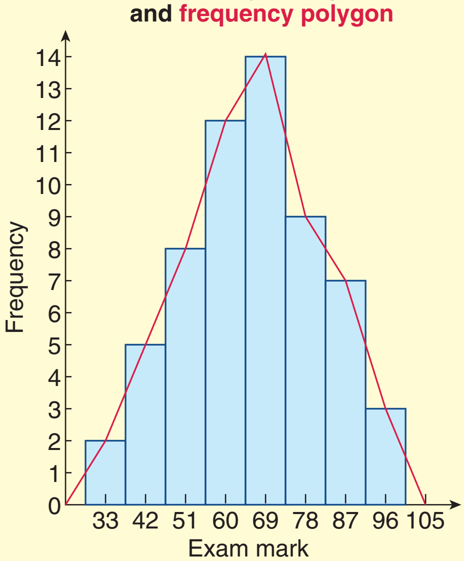
Here, $$\bar{x} = \frac{3969}{60} = 66{\cdot}15$$

3 The median class is the class containing the middle score. In this example there are 60 scores, so the middle score is the average of the 30th and 31st score. Both lie in the 65–73 class.

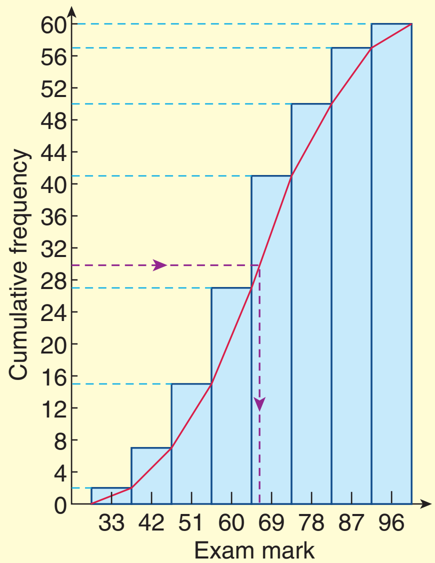
∴ median class is 65–73.

When constructing frequency diagrams for grouped data, the columns are indicated on the horizontal axis by the class centres.

Frequency histogram and frequency polygon



Cumulative frequency histogram
Cumulative frequency polygon



For frequency polygons, we join the middle points at the top of each column of the histogram. For cumulative frequency polygons, we join the right-hand point at the top of each column of the histogram.

The broken line shows the median class to be 65–73 since 30 is half of 60.

A:12 | Simultaneous Equations

A:12A Graphical method of solution

To solve a pair of simultaneous equations graphically, we graph each line. The solution is given by the coordinates of the point of intersection of the lines.

worked example

Solve the following equations simultaneously.

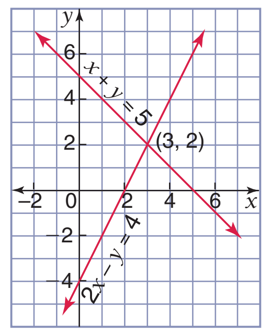
$x + y = 5$

$2x - y = 4$

Solution

You will remember from your earlier work on coordinate geometry that, when the solutions to an equation such as $x + y = 5$ are graphed on a number plane, they form a straight line.

Hence, to solve the equations $x + y = 5$ and $2x - y = 4$ simultaneously, we could simply graph each line and find the point of intersection. Since this point lies on both lines, its coordinates give the solution.



$x + y = 5$

x	0	1	2
y	5	4	2

$2x - y = 4$

x	0	1	2
y	−4	−2	0

- The lines $x + y = 5$ and $2x - y = 4$ intersect at (3, 2).
  Therefore the solution is:
  $x = 3$
  $y = 2$

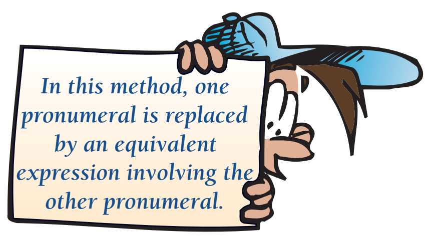
A:12B Substitution method

worked examples

Solve the simultaneous equations:

1 $2x + y = 12$ and $y = 5x - 2$

2 $3a + 2b = 7$, $4a - 3b = 2$



Solutions

When solving simultaneous equations, first 'number' the equations involved.

1 $2x + y = 12$ .... (1)

$y = 5x - 2$ .... (2)

Now from (2) we can see that $5x - 2$ is equal to y. If we substitute this for y in equation (1), we have:

$2x + (5x - 2) = 12$

$7x - 2 = 12$

$7x = 14$

$x = 2$

So the value of x is 2. This value for x can now be substituted into either equation ① or equation ② to find the value for y:

In ①:

$$2(2) + y = 12$$
$$4 + y = 12$$
$$y = 8$$

In ②:

$$y = 5(2) - 2$$
$$= 10 - 2$$
$$= 8$$

So, the total solution is:
$x = 2$, $y = 8$.

■ To check this answer substitute into equations ① and ②.

2 $3a + 2b = 7$ .... ①

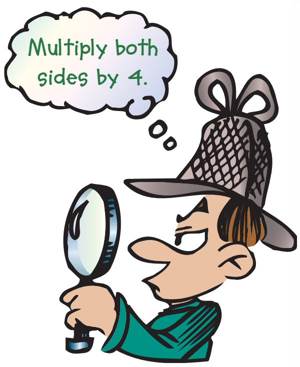
$4a - 3b = 2$ .... ②

Making a the subject of ② gives:

$$a = \frac{2 + 3b}{4}$$

If we substitute this expression for a into equation ①, we get:

$$3\left(\frac{2 + 3b}{4}\right) + 2b = 7$$
$$3(2 + 3b) + 8b = 28$$
$$6 + 9b + 8b = 28$$
$$17b = 22$$
$$b = \frac{22}{17}$$



Substituting this value for b into, say, equation ② gives:

$$4a - 3\left(\frac{22}{17}\right) = 2$$
$$4a - \frac{66}{17} = \frac{34}{17}$$
$$4a = \frac{100}{17}$$
$$a = \frac{25}{17}$$

So the total solution is:

$$a = \frac{25}{17},\ b = \frac{22}{17}.$$

■ To check your answer, substitute $a = \frac{25}{17}$, $b = \frac{22}{17}$ in equations ① and ②.

Check each step!

A:12C Elimination method

worked examples

Solve each pair of simultaneous equations:

1 $5x - 3y = 20$
$2x + 3y = 15$

2 $x + 5y = 14$
$x - 3y = 6$

3 $2x + 3y = 21$
$5x + 2y = 3$

■ In this method, one of the pronumerals is eliminated by adding or subtracting the equations.

Solutions

First, number each equation.

■ You add or subtract the equations, depending upon which operation will eliminate one of the pronumerals.

1 $5x - 3y = 20$ .... (1)

$2x + 3y = 15$ .... (2)

Now if these equations are 'added', the y terms will be eliminated, giving:

$7x = 35$

ie $x = 5$

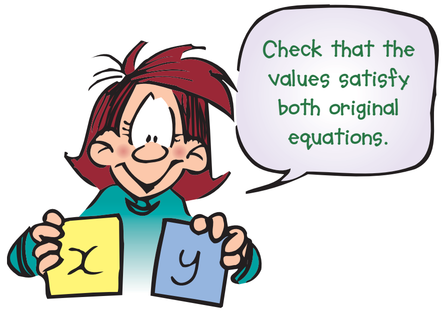
Substituting this value into equation (1) we get:

$5(5) - 3y = 20$

$25 - 3y = 20$

$3y = 5$

$y = \frac{5}{3}$ or $1\frac{2}{3}$.



So the total solution is:

$x = 5,\ y = 1\frac{2}{3}$.

Check in (1): $5(5) - 3(1\frac{2}{3}) = 20$ (true).

Check in (2): $2(5) + 3(1\frac{2}{3}) = 15$ (true).

2 $x + 5y = 14$ .... (1)

$x - 3y = 6$ .... (2)

Now if equation (2) is 'subtracted' from equation (1), the x terms are eliminated and we get:

$8y = 8$

ie $y = 1$

Substituting this value into (1) gives:

$x + 5(1) = 14$

$x + 5 = 14$

$x = 9$

∴ The solution is:

$x = 9,\ y = 1$.

Check in (1): $9 + 5(1) = 14$ (true).

Check in (2): $9 - 3(1) = 6$ (true).

Take one step at a time.

3 $2x + 3y = 21$ .... ①

$5x + 2y = 3$ .... ②

Multiply equation ① by 2

and equation ② by 3.

This gives:

$4x + 6y = 42$ .... ①*

$15x + 6y = 9$............... ②*

Now if ②* is subtracted from ①* the y terms are eliminated and we get:

$-11x = 33$

So $x = -3$

Substituting this value into ① gives:

$2(-3) + 3y = 21$

$-6 + 3y = 21$

$3y = 27$

$y = 9$

So the solution is $x = -3$, $y = 9$

Check in ①: $2(-3) + 3(9) = 21$ (true).

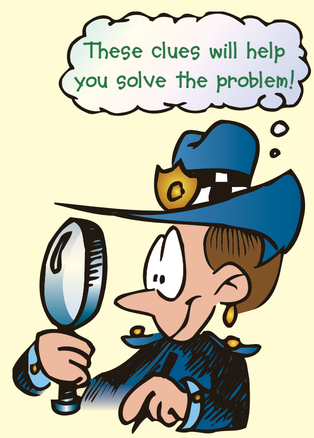
Check in ②: $5(-3) + 2(9) = 3$ (true).

■ *Notice*
To eliminate a pronumeral, the size of the coefficients in each equation must be made the same by multiplying one or both equations by a constant.

■ *Note:* In example 3, x could have been eliminated instead of y, by multiplying ① by 5 and ② by 2.

A:12D Solving problems using simultaneous equations

- Read the question carefully.
- Work out what the problem wants you to find. (These things will be represented by pronumerals.)
- Translate the words of the question into mathematical expressions.
- Form equations by showing how different mathematical expressions are related.
- Solve the equations.
- Finish off with a sentence stating the value of the quantity or quantities that were found.



worked example

Adam is 6 years older than his sister, Bronwyn.
If the sum of their ages is 56 years, find their ages.

Solution

Let Adam's age be x years.

Let Bronwyn's age be y years.

Now, Adam is 6 years older than Bronwyn

$\therefore\ x = y + 6$ .... ①

Also, the sum of their ages is 56 years.

$\therefore\ x + y = 56$.................... ②

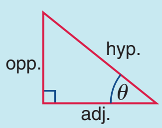
Solving these simultaneously gives:

$x = 31$ and $y = 25$.

$\therefore$ Adam is 31 years old and Bronwyn is 25 years old.

A:13 | Trigonometry

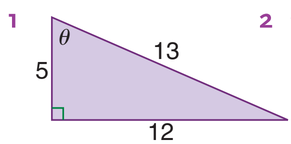
A:13A Trigonometric ratios

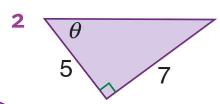


$\sin\theta = \dfrac{\text{opposite}}{\text{hypotenuse}}$ $\quad\cos\theta = \dfrac{\text{adjacent}}{\text{hypotenuse}}$ $\quad\tan\theta = \dfrac{\text{opposite}}{\text{adjacent}}$

worked examples

Find sin θ, cos θ and tan θ for each triangle, and express each as a decimal correct to three decimal places.





Solutions

1 $\sin\theta = \dfrac{\text{opp.}}{\text{hyp.}} = \dfrac{5}{13} \doteqdot 0{\cdot}385$

$\cos\theta = \dfrac{\text{adj.}}{\text{hyp.}} = \dfrac{12}{13} \doteqdot 0{\cdot}923$

$\tan\theta = \dfrac{\text{opp.}}{\text{adj.}} = \dfrac{5}{12} \doteqdot 0{\cdot}417$

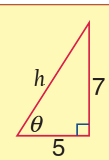
2 First the hypotenuse must be calculated using Pythagoras' theorem. So, then:

$\sin\theta = \dfrac{7}{\sqrt{74}} \doteqdot 0{\cdot}814$

$\tan\theta = \dfrac{7}{5} \doteqdot 1{\cdot}400$

$\cos\theta = \dfrac{5}{\sqrt{74}} \doteqdot 0{\cdot}581$

$h^2 = 5^2 + 7^2$
$= 25 + 49$
$= 74$
ie $h = \sqrt{74}$



A:13B Trig. ratios and the calculator

Finding a ratio given the angle

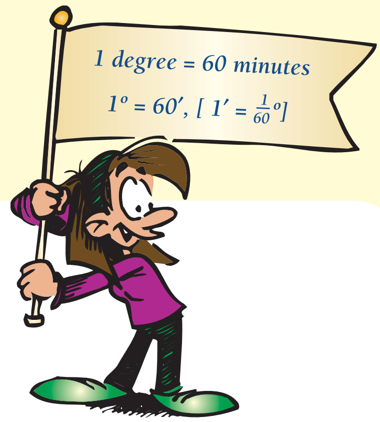
To find tan 31°, ensure your calculator is operating in 'degrees' and then press:

[tan] 31 [=]

The calculator should give tan 31° = 0·600 860 6, correct to seven decimal places.

Degrees and minutes

We can also find the trigonometric ratios of angles given to the nearest minute by using the calculator as shown in the examples below.



worked examples

Find:

1 sin 25°41′ **2** tan 79°05′

Give your answers correct to four decimal places.

Solutions

Two methods are shown, one for each solution. Choose the one that best suits your calculator.

Method 1:

1 For calculators with a *Degrees/Minutes/Seconds* button.
This is usually marked in either of two ways.

[DMS] or [°′″]

Press: [sin] 25 [DMS] 41 [=]

The calculator gives 0·433 396 953.

> ■ *Warning:*
> Your calculator may work differently to the one used here.

Method 2:

2 We convert 79°05′ into decimal degrees by realising that 05′ is $\frac{5}{60}$ of one degree.

Press [tan] [(] 79 [+] 5 [÷] 60 [)] [=]

The calculator gives 5·184 803 521.

Finding an angle, given the ratio

If the value of the trigonometric ratio is known and you want to find the size of the angle to the nearest minute, follow the steps in the examples below.

worked examples

1 If sin $\theta = 0{\cdot}632$, find θ to the nearest minute.

2 If cos $\theta = 0{\cdot}2954$, find θ to the nearest minute.

Solutions

Note: One minute may be divided further, into 60 seconds, and this fact will be used to round off answers to the nearest minute.

Again two methods are shown that correspond to the two methods on the previous page.

1 If sin $\theta = 0{\cdot}632$, press: [2nd F] [sin] 0·632 [=]

The calculator now displays 39·197 833 53°. To convert this to degrees/minutes/seconds mode, press [DMS]. The calculator gives 39°11′52·2″.

$\therefore\ \theta = 39°12'$ (to the nearest minute)

2 If cos $\theta = 0{\cdot}2954$, press [2nd F] [cos] 0·2954 [=]

The answer on the screen is 72·818 475 degrees. The alternative method of converting this to degrees and minutes is to find what 0·818 475 of one degree is, in minutes; ie 0·818 475 × 60′, which gives an answer of 49·1085 minutes, ie 49′ (to the nearest minute).

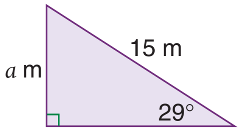
$\therefore\ \theta = 72°49'$.

A:13C Finding an unknown side

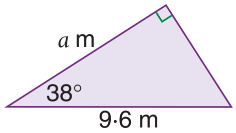
worked examples

1 Find a in these triangles, correct to one decimal place.

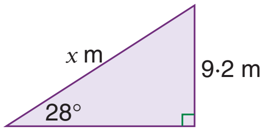
a



b



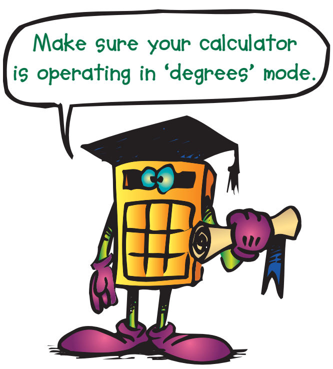
c



2 A ladder that is 8 metres long leans against a wall, and makes an angle of 21° with the wall. How far does the ladder reach up the wall, to the nearest centimetre?

Solutions

Use the trig. button on your calculator.



1 **a** $\dfrac{a}{15} = \sin 29°$

$\therefore\ a = (\sin 29°) \times 15 \leftrightarrow$ [sin] 29 [×] 15 [=]

$= 7{\cdot}272\ 144\ 3$

So $a = 7{\cdot}3$ (to one decimal place)

b $\dfrac{a}{9 \cdot 6} = \cos 38°$

$\therefore\ a = (\cos 38°) \times 9{\cdot}6$ ↔ [cos] 38 [×] 9·6 [=]

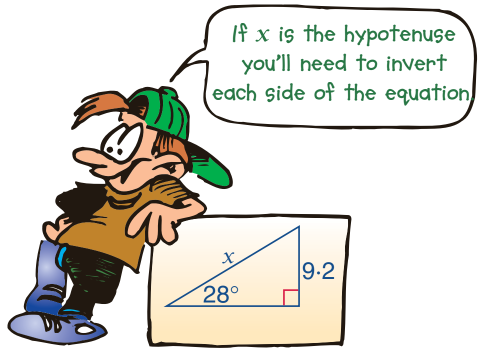
$= 7{\cdot}564\ 903\ 2$

$= 7{\cdot}6$ (to one decimal place)

c $\dfrac{9 \cdot 2}{x} = \sin 28°$

$\dfrac{x}{9 \cdot 2} = \dfrac{1}{\sin 28°}$

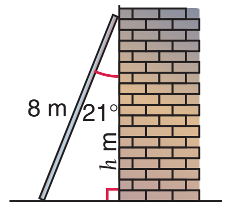
(Note that x is the denominator of the fraction, not the numerator.)



$\therefore\ x = \dfrac{9 \cdot 2}{\sin 28°}$ ↔ 9·2 [÷] 38 [sin] 28 [=]

$= 19{\cdot}6$ (to one decimal place)

2 From the information in the question, a diagram like the one to the right can be drawn. Let the height up the wall be h m.



So: $\dfrac{h}{8} = \cos 21°$

$h = 8 \times \cos 21°$ ↔ 8 [×] [cos] 21 [=]

$= 7{\cdot}468\ 643\ 4$

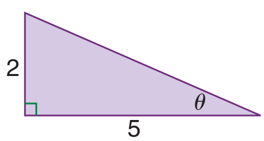
$= 7{\cdot}47$ (to the nearest centimetre)

$\therefore$ The ladder reaches 7·47 m up the wall.

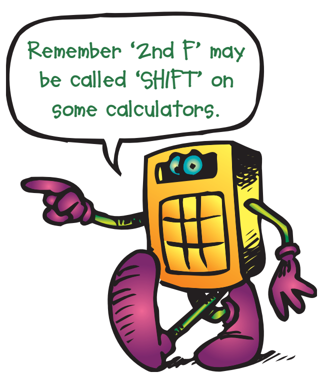
A:13D Finding an unknown angle

worked examples

1 Find the size of angle θ.
Answer to the nearest degree.



2 What angle, to the nearest minute, does the diagonal of a rectangle make with its length, if the dimensions of the rectangle are 12·6 cm by 8·9 cm?



Solutions

1 In the triangle,

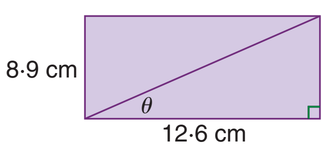
$\tan \theta = \dfrac{2}{5}$

$= 0{\cdot}4$ ↔ [2nd F] [tan] 0·4 [=]

$\therefore\ \theta = 21{\cdot}801\ 409°$

so $\theta = 22°$ (to the nearest degree).

2



Let the required angle be θ. Then:

$\tan \theta = \dfrac{8 \cdot 9}{12 \cdot 6}$ [2nd F] [tan] [(] 8·9 [÷] 12·6 [)] [=] [2nd F] [DMS]

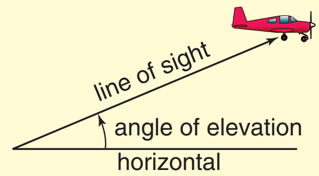
$\therefore\ \theta = 35°14'7{\cdot}59''$

$\therefore\ \theta = 35°14'$ (to the nearest minute).

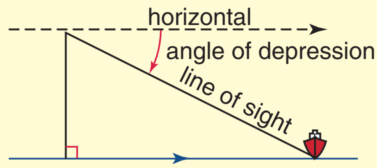
A:13E Solving problems using trigonometry

Angles of elevation and depression

When looking upwards towards an object, the **angle of elevation** is defined as the angle between the line of sight and the horizontal.



When looking downwards towards an object, the **angle of depression** is defined as the angle between the line of sight and the horizontal.



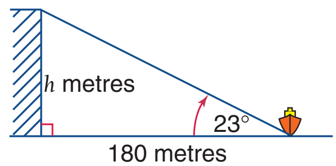
worked examples

1 The angle of elevation of the top of a vertical cliff is observed to be 23° from a boat 180 m from the base to the cliff. What is the height of the cliff? (Answer correct to one decimal place.)

2 An observer stands on the top of a 40-metre cliff to observe a boat that is 650 metres out from the base of the cliff. What is the angle of depression from the observer to the boat? (Answer to the nearest minute.)

Solutions

1 For this example, the diagram would look like the one on the right.



Let the height of the cliff be h metres.

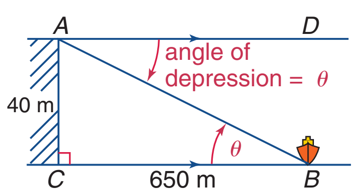
Then: $\dfrac{h}{180} = \tan 23°$

ie $h = (\tan 23°) \times 180$

$= 76{\cdot}405\,467$ (from calculator)

$\therefore$ Height of cliff = 76·4 m (to one decimal place).

2



Note: The angle of depression $\angle DAB = \angle ABC$ (alternate angles and parallel lines).

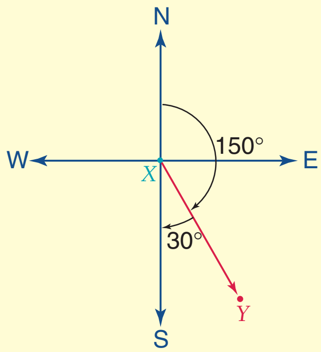
$\tan\theta = \dfrac{40}{650}$ [2nd F] [tan] [(] 40 [÷] 650 [)] [=] [DMS]

ie $\theta = 3°31'17{\cdot}23''$

$= 3°31'$ (to the nearest minute).

Compass bearings

The direction of a point *Y* from an original point *X* is known as the **bearing** of *Y* from *X*. This is mainly expressed in one of two ways. Examine the diagram below.

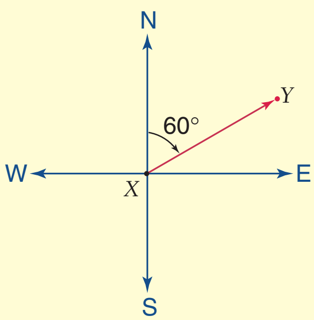


The bearing of *Y* from *X* can be given as:

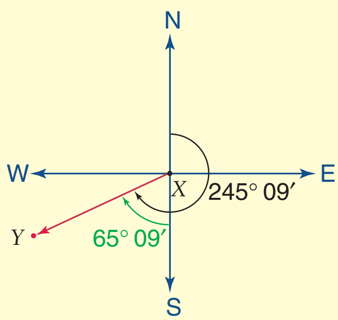
1. 150° (the angle between the interval *XY* and the north line measured in a clockwise direction), or,
2. south 30° east (S30°E).

Sometimes, only letters are used. So SE (or south-east) is halfway between south (180°) and east (90°); that is, 135° or S45°E.

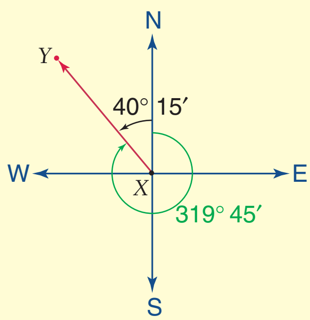
Other examples would look like these.



060° or N60°E



245°09′ or S65°09′W



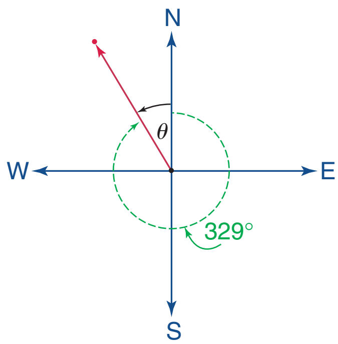
319°45′ or N40°15′W

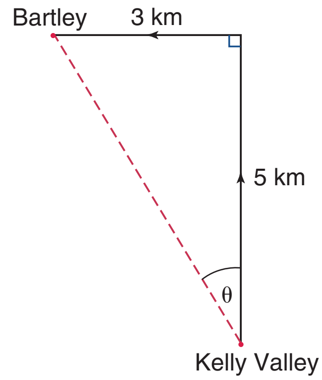
worked examples

1. If the town of Bartley is 5 km north and 3 km west of Kelly Valley, find the bearing of Bartley from Kelly Valley.
2. Two people start walking from the same point. The first walks due east for 3·5 km and the second walks in the direction 123° until the second person is due south of the first person. How far did the second person walk (to the nearest metre)?

Solutions

1 The diagram for this question would look like the one on the right.





Let the angle indicated in the diagram be θ.

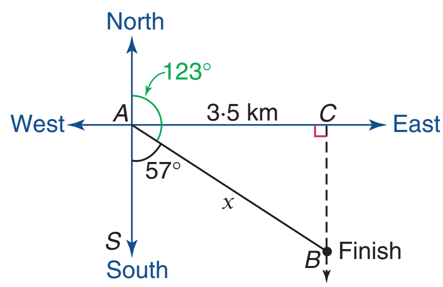
Thus: $\tan \theta = \frac{3}{5}$

$= 0{\cdot}6$

So: $\theta = 31°$ (to the nearest degree)

So the bearing of Bartley from Kelly Valley would be N31°W or simply 329°.

2 This diagram shows the information in the question above.



Since $\angle SAB = \angle CBA$
(alternate angles, $AS \parallel CB$)

then $\angle CBA = 57°$

So: $\frac{3{\cdot}5}{x} = \sin 57°$

ie $\frac{x}{3{\cdot}5} = \frac{1}{\sin 57°}$

$x = \frac{3{\cdot}5}{\sin 57°}$

$= 4{\cdot}173$ km

Press: 3·5 [÷] [sin] 57 [=]

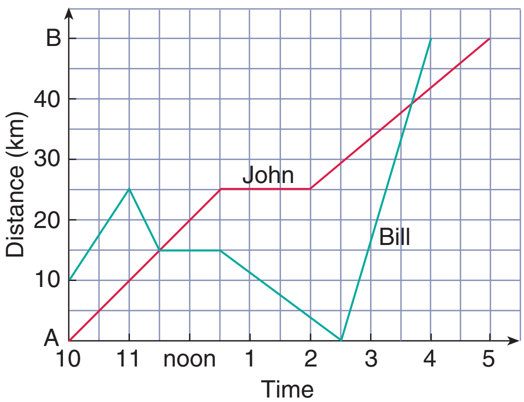
A:14 | Graphs of Physical Phenomena

A:14A Distance–time graphs

- A distance–time graph (or travel graph) is a special type of line graph used to describe one or more trips or journeys.
- The vertical axis represents distance from a certain point, while the horizontal axis represents time.
- The formulae that connect distance travelled (D), time taken (T) and average speed (S) are given here.

$$D = S \times T \qquad S = \frac{D}{T} \qquad T = \frac{D}{S}$$

worked example

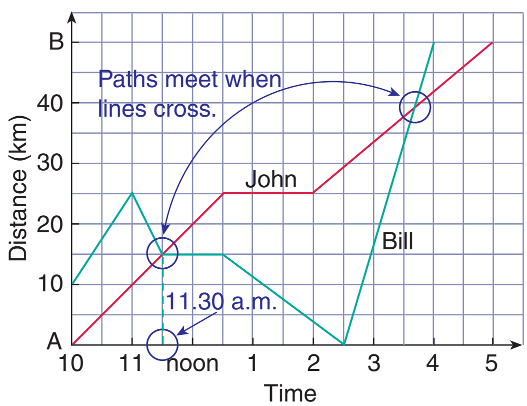


This travel graph shows the journeys of John and Bill between town A and town B. (They travel on the same road.)

a How far from A is Bill when he commences his journey?
b How far is John from B at 2:30 pm?
c When do John and Bill first meet?
d Who reaches town B first?
e At what time does Bill stop to rest?
f How far does John travel?
g How far apart are John and Bill when Bill is at town A?
h How far does Bill travel?

Solutions

a Bill commences his journey at 10:00 am. At that time he is 10 km from town A.



b At 2:00 pm John is 20 km from B (because he is 30 km from A).

c John and Bill first meet at 11:30 am.

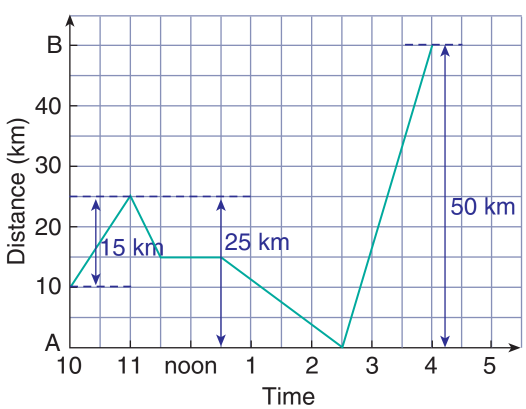
d Bill reaches town B at 4:00 pm.
John reaches town B at 5:00 pm.
∴ Bill reaches town B first.

e The horizontal section indicates a rest.
∴ Bill stops at 11:30 am.

f John travels from town A to town B without backtracking.
∴ John travels 50 km.

g Bill is at town A at 2:30 pm. At that time John is about 30 km from A.
∴ They are about 30 km apart when Bill is at A.

h



Bill's journey involves backtracking. He moves towards B, then returns to A and then moves to B.

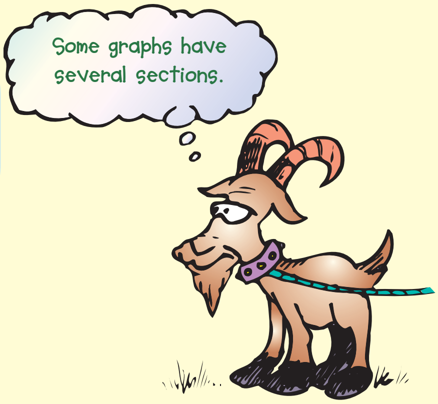
Distance travelled (10:00 am–11:00 am)
$= 25 - 10 = 15$ km

Distance travelled (11:00 am–2:30 pm)
$= 25 - 0 = 25$ km

Distance travelled (2:30 pm–4:00 pm)
$= 50 - 0 = 50$ km

Total distance travelled $= (15 + 25 + 50)$ km
$= 90$ km.

Summary
- **A change in steepness means a change in speed.**
- **The steeper the line, the faster the journey. The flatter the line, the slower the journey.**
- **A horizontal line indicates that the person or object is stationary.**



A:14B Relating graphs to physical phenomena

Graphs provide an excellent means of exploring the relationship between variables.

They give an immediate 'picture' of the relationship, from which we can see such things as:
- whether a variable is increasing or decreasing with respect to the other variable
- when a variable has its highest or lowest value
- whether a variable is increasing quickly or slowly with respect to the other variable.

Graphs can be used to show relationships between data such as:
- temperature and time of day (or year)
- height and weight
- water level before, during and after a bath
- distance and speed
- light brightness and proximity
- tidal movements over time.

worked examples

Example 1

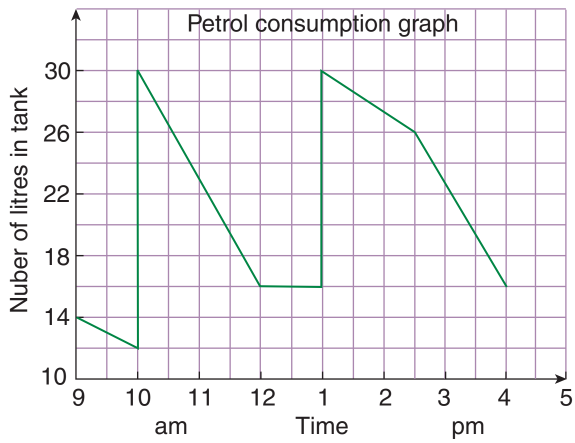
a How much petrol was in the tank at 9:00 am?

b How much petrol was used from 9:00 am to 10:00 am?

c When did the driver fill the car?

d How much petrol was used from 12 noon to 1:00 pm? What does this tell us about the car?

e How much petrol was used altogether?



Solution 1

a At 9:00 am, the line graph starts at 14 on the vertical axis. Therefore the amount of petrol was 14 litres.

b At 10 am, there is 12 litres in the tank. Therefore, between 9:00 am and 10:00 am, 2 litres was used.

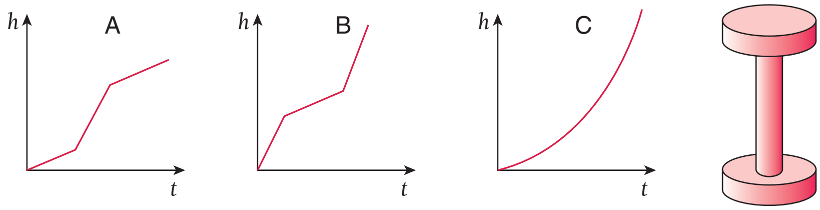
c The car was filled at 10:00 am and 1:00 pm.

d From 12 noon to 1:00 pm, the line graph is horizontal. Therefore, no petrol was used.

e 2 litres was used from 9:00 am to 10:00 am, 14 litres was used from 10:00 am to 12 noon, 14 litres was used from 1:00 pm to 4:00 pm. Therefore 30 litres was used altogether.

Example 2

Water is added to the tank shown at a steady rate. Which graph best represents the increase in the water level h?



Solution 2

Looking at the tank, we notice that the middle part is skinnier than the other parts. Therefore, if water is poured in at a steady rate, it will fill up faster in the middle part than in the other two sections. Hence, in our graph, the water level, h, will increase more quickly for this section of the tank than for the others. Hence, the correct graph must consist of three sections, with the steepest section in the middle. Hence, graph A is the best representation.

APPENDIX B
Working Mathematically

B:01 | Solving Routine Problems

In mathematics the learning of new skills and concepts is usually followed by the use of that newly acquired knowledge in the solving of problems.

These problems are generally routine in nature as the mathematical knowledge and skills needed are fairly obvious. The problem may still be hard to do but at least what the problem is about is clear.

Hence, problems on percentages or measurement or geometry, for instance, are routine in that we know what mathematical knowledge we are trying to use.

No matter what type of problem we are trying to solve, the following steps are important.

Steps for solving problems
Step 1 Read the question carefully.
Step 2 Decide what you are being asked to find.
Step 3 Look for information that might be helpful.
Step 4 Decide on the method you will use.
Step 5 Set out your solution clearly.
Step 6 Make sure that your answer makes sense.

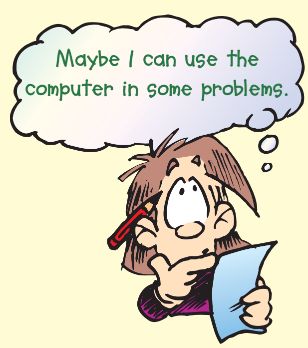
Completing exercises from the text that focus on a skill or concept that has just been presented would mostly involve solving routine problems.

B:02 | Solving Non-Routine Problems

Often in mathematics (as well as in real life) we get a problem that is unlike any we have seen before. We need to reflect on what we already know and see how our existing knowledge can be used. Sometimes the problem will need us to develop new skills, or we may need to look at the problem in a different way.

Applying strategies is one of the processes involved in working mathematically.

Some useful strategies for problem-solving are:
- Eliminating possibilities
- Working backwards
- Acting it out
- Looking for patterns
- Solving a simpler problem
- Trial and error
- Making a drawing, diagram or model
- Using algebra
- Using technology



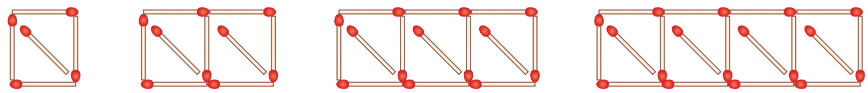
Problems of this type can be found at the end of each chapter of the text in the Working Mathematically assignment.

As the strategies applicable in each problem will vary, worked examples are not always helpful. The examples below, however, involve looking for patterns, making a diagram, using algebra and, perhaps, trial and error.

worked examples

Example 1

Complete the table of values for this pattern of matchsticks and hence find a rule linking the number of squares (s) to the number of matchsticks (M).



Solution

s	1	2	3	4
M	5	9	13	17

4 4 4

It is easy to see that the next number for M would be 21; we simply add 4.

There is a common difference of 4 for consecutive values of M.
Thus, the rule will be $M = 4 \times s$ **plus a constant**.
Since, when $s = 1$, $M = 5$, this constant must be 1.
So the rule is $M = 4s + 1$.

Example 2

Not all number patterns have a common difference.
Find the rule for the pattern 2, 8, 18, 32, … and hence find the 10th term.

Solution

Completing a table may help, matching each term (T) with the number (n) of its position in the pattern.

n	1	2	3	4
T	2	8	18	32

Examining the values of T, we can see that they are double the sequence of numbers 1, 4, 9, 16, … and these are the values of n squared.

Thus, the rule is $T = 2n^2$.
$\therefore$ The 10th term is $2(10)^2$ or 200.

Foundation Worksheet

2:01 | Quadratic Equations

Name: ______________________ Class: ____________

Examples

1 Solve.

a $x(x + 5) = 0$

$x = 0$ or $x + 5 = 0$

$\therefore\ x = 0$ or -5

b $(a + 5)(a - 3) = 0$

$a + 5 = 0$ or $a - 3 = 0$

$\therefore\ a = -5$ or 3

2 Factorise, then solve.

a $y^2 - 6y = 0$

$y(y - 6) = 0$

$y = 0$ or $y - 6 = 0$

$\therefore\ y = 0$ or 6

b $h^2 + 7h - 8 = 0$

$(h + 8)(h - 1) = 0$

$h + 8 = 0$ or $h - 1 = 0$

$\therefore\ h = -8$ or 1

Exercise

1 Factorise.

a $x^2 - 3x$ **b** $x^2 + 3x + 2$ **c** $y^2 + 6y + 5$

d $d^2 - 9$ **e** $3k^2 + 6k$ **f** $t^2 - t - 6$

g $a^2 + 7a + 6$ **h** $x^2 + 2x - 24$ **i** $y^2 + 7y$

2 Solve.

a $x(x - 4) = 0$ **b** $(x - 1)(x + 2) = 0$ **c** $(m + 8)(m - 4) = 0$

d $t(t + 6) = 0$ **e** $(y + 3)(y + 4) = 0$ **f** $(a - 4)(a - 5) = 0$

g $2c(c - 14) = 0$ **h** $(x - 6)(x + 1) = 0$ **i** $(h + 1)(h - 1) = 0$

j $(y - 4)(y + 4) = 0$ **k** $(n - 6)(n - 8) = 0$ **l** $(x + 10)(x - 11) = 0$

3 Factorise, then solve. (*Hint:* Use your answers from Question 1 to start parts **a** to **f**.)

a $x^2 - 3x = 0$ **b** $x^2 + 3x + 2 = 0$ **c** $y^2 + 6y + 5 = 0$

d $d^2 - 9 = 0$ **e** $3k^2 + 6k = 0$ **f** $t^2 - t - 6 = 0$

g $f^2 - f - 12 = 0$ **h** $x^2 - 7x + 12 = 0$ **i** $m^2 - 25 = 0$

j $a^2 + 3a - 4 = 0$ **k** $y^2 - 2y - 8 = 0$ **l** $n^2 + 10n + 16 = 0$

Fun Spot 2:01 | What do ghosts put in their hair?

Find the missing factor in each expression, and match its letter with the answer below.

A $5x^2 - 10x = 5x(\ldots\ldots)$ **C** $x^2 - 4 = (x - 2)(\ldots\ldots)$ **E** $x^2 + 5x + 6 = (x + 2)(\ldots\ldots)$

P $x^2 + 5x = x(\ldots\ldots)$ **R** $x^2 - 3x - 10 = (x + 2)(\ldots\ldots)$ **S** $x^2 + 6x - 7 = (x + 7)(\ldots\ldots)$

Y $x^2 - 1 = (x - 1)(\ldots\ldots)$

										!
$(x - 1)$	$(x + 2)$	$(x - 2)$	$(x - 5)$	$(x + 3)$		$(x - 1)$	$(x + 5)$	$(x - 5)$	$(x - 2)$	$(x + 1)$

Answers can be found in the Interactive Student CD.

Foundation Worksheet

2:03 | The Quadratic Formula

Name: ______________________ **Class:** ____________

Examples

Solve, using the quadratic formula.

1 $x^2 - 7x - 8 = 0 \quad a = 1, b = -7, c = -8$

$$\therefore x = \frac{-(-7) \pm \sqrt{(-7)^2 - 4 \times 1 \times -8}}{2 \times 1}$$

$$= \frac{7 \pm \sqrt{49 + 32}}{2} = \frac{7 \pm \sqrt{81}}{2}$$

$$= \frac{7 \pm 9}{2} = 8 \text{ or } -1$$

2 $x^2 + 8x + 5 = 0 \quad a = 1, b = 8, c = 5$

$$\therefore x = \frac{-8 \pm \sqrt{8^2 - 4 \times 1 \times 5}}{2 \times 1}$$

$$= \frac{-8 \pm \sqrt{64 - 20}}{2} = \frac{-8 \pm \sqrt{44}}{2}$$

$$\doteqdot -0{\cdot}68, -7{\cdot}32$$

Note: $\sqrt{44}$ has no exact value.

The solutions to the quadratic equation $ax^2 + bx + c = 0$ can be found using the formula

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

3 $4x^2 + 2x - 3 = 0 \quad a = 4, b = 2, c = -3$

$$\therefore x = \frac{-2 \pm \sqrt{2^2 - 4 \times 4 \times -3}}{2 \times 4}$$

$$= \frac{-2 \pm \sqrt{4 + 48}}{8}$$

$$= \frac{-2 \pm \sqrt{52}}{8}$$

$$\doteqdot 0{\cdot}65, -1{\cdot}15$$

Exercise

1 Evaluate $\dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ for each set of a, b and c values.

a $a = 1, b = 3, c = 2$
b $a = 2, b = 5, c = -2$
c $a = 1, b = -2, c = -3$
d $a = 4, b = 10, c = 5$
e $a = 1, b = -6, c = 3$
f $a = 1, b = 2, c = -8$
g $a = 2, b = 3, c = -2$
h $a = 1, b = -7, c = 6$
i $a = 3, b = 1, c = -1$
j $a = 5, b = 7, c = 2$
k $a = 5, b = 7, c = -2$
l $a = 4, b = 8, c = 4$

2 Solve, using the quadratic formula.

a $x^2 + 5x + 2 = 0$
b $x^2 - 3x - 1 = 0$
c $x^2 - 7x - 18 = 0$
d $2x^2 + 5x + 2 = 0$
e $3x^2 - x - 2 = 0$
f $3x^2 - 7x + 2 = 0$
g $x^2 + x - 1 = 0$
h $x^2 + 9x - 22 = 0$
i $2x^2 + 13x + 5 = 0$
j $x^2 - 4x - 12 = 0$
k $x^2 - 4x - 2 = 0$
l $x^2 - 4x + 2 = 0$
m $2x^2 - x - 9 = 0$
n $4x^2 + 3x - 3 = 0$
o $x^2 - 6x + 9 = 0$

Fun Spot 2:03 | What do ghosts eat for dinner?

Solve each equation, and match its letter with the answer below.

E $2x = 0$ **I** $a - 4 = 0$ **K** $2k - 3 = 0$ **O** $3n + 4 = 0$
P $3h + 2 = 0$ **S** $e + 3 = 0$ **T** $2c - 1 = 0$

					-					!
-3	$-\frac{2}{3}$	$-\frac{4}{3}$	$-\frac{4}{3}$	$\frac{3}{2}$		0	$\frac{1}{2}$	$\frac{1}{2}$	4	

Answers can be found in the Interactive Student CD.

Foundation Worksheet

3:01 | Probability Review

Name: ______________________ **Class:** ____________

The probability of an event is the number of times it may occur divided by the total number of possibilities.

Exercise

1 There are ten cards, labelled 1 to 10. Find the probability of choosing, at random:

a the number 7 **b** a number less than 7 **c** a prime number.

2 Amy, Vicki and Veronica have birthdays on different days in June. If a date is chosen at random, find the probability that it is the birthday of:

a Vicki **b** Vicki or Veronica **c** any of the three girls.

3 Jack numbers discs from 1 to 20. He then chooses one at random. What is the probability that it is:

a an even number? **b** a multiple of 3? **c** a number less than 10?

4 Jesse has 36 CDs: 12 are compilations, 8 are by bands, 6 are by females and the rest are by males. If Jesse chooses a CD at random, find the probability that it is:

a by a band **b** a compilation

c not a compilation **d** by a male or a female.

5 A black bag contains 60 coloured counters: 20 are blue, 15 red, 10 white, 8 green and the rest yellow. If Keira chooses one, find the probability that it is:

a blue **b** red or white **c** red, white or blue

d green **e** not green **f** not white or green.

6 Cards are marked with each letter of the alphabet, then placed face-down in a basket. One card is then chosen. Find the probability that:

a a vowel is chosen

b a consonant is chosen

c Holly chooses a letter in her name

d Briana chooses a letter in her name

e a letter from the word PARRAMATTA is chosen.

7 In Xander's class there are 14 girls and 10 boys. A student is chosen at random to run a message. What is the probability that the student is:

a a boy?

b a girl?

c Xander?

Answers can be found in the Interactive Student CD.

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Foundation Worksheet

3:02 | Organising Outcomes of Compound Events

Name: ____________________ **Class:** ____________

The sample space gives all the possible outcomes.

Exercise

1 Show the possible outcomes as a list for:

- **a** having boys and girls in a family of 3 children
- **b** selecting 2 cards together from 5 cards labelled A, B, C, D, E
- **c** tossing a coin and rolling a normal die
- **d** spinning a red spinner labelled 1 to 4 and a blue spinner labelled 1 to 3
- **e** throwing a 4-faced die, marked with 1, 2, 3, 4 dots, twice
- **f** choosing an ice-cream (flavoured strawberry, vanilla or choc-chip) and a topping (flavoured strawberry, caramel or lime)
- **g** selecting a mixed doubles tennis pair from Vicki, Jools, Sarah and Jemma, and Ben and Roy
- **h** making a three-letter 'word' with: first letter p, t or w; second letter e or u; and third letter g or d
- **i** Josie, Veronica and Xander standing in a line.

2 Show the possible outcomes for each part of Question 1 as a tree diagram.

Answers can be found in the Interactive Student CD.

Foundation Worksheet

4:02 | Simple Interest

Name: ______________________ **Class:** ____________

Examples

% means out of 100 or move 2 decimal places left,
e.g. $82\% = \frac{82}{100} = 0{\cdot}82$, $7\% = \frac{7}{100} = 0{\cdot}07$

1 Calculate each value.

a 54% of $\$70 = 0{\cdot}54 \times \$70$
$= \$37.80$

b 4% of $\$95 = 0{\cdot}04 \times \$95$
$= \$3.80$

2 Find the simple interest on \$300 at 17% p.a. for 5 years.

$$\begin{aligned} I &= PRT \\ &= \$300 \times 17\% \times 5 \\ &= \$300 \times 0{\cdot}17 \times 5 \\ &= \$255 \end{aligned}$$

Exercise

1 Calculate each value.

a 15% of \$60
b 9% of \$80
c 2% of \$120
d 30% of \$55
e 70% of \$620
f 40% of \$900
g 8% of \$700
h 18% of \$750
i 65% of \$30
j 22% of \$300
k 45% of \$1000
l 90% of \$250

2 Find the simple interest on:

a \$200 at 10% p.a. for 5 years
b \$500 at 8% p.a. for 10 years
c \$600 at 3% p.a. for 4 years
d \$400 at 12% p.a. for 5 years
e \$800 at 7% p.a. for 10 years
f \$750 at 11% p.a. for 4 years
g \$1000 at 5% p.a. for 6 years
h \$440 at 10% p.a. for 3 years
i \$525 at 4% p.a. for 5 years
j \$400 at 6% p.a. for 7 years
k \$780 at 2% p.a. for 4 years
l \$320 at 18% p.a. for 5 years.

Fun Spot 4:02 | What do you give a sick car?

Change each value to a decimal and match the letters with the answers below.

A 7% **C** $\frac{3}{5}$ **E** $\frac{11}{20}$ **F** $\frac{1}{4}$ **I** 6% **J** 11%

L 10% **N** $\frac{9}{10}$ **O** $\frac{9}{20}$ **T** 1% **U** 70%

0·07	0·25	0·7	0·55	0·1	0·06	0·9	0·11	0·55	0·6	0·01	0·06	0·45	0·9	!

Answers can be found in the Interactive Student CD.

Foundation Worksheet

4:04 | Compound Interest

Name: ______________________ **Class:** ____________

Example

How much will $800 grow to after 3 years at 5% p.a. compound interest?
How much interest is earned?

Method 1

After 1 year, total = $800 + 5% of $800
= $800 × 105%
= $800 × 1·05
= $840

After 2 years, total = $840 × 1·05
= $882

After 3 years, total = $882 × 1·05
= $926.10

Method 2

After 3 years, total = $800 × 1·05 × 1·05 × 1·05
= $800 × $1{\cdot}05^3$
= $926.10

Interest = total − original amount
= $926.10 − $800
= $126.10

Exercise

1 How much will $10 000 grow to at 7% p.a. compound interest after:
a 1 year? **b** 2 years? **c** 3 years?

2 Find the compound interest for each part of Question 1.

3 What is the total value of $4000 at 5% p.a. compound interest after:
a 1 year? **b** 2 years? **c** 3 years?

4 Find the compound interest earned for each part of Question 3.

5 How much will $2000 grow to at 8% p.a. compound interest after:
a 1 year? **b** 2 years? **c** 3 years? **d** 4 years?

6 Find the compound interest for each part of Question 5.

7 Calculate the value of $5000 after 4 years at 4% p.a. compound interest. How much interest is earned?

8 How much will $20 000 grow to after 10 years at 6% p.a. compound interest. What interest will be earned?

9 What will be the value of $12 000 compounded at 4% p.a. for 5 years? What will the interest be?

10 Calculate the interest earned when $15 000 is invested at 4% p.a. and compounded for 5 years.

11 Find the interest when $3000 is compounded at 7% p.a. for 10 years.

12 $800 was invested at 5% p.a. compound interest over 8 years. How much is the investment now worth? What interest has been gained?

Answers can be found in the Interactive Student CD.

Foundation Worksheet

4:06 | Compound Interest Formula

Name: ______________________ **Class:** ____________

Examples

1 Find the amount received, and interest earned, when \$2000 is invested at 3% p.a. compound interest for 6 years.

Use $A = P(1 + r)^n$

where $P = \$2000$, $n = 6$, $r = 3\% = 0{\cdot}03$

$\therefore A = \$2000(1 + 0{\cdot}03)^6$

$\quad \doteqdot \$2388.10$

Interest $= \$2388.10 - \$2000$

$\quad = \$388.10$

2 Use the formula $A = P(1 - r)^n$ to find the value of an item worth \$25 000 after it has depreciated at 10% p.a. for 5 years.

Now $P = \$25\,000$, $n = 5$, $r = 10\% = 0{\cdot}1$

$\therefore A = \$25\,000(1 - 0{\cdot}1)^5$

$\quad = \$25\,000 \times 0{\cdot}9^5$

$\quad = \$14\,762.25$

Exercise

1 Use the formula $A = P(1 + r)^n$ to find the amount received if:

a \$10 000 is invested at 6% p.a. for 7 years

b \$4000 is invested at 5% p.a. for 4 years

c \$5000 is invested at 10% p.a. for 6 years

d \$6000 is invested at 13% p.a. for 5 years

e \$8000 is invested at 8% p.a. for 10 years

f \$1500 is invested at 4% p.a. for 8 years.

2 Find the interest earned for each part of Question 1.

3 Use the formula $A = P(1 - r)^n$ to find the value of an item worth:

a \$20 000 after it has depreciated 5% p.a. for 6 years

b \$8000 after it has depreciated 4% p.a. for 5 years

c \$3000 after it has depreciated 10% p.a. for 4 years

d \$5000 after it has depreciated 15% p.a. for 6 years

e \$40 000 after it has depreciated 20% p.a. for 5 years

f \$2000 after it has depreciated 8% p.a. for 10 years.

Fun Spot 4:06 | What kind of car did Elvis drive?

Match the letters with the answers below.

What decimal do I multiply by for compound interest of:

A 4% **C** 10% **E** 6% **K** 5% **L** 15%?

What decimal do I multiply by to depreciate by:

N 6% **O** 4% **R** 15% **S** 10% **Y** 5%?

1·04		0·85	0·96	1·1	1·05		0·94		0·85	0·96	1·15	1·15	0·9		0·85	0·96	0·95	1·1	1·06	!

Answers can be found in the Interactive Student CD.

Foundation Worksheet

5:02 | The Parabola $y = ax^2 + bx + c$

Name: ______________________ **Class:** ____________

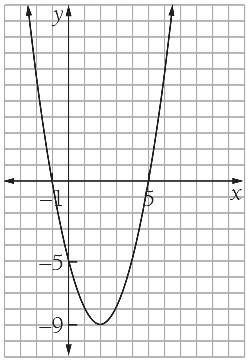
Examples

For each parabola, find the:

i y-intercept **ii** x-intercepts

iii equation of the axis of symmetry **iv** coordinates of the vertex.

a



i $y = -5$ **ii** $x = -1, 5$

iii $x = 2$ **iv** $(2, -9)$

b $y = x^2 - 8x + 12$

i When $x = 0$, $y = 12$

$\therefore$ y-intercept $= 12$

ii When $y = 0$, $x^2 - 8x + 12 = 0$

$(x - 2)(x - 6) = 0$

$x = 2, 6$

iii Axis is halfway between x-intercepts,

i.e. $x = \dfrac{2 + 6}{2} = 4$

iv For vertex, $x = 4$, $y = 4^2 - 8(4) + 12$

$= -4$

$\therefore$ Vertex $= (4, -4)$

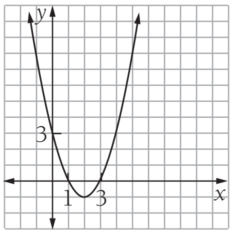
Exercise

1 For each graph, find the:

i y-intercept **ii** x-intercepts

iii equation of the axis of symmetry **iv** coordinates of the vertex.

a



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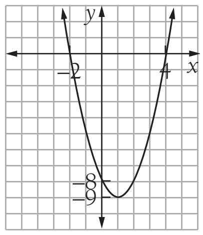

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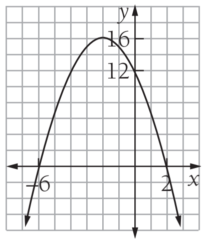

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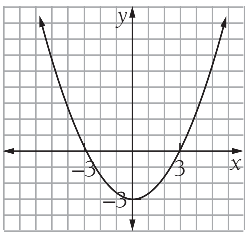

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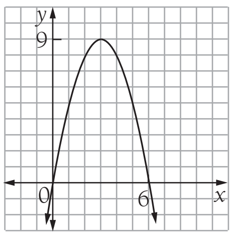

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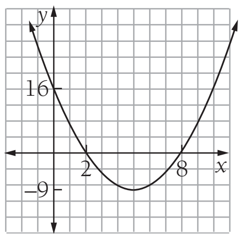


2 For each of these parabolas, find the:

i y-intercept **ii** x-intercepts

iii equation of the axis of symmetry **iv** coordinates of the vertex.

a $y = x^2 + x - 2$ **b** $y = x^2 - 8x$ **c** $y = x^2 + 2x - 8$

d $y = 24 - 2x - x^2$ **e** $y = x^2 - 8x + 15$ **f** $y = 4x + x^2$

Answers can be found in the Interactive Student CD.

Foundation Worksheet

5:05 | The Circle

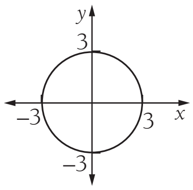
Name: ______________________ **Class:** ____________

Examples

The equation of a circle with centre the origin (0, 0) and radius r is $x^2 + y^2 = r^2$.

1 For this circle, write down the:

a radius **b** equation.



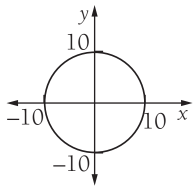
a $r = 3$ **b** $x^2 + y^2 = 3^2$

$\therefore x^2 + y^2 = 9$

2 Sketch the circle $x^2 + y^2 = 100$.

Centre = (0, 0)

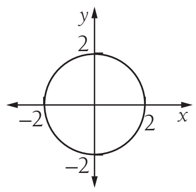
Radius = $\sqrt{100} = 10$



Exercise

1 For each circle, write down its: **i** radius **ii** equation.

a



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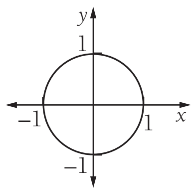

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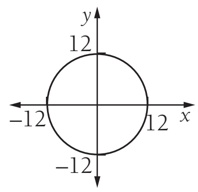

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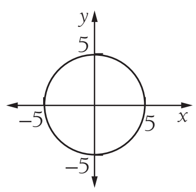

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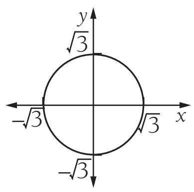

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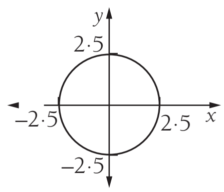

2 Sketch the circle represented by:

a $x^2 + y^2 = 36$ **b** $x^2 + y^2 = 4$ **c** $x^2 + y^2 = 64$

d $x^2 + y^2 = 81$ **e** $x^2 + y^2 = 49$ **f** $x^2 + y^2 = 121$

g $x^2 + y^2 = 20$ **h** $x^2 + y^2 = 256$ **i** $x^2 + y^2 = 12{\cdot}25$

j $x^2 + y^2 = 17$ **k** $x^2 + y^2 = 400$ **l** $x^2 + y^2 = 1{\cdot}21$

Answers can be found in the Interactive Student CD.

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Foundation Worksheet

5:08 | Coordinate Geometry

Name: ______________________ **Class:** ____________

In this work you apply the formulas previously used, to solve geometry problems.

Distance, $d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$

Gradient, $m = \dfrac{y_2 - y_1}{x_2 - x_1}$

Midpoint $= \left(\dfrac{x_1 + x_2}{2}, \dfrac{y_1 + y_2}{2}\right)$

Exercise

1 **a** Show that the triangle formed by the points $A(-2, 0)$, $B(0, 4)$ and $C(2, 0)$ is isosceles.

b Show that $\triangle JKL$ is a right-angled triangle if $J = (-2, 5)$, $K = (1, 3)$ and $L = (-1, 0)$.

c Find the perimeter of $\triangle XYZ$ if $X = (6, 0)$, $Y = (2, -3)$ and $Z = (-4, 5)$.

2 **a** Show that the quadrilateral with its vertices at $A(-1, 1)$, $B(2, 1)$, $C(3, -1)$ and $D(0, -1)$ is a parallelogram.

b Show that the diagonals of *ABCD* in part **a** bisect each other.

c $\triangle DEF$ has vertices $D(0, 16)$, $E(8, 4)$ and $F(2, 2)$.

i Find the midpoints *G* and *H* of *DE* and *DF* respectively.

ii Show that $EF \parallel GH$.

3 **a** $A(-2, 0)$, $B(0, 3)$, $C(2, 0)$ and $D(0, -3)$ are the vertices of a quadrilateral. Show that *ABCD* is a rhombus.

b Prove that the points $W(1, 8)$, $X(5, 5)$, $Y(2, 1)$ and $Z(-2, 4)$ are the vertices of a square.

c Show that the diagonals of *WXYZ* in part **b** meet at right angles.

Answers can be found in the Interactive Student CD.

Foundation Worksheet

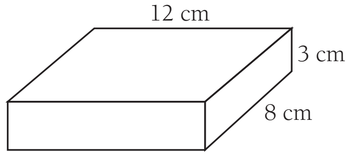
6:01 | Surface Area Review

Name: ______________________ **Class:** ____________

Exercise

1 Find the surface area of each of these rectangular prisms.

a



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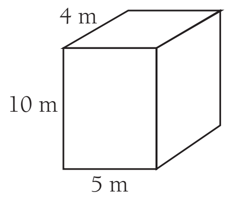

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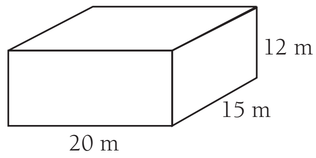

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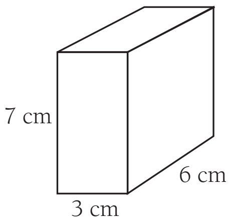

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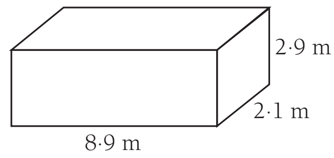

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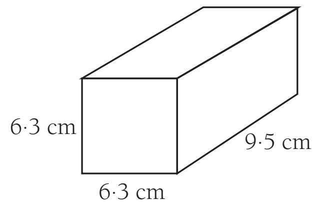
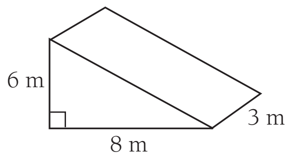


2 Find the surface area of each of these triangular prisms.

a



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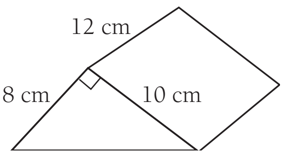

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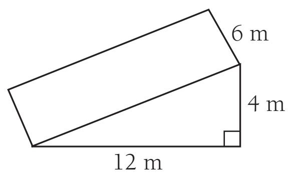

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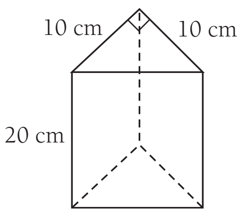

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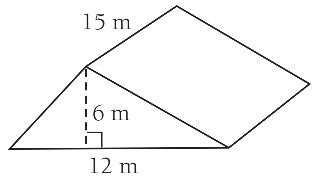

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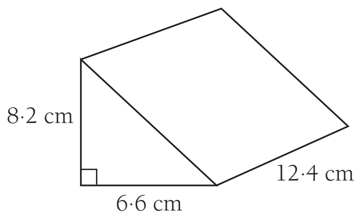
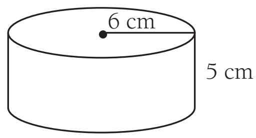


3 Find the surface area of each of these solid cylinders. Answer correct to 1 decimal place.

a



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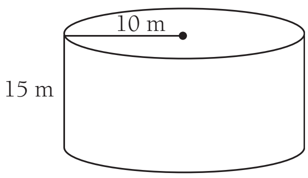

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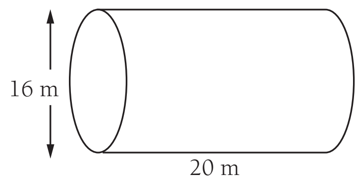

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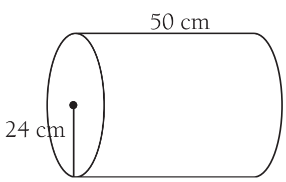

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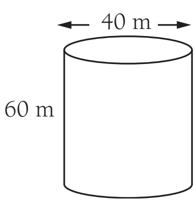

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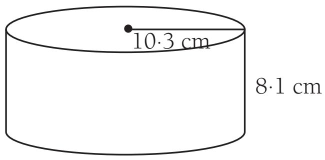

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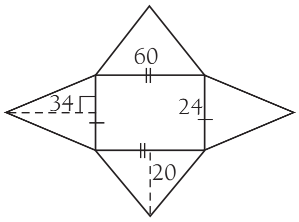
Foundation Worksheet

6:02 | Surface Area of a Pyramid

Name: ____________________ **Class:** __________

Examples

1 Calculate the area of this net.



Base area = $60 \times 24 = 1440$ units^2

Area of triangles = $2 \times (\frac{1}{2} \times 60 \times 20)$

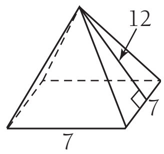
$+ 2 \times (\frac{1}{2} \times 24 \times 34)$

= 2016 units^2

$\therefore$ Area of net = 1440 + 2016

= 3456 units^2

2 Find the surface area of this square pyramid.



Base area = $7 \times 7 = 49$ units^2

Area of 4 triangles = $4 \times (\frac{1}{2} \times 7 \times 12)$

= 168 units^2

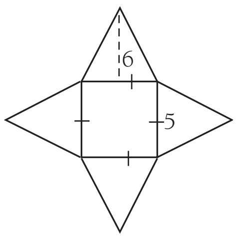
$\therefore$ Surface area = 49 + 168

= 217 units^2

Exercise

1 Calculate the area of each net.

a



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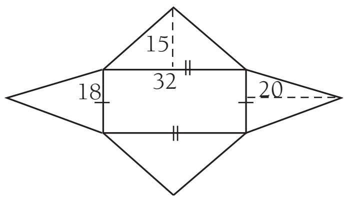

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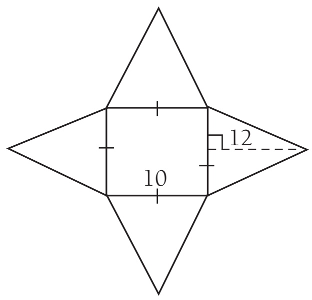
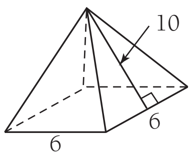


2 Find the surface area of each pyramid.

a



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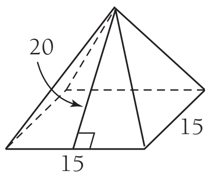

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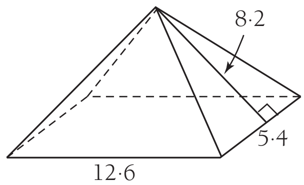

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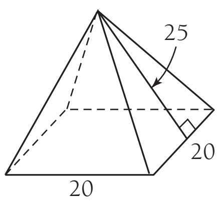

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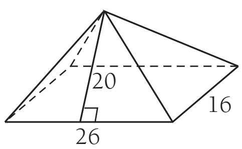

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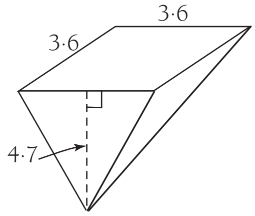

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Foundation Worksheet

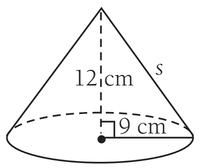
6:03 | Surface Area of a Cone

Name: ______________________ **Class:** ____________

Example

Surface area of cone = area of base + area of curved surface
$= \pi r^2 + \pi rs$
where r = radius of base and s = slant height.

Find the surface area of this cone.

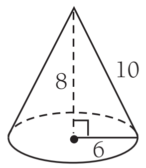


Slant height, $s = \sqrt{9^2 + 12^2}$
$= 15$ cm
Radius $= 9$ cm
$SA = \pi \times 9^2 + \pi \times 9 \times 15$
$\doteqdot 678{\cdot}6 \text{ cm}^3$

Exercise

1 For each cone, identify the radius, height and slant height.

a



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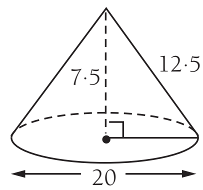

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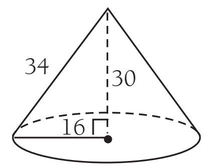
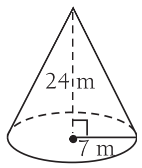


2 For each cone in Question 1, find the:
i curved surface area **ii** total surface area.
Answer correct to 1 decimal place.

3 Use Pythagoras' theorem to find the slant height of a cone if:
a radius = 3 cm, height = 4 cm
b diameter = 16 cm, height = 15 cm
c radius = 7 cm, height = 24 cm
d diameter = 36 cm, height = 24 cm.

4 For each cone below, find the:
i slant height **ii** total surface area.
Answer correct to 1 decimal place where appropriate.

a



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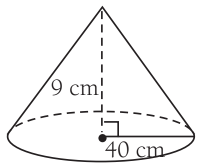

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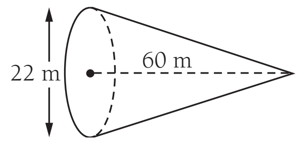

d Radius = 3·3 cm, height = 4·4 cm
e Diameter = 10 cm, height = 12 cm
f Radius = 2 cm, height = 4·8 cm

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Foundation Worksheet

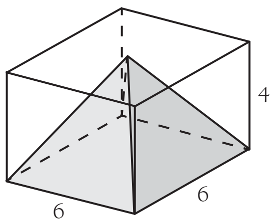
6:05 | Volume of a Pyramid

Name: ______________________ **Class:** ____________

Exercise

1 For each of the following, calculate the volume of the prism, then divide by 3 to find the volume of the pyramid.

a



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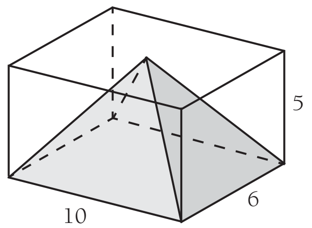

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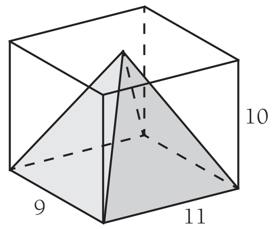

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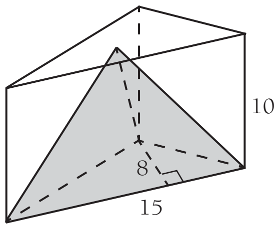

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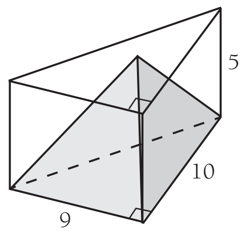

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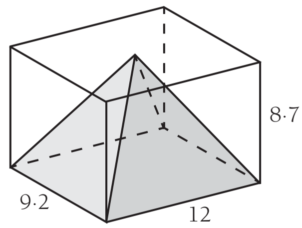
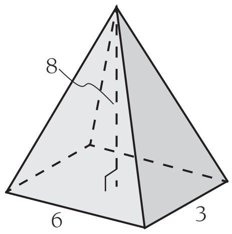


2 For each pyramid, find the value of A and h, then use the formula $V = \frac{Ah}{3}$ to find the volume.

a



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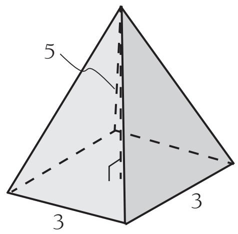
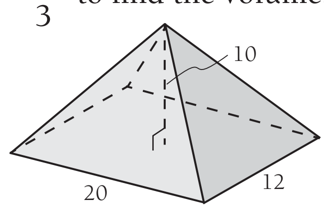


c



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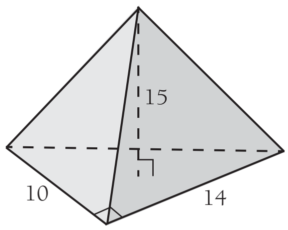

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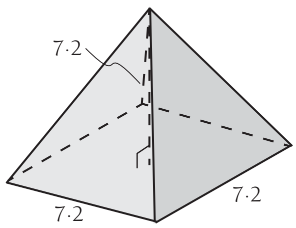
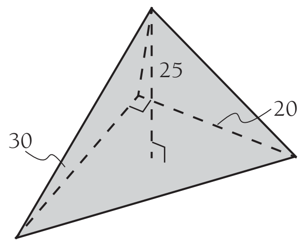


f



Fun Spot 6:05 | What wobbles when it flies?

Calculate the volume of each pyramid using the formula $V = \frac{Ah}{3}$, correct to the nearest whole. Match the letters with the answers below.

A $A = 40$, $h = 9$ **C** $A = 36{\cdot}9$, $h = 12$ **E** $A = 100$, $h = 15$

J $A = 150$, $h = 20$ **L** $A = 25{\cdot}8$, $h = 21$ **O** $A = 34{\cdot}71$, $h = 8$

P $A = 28{\cdot}4$, $h = 4{\cdot}2$ **R** $A = 63$, $h = 11$ **T** $A = 50$, $h = 14{\cdot}1$

Y $A = 5{\cdot}3$, $h = 42$

											!
120	1000	500	181	181	74	148	93	40	235	500	231

Answers can be found in the Interactive Student CD.

Foundation Worksheet

7:02 | Inter-quartile Range

Name: ______________________ **Class:** ____________

Examples

For each set of scores, find the median (Q_2), the first and third quartiles (Q_1, Q_3) and the inter-quartile range.

1 5, 9, 12, 14, 15, 16, 16, 17, 18, 19, 20, 20

Divide the scores into 4 groups.

5, 9, 12, | 14, 15, 16, | 16, 17, 18, | 19, 20, 20

$$\text{Median } (Q_2) = \text{2nd quarter} = 16$$

$$Q_1 = \frac{12 + 14}{2} = 13$$

$$Q_3 = \frac{18 + 19}{2} = 18{\cdot}5$$

$$\text{Inter-quartile range} = Q_3 - Q_1 = 5{\cdot}5$$

2 6, 8, 4, 3, 6, 2, 4, 7

Write the scores in order, then split into 4 groups.

2, 3, | 4, 4, | 6, 6, | 7, 8

$$\text{Median } (Q_2) = \frac{4 + 6}{2} = 5$$

$$Q_1 = \frac{3 + 4}{2} = 3{\cdot}5$$

$$Q_3 = \frac{6 + 7}{2} = 6{\cdot}5$$

$$\text{Inter-quartile range} = Q_3 - Q_1 = 3$$

Exercise

1 These sets of scores have been arranged in order and divided into quartiles. For each set, find:

- **i** the median (Q_2)
- **ii** the first and third quartiles (Q_1, Q_3)
- **iii** the inter-quartile range.

a 1, 2, | 2, 4, | 5, 6, | 6, 8
b 1, 2, | 3, 4, | 5, 6, | 9, 12
c 1, 2, 3, | 5, 5, 6, | 7, 8, 10, | 11, 12, 18
d 10, 10, 11, | 11, 12, 14, | 15, 17, 18, | 20, 20, 20
e 3, 7, | 8, 9, | 11, 13, | 16, 18
f 8, 12, 13, | 17, 18, 20, | 20, 22, 23, | 24, 26, 28

2 Find the inter-quartile range for each set of scores.

a 5, 6, 7, 7, 9, 10, 10, 11, 15, 17, 18, 20, 20, 21, 23, 25
b 18, 20, 30, 34, 36, 38, 40, 45
c 17, 20, 12, 15, 8, 10, 16, 12
d 4, 5, 15, 4, 6, 4, 5, 9, 8, 8, 9, 8
e 11, 13, 8, 10, 4, 6, 7, 9 10, 5, 6, 4
f 16, 4, 8, 10, 8, 12, 11, 9
g 1, 5, 7, 9, 6, 4, 8, 7, 6, 3, 2, 3, 5, 5, 4, 7
h 5, 9, 7, 3, 4, 2, 8, 6, 5, 3, 4, 10
i 18, 22, 19, 33, 17, 16, 28, 36
j 100, 120, 420, 210, 190, 230, 270, 300, 320, 350, 140, 250

Answers can be found in the Interactive Student CD.

Foundation Worksheet

7:04 | Standard Deviation

Name: ______________________________ **Class:** ______________

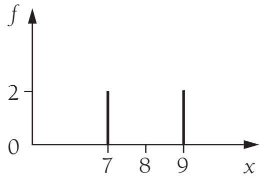
- **The standard deviation measures how spread out the scores are from the mean.**
- **To calculate the standard deviation, you use the $\boxed{\sigma_n}$ key after entering the scores on your calculator.**

Exercise

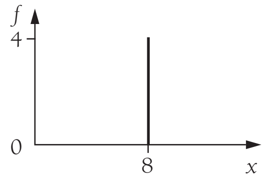
1 These sets of scores all have a mean of 8. By looking at the spread of scores, find which set (A, B or C) in each case has:

i the smallest standard deviation **ii** the largest deviation.

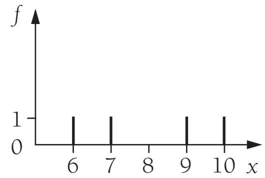
a **A**



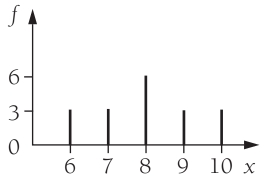
B



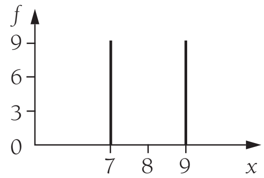
C



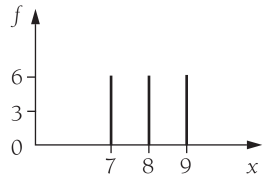
b **A**



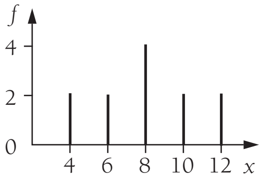
B



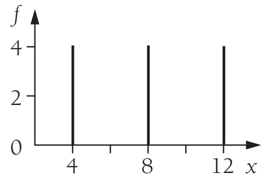
C



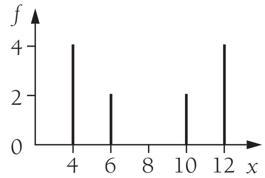
c **A**



B



C



2 Use your calculator to find the standard deviation of each set of scores, to 1 decimal place.

a 1, 2, 3, 4, 5
b 2, 2, 3, 4, 4
c 1, 1, 3, 5, 5
d 8, 10, 11, 12, 13, 14, 16
e 6, 6, 8, 12, 16, 18, 18
f 18, 13, 14, 12, 17, 18, 16, 15, 19, 20, 14, 14
g 20, 30, 40, 50, 30, 20, 60, 40, 20, 10, 80, 70, 40, 50, 70

Fun Spot 7:04 | What's a cheap name for an expensive car?

Calculate the mean and standard deviation for each set of scores (to 1 decimal place). Match the letters with the answers below.

5, 4, 13, 7, 9, 8:	**A** = mean	**C** = standard deviation
5, 6, 7, 9, 12, 15:	**E** = mean	**H** = standard deviation
4, 8, 5, 9, 6:	**O** = mean	**P** = standard deviation
7, 8, 8, 10, 14, 9, 7, 6:	**R** = mean	**S** = standard deviation

						-					!
7·7		1·9	6·4	6·4	8·6		2·3	2·9	3·5	9	

Foundation Worksheet

8:03 | Finding Unknown Sides in Similar Triangles

Name: ______________________ **Class:** ____________

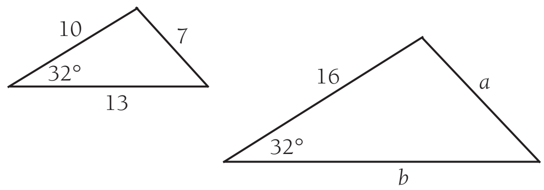
Example

Find the value of the pronumerals in these similar triangles.

16 matches 10. $\therefore$ Enlargement factor $= \frac{16}{10} = 1{\cdot}6$

a matches 7. $\therefore a = 7 \times 1{\cdot}6 = 11{\cdot}2$

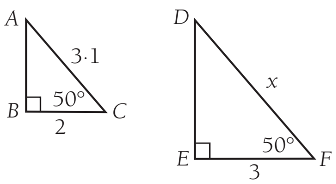
b matches 13. $\therefore b = 13 \times 1{\cdot}6 = 20{\cdot}8$



Exercise

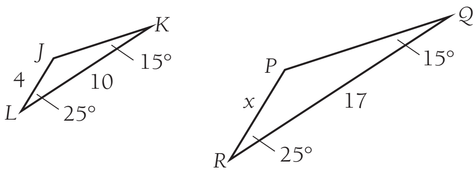
1 Complete the working for each pair of similar triangles.

a



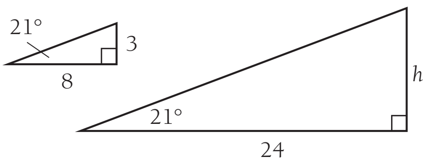
EF matches …
Enlargement factor = …
$x = \ldots$

b



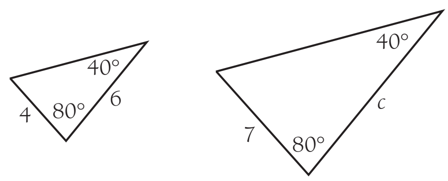
QR matches …
Enlargement factor = …
$x = \ldots$

c



Side 24 matches …
Enlargement factor = …
$h = \ldots$

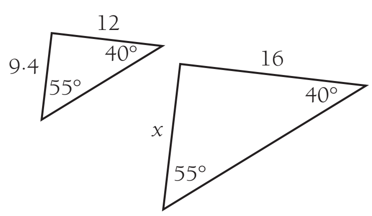
d



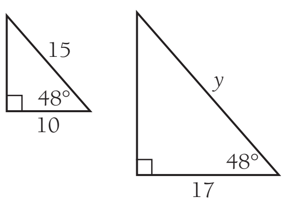
Side 7 matches …
Enlargement factor = …
$c = \ldots$

2 Find the values of the pronumerals.

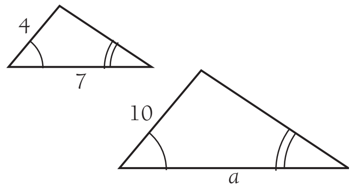
a



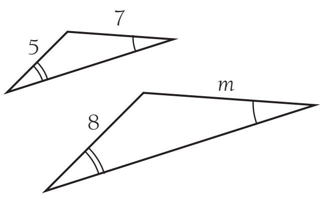
b



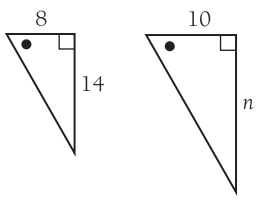
c



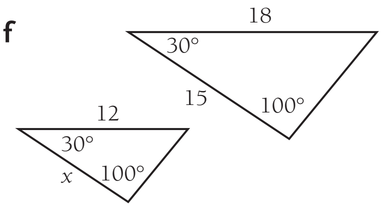
d



e



f



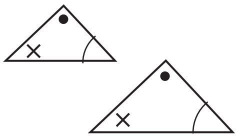
Answers can be found in the Interactive Student CD.

Foundation Worksheet

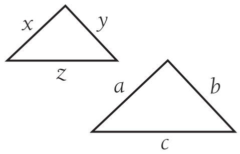
8:04 | Similar Triangles Proofs

Name: ______________________________ **Class:** ______________

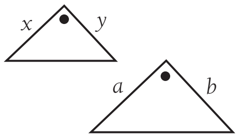
Tests for similar triangles:



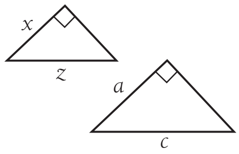
AAA



SSS $\frac{a}{x} = \frac{b}{y} = \frac{c}{z}$



SAS $\frac{a}{x} = \frac{b}{y}$

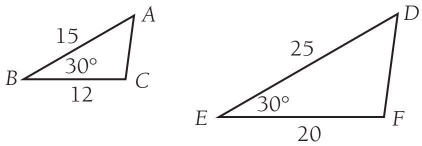


RHS $\frac{a}{c} = \frac{x}{z}$

Exercise

In each question, complete the proof to show that the triangles are similar.

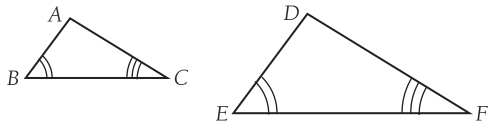
1



1. $\angle B =$ ______ (given)
2. $\frac{DE}{AB} = \frac{25}{15} =$ ______
3. $\frac{FE}{CB} =$ ______ $=$ ______

$\therefore \triangle ABC\ |||\ \triangle DEF$ (SAS test)

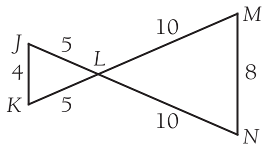
2



1. $\angle B =$ ______ (______)
2. $\angle C =$ ______ (______)
3. $\angle A =$ ______ (angle sum of triangle)

$\therefore \triangle ABC\ |||\ \triangle DEF$ ______ ______)

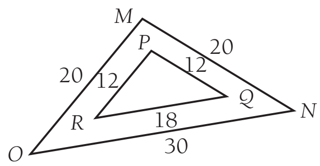
3



1. $\frac{MN}{JK} = \frac{8}{4} =$ ______
2. $\frac{ML}{JL} =$ ______ $= \frac{10}{5} =$ ______

$\therefore \triangle JKL\ |||\ \triangle MNL$ (______ ______)

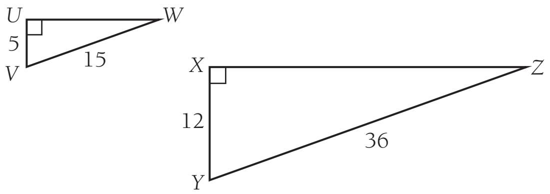
4



1. $\frac{MN}{PQ} =$ ______ $=$ ______ $= \frac{5}{3}$
2. $\frac{NO}{QR} =$ ______ $=$ ______

$\therefore \triangle MNO\ |||\ \triangle PQR$ (______ ______)

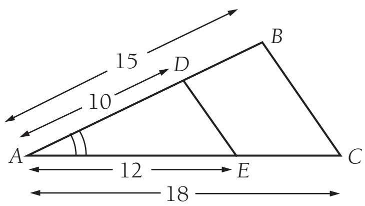
5



1. $\angle U = \angle X =$ ______ (______)
2. $\frac{UV}{WV} = \frac{5}{15} =$ ______
3. $\frac{XY}{ZY} =$ ______ $=$ ______

$\therefore \triangle UVW\ |||\ \triangle XYZ$ (______ ______)

6



1. $\angle A$ is ______
2. $\frac{AB}{AD} = \frac{15}{10} =$ ______
3. $\frac{AC}{AE} =$ ______ $=$ ______

$\therefore \triangle ADE\ |||\ \triangle ABC$ (______ ______)

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Foundation Worksheet

9:02 | Trig Ratios of Obtuse Angles

Name: ______________________ **Class:** ____________

Examples

For obtuse angles, sin is +, cos is –, tan is –.

If θ is between 0° and 180°, find θ to the nearest degree.

1 $\cos\theta = -0{\cdot}74$

$\therefore \theta = \cos^{-1}(-0{\cdot}74)$

Press [2nd F] [cos] 0·74 [=].

This gives 42°.

Since $\cos\theta < 0$, θ is obtuse.

$\therefore \theta = 180° - 42°$

$= 138°$

2 $\sin\theta = 0{\cdot}6$

$\therefore \theta = \sin^{-1} 0{\cdot}6$

Press [2nd F] [sin] 0·6 [=].

This gives 37°.

Since $\sin\theta > 0$, θ is acute or obtuse.

$\therefore \theta = 37°$ or $180° - 37°$

$= 37°$ or $143°$

3 $\tan\theta = -1{\cdot}45$

$\therefore \theta = \tan^{-1}(-1{\cdot}45)$

Press [2nd F] [tan] 1·45 [=].

This gives 55°.

Since $\tan\theta < 0$, θ is obtuse.

$\therefore \theta = 180° - 55°$

$= 125°$

Note: Some calculators use [SHIFT] instead of [2nd F].

Exercise

1 Use a calculator to find these sine ratios, correct to 3 decimal places.

a sin 40°, sin 140° **b** sin 60°, sin 110° **c** sin 80°, sin 100°
d sin 95°, sin 85° **e** sin 25°, sin 175° **f** sin 60°, sin 120°
g sin 170°, sin 10° **h** sin 45°, sin 135° **i** sin 66°, sin 124°

2 **a** In which parts of Question 1 were the ratios equal?
b For the ratios that were equal, what did you notice about the angles?

3 Which acute angle has the same sine ratio as:

a 120°? **b** 135°? **c** 110°? **d** 150°?
e 95°? **f** 130°? **g** 165°? **h** 107°?

4 Which obtuse angle has the same sine ratio as:

a 80°? **b** 20°? **c** 40°? **d** 10°?
e 75°? **f** 36°? **g** 61°? **h** 54°?

5 If θ is between 0° and 180°, find θ to the nearest degree.

a $\sin\theta = 0{\cdot}8$ **b** $\cos\theta = -0{\cdot}5$ **c** $\tan\theta = -2$
d $\cos\theta = -0{\cdot}111$ **e** $\tan\theta = -0{\cdot}7$ **f** $\sin\theta = 0{\cdot}83$
g $\tan\theta = -1{\cdot}5$ **h** $\cos\theta = -0{\cdot}4$ **i** $\sin\theta = 0{\cdot}2$

Answers can be found in the Interactive Student CD.

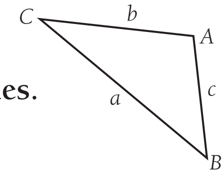
Foundation Worksheet

9:03 | The Sine Rule

Name: ______________________ **Class:** ____________

Examples

- Sine rule: $\dfrac{a}{\sin A} = \dfrac{b}{\sin B} = \dfrac{c}{\sin C}$
- Always match angles with opposite sides.
- Any two angles must add to less than the angle sum of a triangle, 180°.



1 Solve $\dfrac{a}{0{\cdot}72} = \dfrac{5{\cdot}2}{0{\cdot}31}$.

$a = \dfrac{5{\cdot}2}{0{\cdot}31} \times 0{\cdot}72$

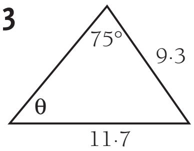
$\doteqdot 12{\cdot}1$

2 Find a when $\dfrac{a}{\sin 20°} = \dfrac{32}{\sin 70°}$.

$a = \dfrac{32}{\sin 70°} \times \sin 20°$

$\doteqdot 11{\cdot}6$

3



Find θ.

$\dfrac{\sin \theta}{9{\cdot}3} = \dfrac{\sin 75°}{11{\cdot}7}$

$\sin \theta = \dfrac{\sin 75°}{11{\cdot}7} \times 9{\cdot}3$

$= 0{\cdot}767\ \ldots$

$\therefore \theta = \sin^{-1} 0{\cdot}767\ \ldots$

$\doteqdot 50°$

(*not* $180° - 50° = 130°$)

Exercise

Answer to the nearest degree for angles, or correct to 1 decimal place for sides.

1 Solve these equations.

a $\dfrac{x}{6} = 8$ **b** $\dfrac{a}{0{\cdot}756} = 4{\cdot}654$ **c** $\dfrac{\sin\theta}{7{\cdot}3} = 0{\cdot}12$

d $\dfrac{h}{0{\cdot}57} = \dfrac{8{\cdot}3}{0{\cdot}29}$ **e** $\dfrac{c}{0{\cdot}8} = \dfrac{12}{0{\cdot}7}$ **f** $\dfrac{\sin\theta}{2{\cdot}5} = \dfrac{0{\cdot}37}{3{\cdot}6}$

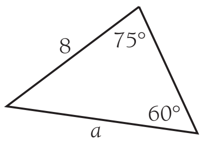
2 If $\dfrac{a}{\sin A} = \dfrac{b}{\sin B}$, find a when:

a $A = 30°, B = 70°, b = 15$ **b** $A = 22°, B = 46°, b = 9{\cdot}2$ **c** $A = 50°, B = 60°, b = 8$

d $A = 65°, B = 42°, b = 6{\cdot}8$ **e** $A = 50°, B = 100°, b = 18{\cdot}4$ **f** $A = 38°, B = 32°, b = 120$

3 Write the sine rule substitution for each triangle.

a



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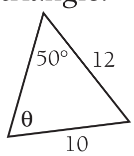

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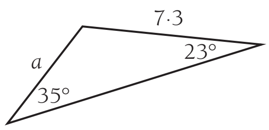

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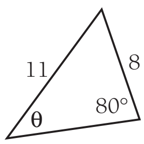

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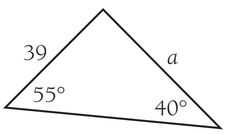

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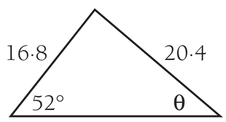

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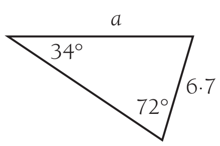

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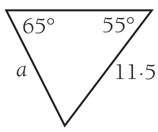

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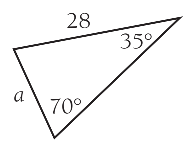


4 Now solve each part of Question 3 to find the values of the pronumerals.

Answers can be found in the Interactive Student CD.

Foundation Worksheet

9:04 | Sine Rule—the Ambiguous Case

Name: ______________________ **Class:** ______________

Examples

1 Find the acute and obtuse solutions for θ if $\sin\theta = 0{\cdot}37$.

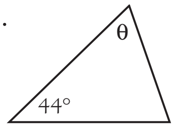
$\sin\theta = 0{\cdot}37$

$\therefore\ \theta = \sin^{-1} 0{\cdot}37$

$\doteqdot 22°$

or $180° - 22° = 158°$

2 Show whether θ has one or two solutions.



θ = 65° or 115°

If $\theta = 65°$:

$\theta + 44° = 109°$

$\therefore$ 3rd angle $= 71°$

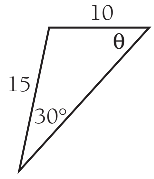
If $\theta = 115°$:

$\theta + 44° = 159°$

$\therefore$ 3rd angle $= 21°$

$\therefore$ Both solutions possible.

3 Find the acute and obtuse solutions for θ.



$$\frac{\sin\theta}{15} = \frac{\sin 30°}{10}$$

$$\sin\theta = \frac{\sin 30°}{10} \times 15$$

$$= 0{\cdot}75$$

$$\therefore\ \theta = \sin^{-1} 0{\cdot}75$$

$$\doteqdot 48°$$

or $180° - 48° = 132°$

Exercise

Give all solutions to the nearest degree.

1 **a** Find θ if $\sin\theta = 0{\cdot}8$ and θ is: **i** acute **ii** obtuse.

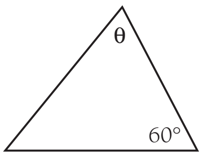
b Find θ if $\sin\theta = 0{\cdot}28$ and θ is: **i** acute **ii** obtuse.

c Find θ if $\sin\theta = 0{\cdot}123$ and θ is: **i** acute **ii** obtuse.

d Find θ if $\sin\theta = 0{\cdot}65$ and θ is: **i** acute **ii** obtuse.

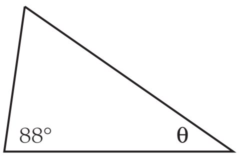
2 From the information given, show whether one or two solutions are possible for θ.

a



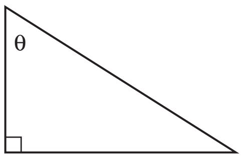
θ = 70° or 110°

b



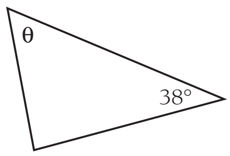
θ = 30° or 150°

c



θ = 60° or 120°

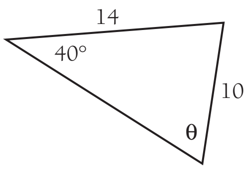
d



θ = 55° or 125°

3 Find the acute and obtuse solutions for θ in each triangle.

a



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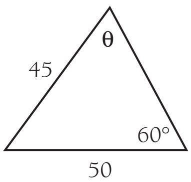

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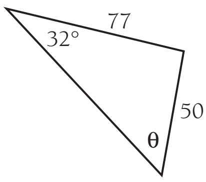

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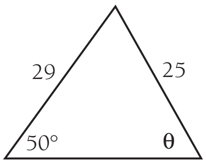

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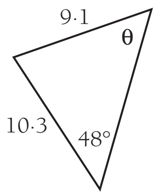

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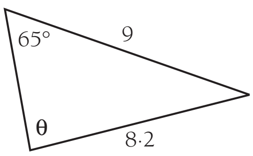

Answers can be found in the Interactive Student CD.

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Foundation Worksheet

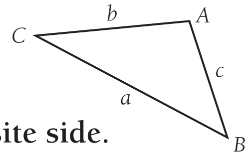
9:05 | The Cosine Rule

Name: ______________________ **Class:** ____________

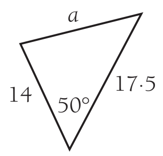
Examples

- **Cosine rule: $a^2 = b^2 + c^2 - 2bc \cos A$**

  or $\cos A = \dfrac{b^2 + c^2 - a^2}{2bc}$
- **Always match the angle with its opposite side.**



1 Find the length a.



$a^2 = b^2 + c^2 - 2bc \cos A$

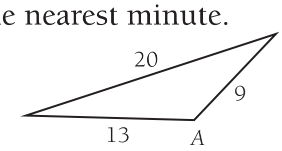
$= 14^2 + 17{\cdot}5^2 - 2 \times 14 \times 17{\cdot}5 \cos 50°$

$= 187{\cdot}284 \ldots$

$a \doteqdot 13{\cdot}7$

Note: Some calculators use [SHIFT] not [2nd F], or [° ’ ”] not [DMS].

2 Find the size of A, to the nearest minute.



$\cos A = \dfrac{b^2 + c^2 - a^2}{2bc}$

$= \dfrac{13^2 + 9^2 - 20^2}{2 \times 13 \times 9}$

$= -0{\cdot}641 \ldots$

$\therefore A = \cos^{-1}(-0{\cdot}641)$

Press [2nd F] [cos] 0·641 … [=] [DMS].

This gives 50° 8′. Since $\cos A < 0$, A is obtuse.

$\therefore A = 180° - 50°8' = 129°52'$

Exercise

1 Find a, using $a^2 = b^2 + c^2 - 2bc \cos A$ if:

a $b = 10$, $c = 12$, $A = 60°$ **b** $b = 8{\cdot}6$, $c = 7{\cdot}4$, $A = 47°$

c $b = 70$, $c = 64$, $A = 28°$ **d** $b = 22$, $c = 44$, $A = 71°$.

2 Find A, to the nearest degree, if A is acute or obtuse and:

a $\cos A = 0{\cdot}5$ **b** $\cos A = -0{\cdot}5$ **c** $\cos A = 0{\cdot}7071$

d $\cos A = -0{\cdot}7071$ **e** $\cos A = 0{\cdot}531$ **f** $\cos A = -0{\cdot}531$

g $\cos A = -0{\cdot}08$ **h** $\cos A = -0{\cdot}66$ **i** $\cos A = -0{\cdot}2$.

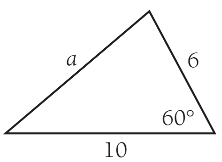
3 Find A, to the nearest minute, using $\cos A = \dfrac{b^2 + c^2 - a^2}{2bc}$ if:

a $a = 15$, $b = 14$, $c = 13$ **b** $a = 7$, $b = 15$, $c = 15$

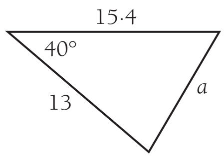
c $a = 7{\cdot}5$, $b = 17{\cdot}4$, $c = 23{\cdot}7$ **d** $a = 5$, $b = 4$, $c = 6$.

4 Write the cosine rule substitution for each triangle.

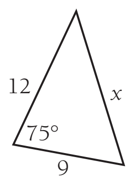
a



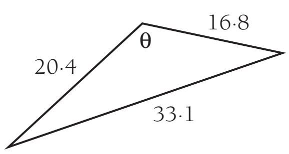
b



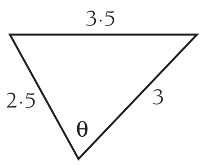
c



d



e



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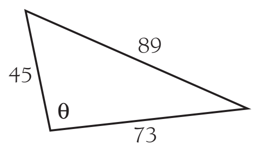


5 Find the values of the pronumerals in Question 4. For angles, answer to the nearest minute.

Answers can be found in the Interactive Student CD.

Foundation Worksheet

9:07 | Sine Rule or Cosine Rule?

Name: ______________________ **Class:** ____________

One way to decide whether to use the sine rule or cosine rule is to use tests similar to those for congruent triangles.

Use the sine rule if

AAS (you have two angles and any side)

SSA (you have two sides and a non-included angle).

Use the cosine rule if

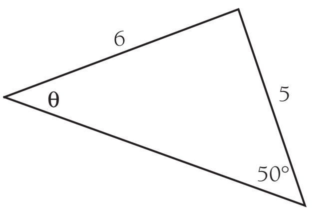
SSS (you have three sides)

SAS (you have two sides and the included angle).

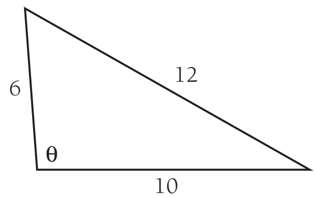
Exercise

1 Use the AAS, SSA, SSS, SAS tests to decide whether to use the sine rule or cosine rule for each of the following problems.

a



b



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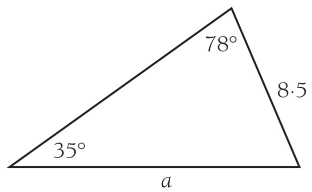

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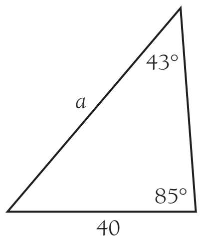

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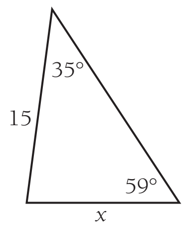

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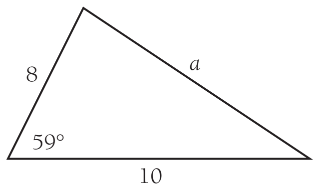

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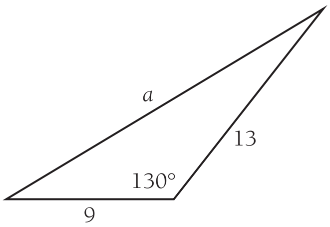

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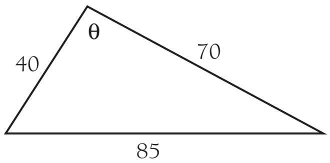

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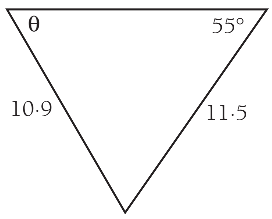

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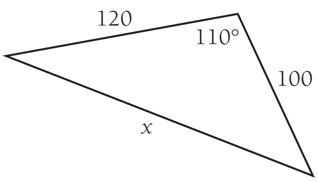

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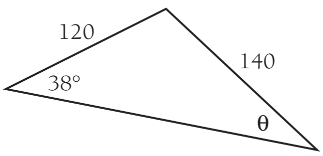

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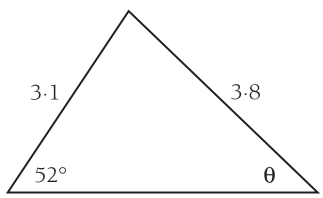

2 Find the values of the pronumerals in Question 1. Express angles to the nearest minute, and sides correct to 1 decimal place.

Answers can be found in the Interactive Student CD.

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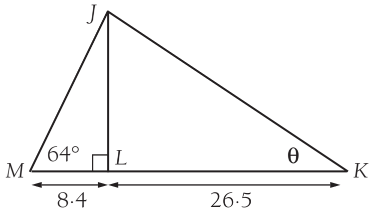
Foundation Worksheet

9:08 | Problems with More than One Triangle

Name: ______________________ **Class:** ____________

Examples

1



a Use $\triangle JLM$ to find JL.
b Use $\triangle JKL$ to find θ.

Solution

a $\tan 64° = \dfrac{JL}{8{\cdot}4}$

$JL = 8{\cdot}4 \times \tan 64°$

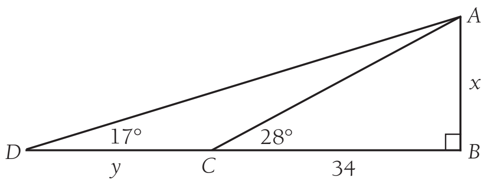
$\doteqdot 17{\cdot}2$

b $\tan \theta = \dfrac{17{\cdot}2}{26{\cdot}5}$

$\theta = \tan^{-1}\left(\dfrac{17{\cdot}2}{26{\cdot}5}\right)$

$\doteqdot 33°$

2



a Find x in $\triangle ABC$.
b Find DB in $\triangle ABD$.
c Find y, noting that $DB = y + 34$.

Solution

a $\tan 28° = \dfrac{x}{34}$

$x = 34 \times \tan 28°$

$\doteqdot 18$

b $\tan 17° = \dfrac{x}{DB} = \dfrac{18}{DB}$

$DB = 18 \div \tan 17°$

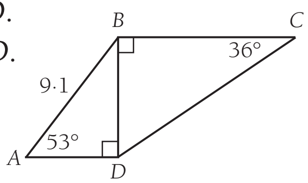
$\doteqdot 59$

c $y + 34 = DB \doteqdot 59$

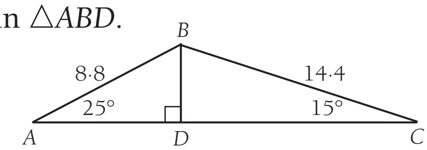
$y = 25$

Exercise

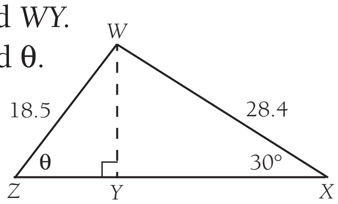
1 **a** Find BD in $\triangle ABD$.
b Find DC in $\triangle BCD$.



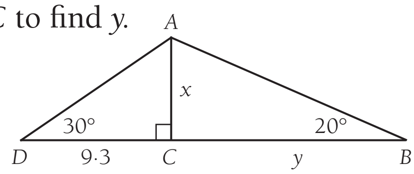
2 **a** Find DC in $\triangle BCD$.
b Find AD in $\triangle ABD$.
c Find AC.



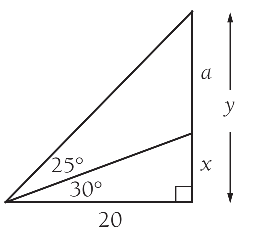
3 **a** Using $\triangle WXY$, find WY.
b Using $\triangle WYZ$, find θ.



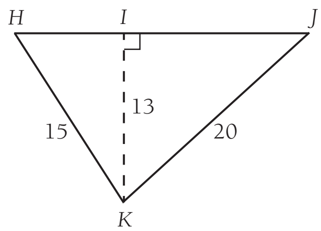
4 **a** Use $\triangle ACD$ to find x.
b Use $\triangle ABC$ to find y.



5 **a** Find x.
b Find y.
c Find a using the fact that $a = y - x$.



6 **a** Use $\triangle HIK$ to find $\angle HKI$.
b Use $\triangle IJK$ to find $\angle IKJ$.
c Find $\angle HKJ$.



Answers can be found in the Interactive Student CD.

Foundation Worksheet

10:02 | Literal Equations

Name: ______________________ **Class:** ____________

Examples

To solve a literal equation, rearrange the equation by using inverse operations.

1 Solve each equation for x.

a

$$y = 5x = 4$$
$$\therefore y + 4 = 5x$$
$$\frac{y+4}{5} = x$$
$$x = \frac{y+4}{5}$$

b

$$y = \frac{7}{5+x}$$
$$\therefore (5+x)y = 7$$
$$5y + xy = 7$$
$$xy = 7 - 5y$$
$$x = \frac{7-5y}{y}$$

c

$$\sqrt{2x-3} = y$$
$$\therefore 2x - 3 = y^2$$
$$2x = y^2 + 3$$
$$x = \frac{y^2+3}{2}$$

2 Check whether $x = 4$ can be a value for each formula.

a $y = \dfrac{3}{x-4}$

Substitute $x = 4$.

$y = \dfrac{3}{0}$

Not allowed since we can't divide by 0.

b $y = 3\sqrt{x-6}$

Substitute $x = 4$.

$y = 3\sqrt{-2}$

Not allowed since we can't have $\sqrt{\text{negative}}$.

Exercise

1 Solve for x.

a $y = 3x + 2$

b $y = \dfrac{1}{x+2}$

c $y = 6x + 11$

d $y = x^2 + 6$

e $y = \dfrac{x+5}{7}$

f $y = \sqrt{x+5}$

g $y = 7 - 4x$

h $y = \dfrac{8}{x-3}$

i $y = \sqrt{6x-4}$

j $y = \dfrac{2x-1}{4}$

k $y = \dfrac{x^2-2}{4}$

l $y = \dfrac{5}{2x-1}$

2 Can x be the value given for each formula?

a $x = 2$ if $y = \sqrt{x-3}$

b $x = 3$ if $y = \dfrac{1}{x+2}$

c $x = 5$ if $y = \dfrac{6}{5-x}$

d $x = 1$ if $y = \dfrac{3}{x^2-1}$

e $x = -2$ if $y = \sqrt{4-2x}$

f $x = 0$ if $y = \sqrt{4x}$

g $x = 6$ if $y = \dfrac{x-6}{x+6}$

h $x = -1$ if $y = \sqrt{x-1}$

Answers can be found in the Interactive Student CD.

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Foundation Worksheet

10:03 | Understanding Variables

Name: ______________________ **Class:** ____________

Examples

1 Find an expression for $4h + 11$ if h is replaced by the x expression given in each case.

a $7x$

$4h + 11 = 4(7x) + 11$
$= 28x + 11$

b $3x - 1$

$4h + 11 = 4(3x - 1) + 11$
$= 12x - 4 + 11$
$= 12x + 7$

c $x^2 - 4$

$4h + 11 = 4(x^2 - 4) + 11$
$= 4x^2 - 16 + 11$
$= 4x^2 - 5$

2 Solve each equation using the given change of variable.

a $x^4 - 9x^2 + 20 = 0$ if $X = x^2$

$\therefore X^2 - 9X + 20 = 0$
$(X - 4)(X - 5) = 0$
$X = 4$ or 5
$\therefore x^2 = 4$ or 5
$x = \pm 2$ or $\pm\sqrt{5}$

b $x - 9\sqrt{x} + 20 = 0$ if $X = \sqrt{x}$

$\therefore X^2 - 9X + 20 = 0$
$(X - 4)(X - 5) = 0$
$X = 4$ or 5
$\therefore \sqrt{x} = 4$ or 5
$x = 16$ or 25

Exercise

1 Find the new expression for $2a - 5$ if a is replaced by:

a $3x$ **b** $x + 1$ **c** $x^2 + 2$ **d** $7x - 2$.

2 Find an expression for $5b + 9$ if b is replaced by:

a $4x + 7$ **b** $3 - 2x$ **c** $x^2 - 4$ **d** $5x^2$.

3 Find the new expression for $(c + 5)(c - 2)$ if c is replaced by:

a $2x$ **b** $x + 1$ **c** $x - 7$ **d** x^2.

4 Solve each equation using the given change of variable.

a $x^4 - 10x^2 + 9 = 0$ if $X = x^2$

b $x - 8\sqrt{x} + 15 = 0$ if $X = \sqrt{x}$

c $x - 9\sqrt{x} + 14 = 0$ if $X = \sqrt{x}$

d $x^4 - 17x^2 + 16 = 0$ if $X = x^2$

e $x - 7\sqrt{x} + 10 = 0$ if $X = \sqrt{x}$

f $x^4 - 11x^2 + 10 = 0$ if $X = x^2$

g $x^4 - 29x^2 + 100 = 0$ if $X = x^2$

h $x - 9\sqrt{x} + 18 = 0$ if $X = \sqrt{x}$

i $x^4 - 12x^2 + 27 = 0$ if $X = x^2$

j $x - 10\sqrt{x} + 21 = 0$ if $X = \sqrt{x}$

Answers can be found in the Interactive Student CD.

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Foundation Worksheet

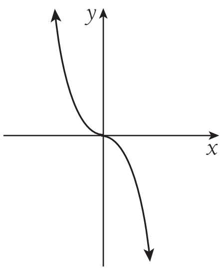
12:01 | Curves of the Form $y = ax^n$ and $y = ax^n + d$

Name: ____________________ **Class:** ____________

Exercise

1 For each equation, by examining the sign of the x term, and the value of the index, choose which of the curves A to D is the best sketch for the equation.

A



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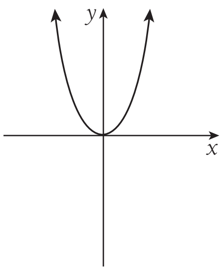
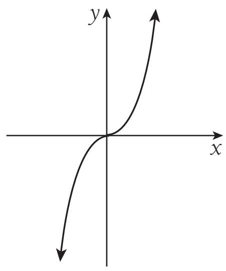
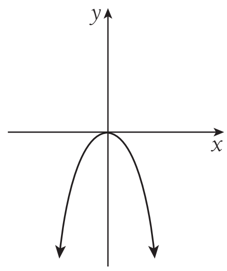


C



D



a $y = x^6$ **b** $y = -2x^4$ **c** $y = 3x^5$

d $y = -x^3$ **e** $y = 7x^2$ **f** $y = -4x^5$

g $y = 10x^7$ **h** $y = -x^8$ **i** $y = -x^{11}$

j $y = \frac{1}{5}x^{10}$ **k** $y = -\frac{x^2}{4}$ **l** $y = \frac{1}{3}x^9$

2 Sketch the following curves on separate number planes.

a $y = x^4 - 4$ **b** $y = 2 - 4x^5$ **c** $y = x^2 - 2$

d $y = x^3 + 3$ **e** $y = 2x^2 + 5$ **f** $y = 10 - x^6$

g $y = -2x^3 - 5$ **h** $y = 8x^{10} + 1$ **i** $y = -3x^4$

j $y = 6x^7 + 3$ **k** $y = -x^8 - 2$ **l** $y = -\frac{1}{3}x^7 - 1$

Fun Spot 12:01 | What do you call a snowman with a suntan?

fun spot

Match the letters of the graph shapes with the equations below.

A Circle **D** Cubic **E** Exponential

L Hyperbola **P** Line **U** Parabola

$x^2 + y^2 = 10$		$y = 7 - x$	$y = 5 - x^2$	$y = 7x^5$	$y = 8 - 5x^3$	$xy = 6$	$y = 4^x$	!

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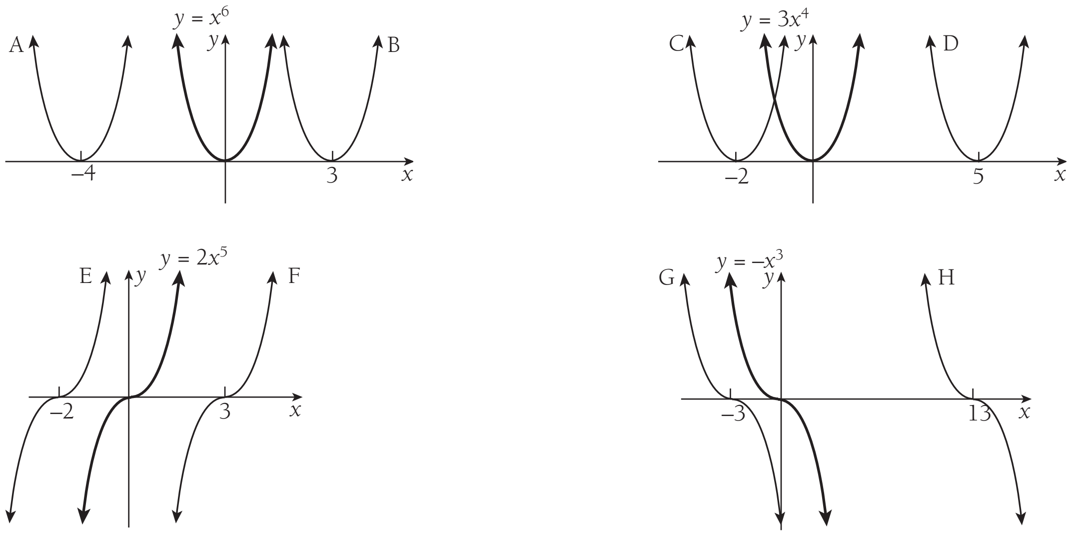
Foundation Worksheet

12:02 | Curves of the Form $y = ax^n$ and $y = a(x - r)^n$

Name: ____________________ **Class:** __________

Exercise

1 On each graph, the 'thick' curve has been translated horizontally to produce the other two curves. Find their equations, given the equation of the 'thick' curve.



2 Write down the equation of the curve obtained when $y = x^3$ is translated:
a 2 units to the left **b** 1 unit to the right.

3 Write down the equation of the curve obtained when $y = 3x^8$ is translated:
a 3 units to the left **b** 2 units to the right.

4 Write down the equation of the curve obtained when $y = -3x^5$ is translated:
a 1 unit to the left **b** 1 unit to the right.

5 Write down the equation of the curve obtained when $y = -x^4$ is translated:
a 2 units to the left **b** 5 units to the right.

6 Write down the equation of the curve obtained when $y = 3(x - 2)^6$ is translated:
a 3 units to the left **b** 3 units to the right.

Foundation Worksheet

12:03 | Curves of the Form $y = a(x - r)(x - s)(x - t)$

Name: ______________________ **Class:** ____________

Example

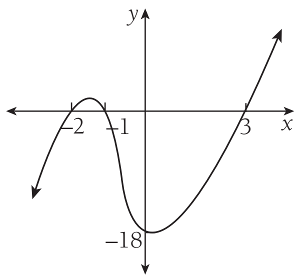
Sketch $y = 3(x + 2)(x + 1)(x - 3)$.

- x-intercepts ($y = 0$) are $x = -2, -1, 3$.
- Complete a sign analysis table for the y values.

x	-3	-2	$-1{\cdot}5$	-1	0	3	4
y	$-$	0	$+$	0	$-$	0	$+$

Curves is below x-axis for y negative.
Curve is above x-axis for y positive.

- y-intercept ($x = 0$) is $y = 3(2)(1)(-3) = -18$.



Exercise

1 Match the curves A to D with these equations.

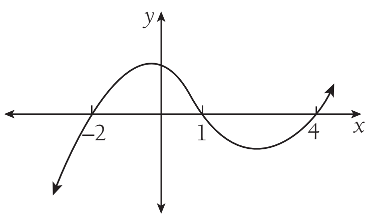
a $y = (x - 1)(x - 2)(x - 4)$

b $y = (x + 1)(x + 2)(x + 4)$

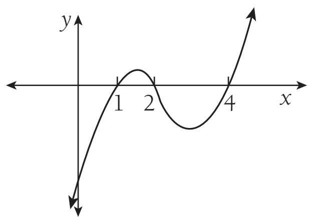
c $y = (x - 1)(x + 2)(x - 4)$

d $y = (x + 1)(x + 2)(x - 4)$

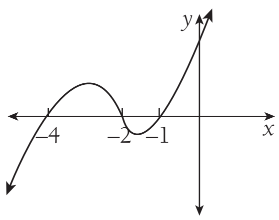
A



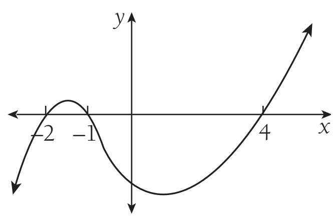
B



C



D



2 Sketch these curves on separate number planes.

a $y = (x - 3)(x + 2)(x - 1)$

b $y = 2(x - 3)(x + 2)(x - 1)$

c $y = (x - 1)(x + 1)(x + 2)$

d $y = x(x - 1)(x - 3)$

e $y = 2x(x - 3)(x + 3)$

f $y = (x + 3)(x + 2)(x - 1)$

g $y = 2(x - 5)(x - 1)(x + 1)$

h $y = 5(x + 2)(x - 2)(x - 4)$

Answers can be found in the Interactive Student CD.

Foundation Worksheets

Answers

2:01 Quadratic Equations

1 **a** $x(x-3)$ **b** $(x+1)(x+2)$ **c** $(y+5)(y+1)$ **d** $(d-3)(d+3)$
e $3k(k+2)$ **f** $(t-3)(t+2)$ **g** $(a+1)(a+6)$ **h** $(x+6)(x-4)$
i $y(y+7)$

2 **a** $x=0, 4$ **b** $x=1, -2$ **c** $m=-8, 4$ **d** $t=0, -6$
e $y=-3, -4$ **f** $a=4, 5$ **g** $c=0, 14$ **h** $x=6, -1$
i $h=-1, 1$ **j** $y=4, -4$ **k** $n=6, 8$ **l** $x=-10, 11$

3 **a** $x=0, 3$ **b** $x=-1, -2$ **c** $y=-1, -5$ **d** $d=3, -3$
e $k=0, -2$ **f** $t=3, -2$ **g** $f=4, -3$ **h** $x=3, 4$
i $m=5, -5$ **j** $a=-4, 1$ **k** $y=4, -2$ **l** $n=-8, -2$

2:03 The Quadratic Formula

1 **a** $-2, -1$ **b** $0\cdot35, -2\cdot85$ **c** $3, -1$ **d** $-0\cdot69, -1\cdot81$
e $5\cdot45, 0\cdot55$ **f** $-4, 2$ **g** $\frac{1}{2}, -2$ **h** $1, 6$
i $0\cdot43, -0\cdot77$ **j** $-\frac{2}{5}, -1$ **k** $-1\cdot64, 0\cdot24$ **l** -1

2 **a** $x=-0\cdot44, -4\cdot56$ **b** $x=3\cdot30, -0\cdot30$ **c** $x=9, -2$ **d** $x=-\frac{1}{2}, -2$
e $x=-\frac{2}{3}, 1$ **f** $x=\frac{1}{3}, 2$ **g** $x=0\cdot62, -1\cdot62$ **h** $x=-11, 2$
i $x=-0\cdot41, -6\cdot09$ **j** $x=6, -2$ **k** $x=4\cdot45, -0\cdot45$ **l** $x=0\cdot59, 3\cdot41$
m $x=2\cdot39, -1\cdot89$ **n** $x=0\cdot57, -1\cdot32$ **o** $x=3$

3:01 Probability Review

1 **a** $\frac{1}{10}$ **b** $\frac{3}{5}$ **c** $\frac{2}{5}$

2 **a** $\frac{1}{30}$ **b** $\frac{1}{15}$ **c** $\frac{1}{10}$

3 **a** $\frac{1}{2}$ **b** $\frac{3}{10}$ **c** $\frac{9}{20}$

4 **a** $\frac{2}{9}$ **b** $\frac{1}{3}$ **c** $\frac{2}{3}$ **d** $\frac{4}{9}$

5 **a** $\frac{1}{3}$ **b** $\frac{5}{12}$ **c** $\frac{3}{4}$ **d** $\frac{2}{15}$ **e** $\frac{13}{15}$ **f** $\frac{7}{10}$

6 **a** $\frac{5}{26}$ **b** $\frac{21}{26}$ **c** $\frac{2}{13}$ **d** $\frac{5}{26}$ **e** $\frac{5}{26}$

7 **a** $\frac{5}{12}$ **b** $\frac{7}{12}$ **c** $\frac{1}{24}$

3:02 Organising Outcomes of Compound Events

1 **a** BBB, BBG, BGB, BGG, GBB, GBG, GGB, GGG
b AB, AC, AD, AE, BC, BD, BE, CD, CE, DE
c H1, H2, H3, H4, H5, H6, T1, T2, T3, T4, T5, T6
d R1B1, R1B2, R1B3, R2B1, R2B2, R2B3, R3B1, R3B2, R3B3, R4B1, R4B2, R4B3
e 11, 12, 13, 14, 21, 22, 23, 24, 31, 32, 33, 34, 41, 42, 43, 44
f ss, sc, sl, vs, vc, vl, c-cs, c-cc, c-cl
g VB, VR, JoB, JoR, SB, SR, JeB, JeR
h peg, ped, pug, pud, teg, ted, tug, tud, weg, wed, wug, wud
i JVX, JXV, VJX, VXJ, XJV, XVJ

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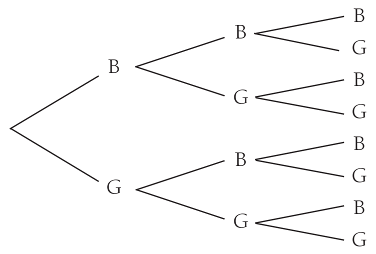
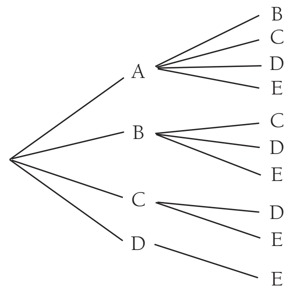
2 a

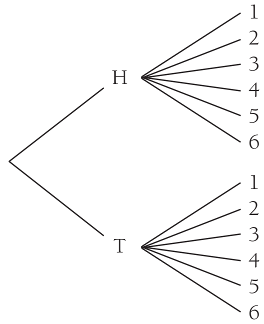


b



Note: AB, BA, etc. yield the same two cards.

c



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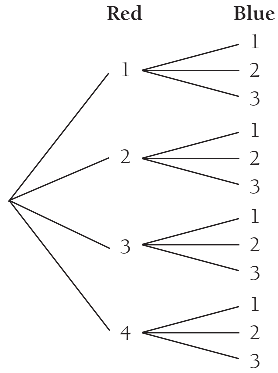

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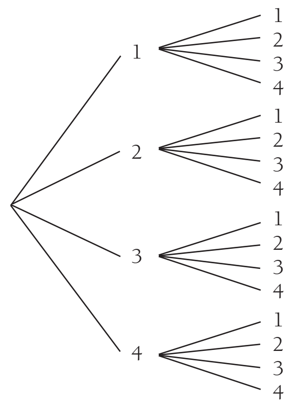

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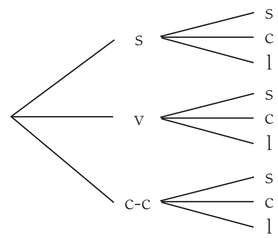

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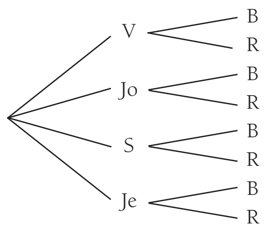

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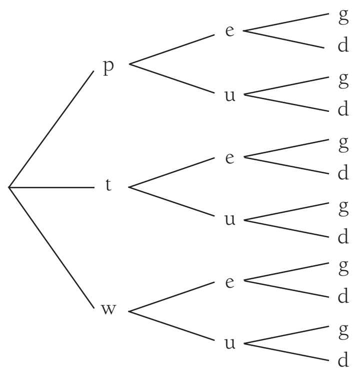
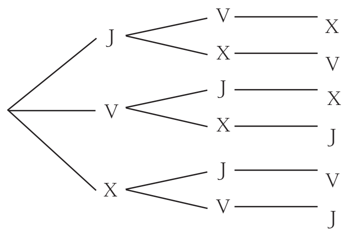


i



4:02 Simple Interest

1	**a** $9	**b** $7.20	**c** $2.40	**d** $16.50	**e** $434	**f** $360
	g $56	**h** $135	**i** $19.50	**j** $66	**k** $450	**l** $225
2	**a** $100	**b** $400	**c** $72	**d** $240	**e** $560	**f** $330
	g $300	**h** $132	**i** $105	**j** $168	**k** $62.40	**l** $288

4:04 Compound Interest

1 **a** $10 700 **b** $11 449 **c** $12 250.43

2 **a** $700 **b** $1449 **c** $2250.43

3 **a** $4200 **b** $4410 **c** $4630.50

4 **a** $200 **b** $410 **c** $630.50

5 **a** $2160 **b** $2332.80 **c** $2519.42 **d** $2720.97

6 **a** $160 **b** $332.80 **c** $519.42 **d** $720.97

7 Value = $5849.29, interest = $849.29

8 Value = $35 816.95, interest = $15 816.95

9 Value = $14 599.83, interest = $2599.83

10 $3249.79

11 $2901.45

12 Value = $1181.96, interest = $381.96

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4:06 Compound Interest Formula

1 **a** \$15 036.30 **b** \$4862.03 **c** \$8857.81 **d** \$11 054.61 **e** \$17 271.40 **f** \$2052.85

2 **a** \$5036.30 **b** \$862.03 **c** \$3857.81 **d** \$5054.61 **e** \$9271.40 **f** \$552.85

3 **a** \$14 701.84 **b** \$6522.98 **c** \$1968.30 **d** \$1885.75 **e** \$13 107.20 **f** \$868.78

5:02 The Parabola $y = ax^2 + bx + c$

1 **a** **i** 3 **ii** 1, 3 **iii** $x = 2$ **iv** (2, –1) **b** **i** –8 **ii** –2, 4 **iii** $x = 1$ **iv** (1, –9)

c **i** 12 **ii** –6, 2 **iii** $x = -2$ **iv** (–2, 16) **d** **i** –3 **ii** –3, 3 **iii** $x = 0$ **iv** (0, –3)

e **i** 0 **ii** 0, 6 **iii** $x = 3$ **iv** (3, 9) **f** **i** 16 **ii** 2, 8 **iii** $x = 5$ **iv** (5, –9)

2 **a** **i** –2 **ii** –2, 1 **iii** $x = -\frac{1}{2}$ **iv** $(-\frac{1}{2}, -2\frac{1}{4})$ **b** **i** 0 **ii** 0, 8 **iii** $x = 4$ **iv** (4, –16)

c **i** –8 **ii** –4, 2 **iii** $x = -1$ **iv** (–1, –9) **d** **i** 24 **ii** –6, 4 **iii** $x = -1$ **iv** (–1, 25)

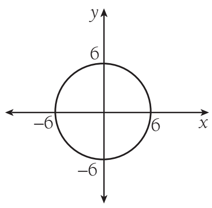
e **i** 15 **ii** 3, 5 **iii** $x = 4$ **iv** (4, –1) **f** **i** 0 **ii** 0, –4 **iii** $x = -2$ **iv** (–2, –4)

5:05 The Circle

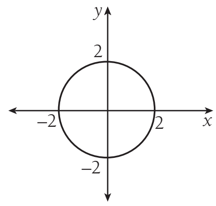
1 **a** **i** 2 **ii** $x^2 + y^2 = 4$ **b** **i** 1 **ii** $x^2 + y^2 = 1$ **c** **i** 12 **ii** $x^2 + y^2 = 144$

d **i** 5 **ii** $x^2 + y^2 = 25$ **e** **i** $\sqrt{3}$ **ii** $x^2 + y^2 = 3$ **f** **i** 2·5 **ii** $x^2 + y^2 = 6{\cdot}25$

2 **a**



b



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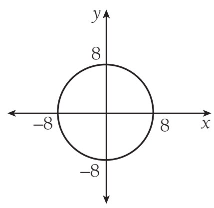

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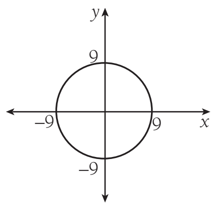

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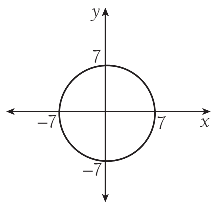

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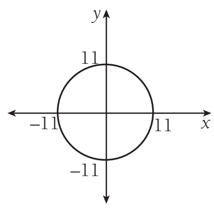

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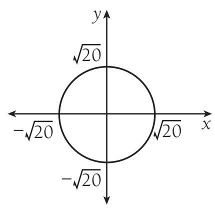

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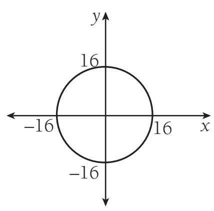

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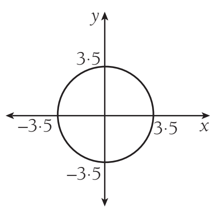

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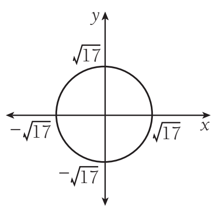

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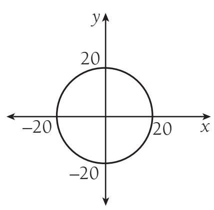

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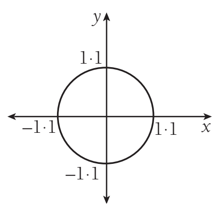

5:08 Coordinate Geometry

1 **a** Length $AB = \sqrt{20}$, length $AC = 4$, length $BC = \sqrt{20}$

$\therefore \triangle ABC$ is isosceles since two sides are equal.

b Length $JK = \sqrt{13}$, length $KL = \sqrt{13}$, length $JL = \sqrt{26}$

$\therefore JK^2 + KL^2 = JL^2$

$\therefore \triangle JKL$ is right-angled by Pythagoras' Theorem.

c Length $XY = 5$, length $YZ = 10$, length $XZ = \sqrt{125}$

$\therefore$ Perimeter $= 15 + \sqrt{125}$

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2 a Gradient $AB = 0$, gradient $BC = -2$, gradient $CD = 0$, gradient $DA = -2$

$\therefore AB \parallel CD$ and $BC \parallel DA$ since they have the same gradient.

$\therefore ABCD$ is a parallelogram since opposite sides are parallel.

b Midpoint diagonal $AC = (1, 0)$, midpoint diagonal $BD = (1, 0)$

$\therefore$ Diagonals bisect since midpoints are the same.

c i $G = (4, 10)$ $H = (1, 9)$

ii Gradient $EF = \frac{1}{3}$, gradient $GH = \frac{1}{3}$

$\therefore EF \parallel GH$ since they have the same gradient.

3 a Length $AB = \sqrt{13}$, length $BC = \sqrt{13}$, length $CD = \sqrt{13}$, length $DA = \sqrt{13}$

$\therefore ABCD$ is a rhombus since all sides are equal.

b Length WX = length XY = length YZ = length $ZW = 5$

Gradient $WX = -\frac{3}{4}$, gradient $XY = \frac{4}{3}$

$\therefore WX \perp XY$ since the product of their gradients is -1.

$\therefore WXYZ$ is a square since all sides are equal and two sides are perpendicular.

c Gradient diagonal $WY = -7$, gradient diagonal $XZ = \frac{1}{7}$

$\therefore$ Gradients are negative reciprocals.

$\therefore$ Diagonals meet at right angles.

6:01 Surface Area Review

1 a 312 cm^2 **b** 220 m^2 **c** 1440 m^2 **d** 162 cm^2 **e** 101·18 m^2 **f** 318·78 cm^2

2 a 120 m^2 **b** 449·7 cm^2 **c** 219·9 m^2 **d** 782·8 cm^2 **e** 510·2 m^2 **f** 368·2 cm^2

3 a 414·7 cm^2 **b** 1570·8 m^2 **c** 1407·4 m^2 **d** 11 158·9 cm^2 **e** 10 053·1 m^2 **f** 1190·8 cm^2

6:02 Surface Area of a Pyramid

1 a 85 $units^2$ **b** 1416 $units^2$ **c** 340 $units^2$

2 a 156 $units^2$ **b** 825 $units^2$ **c** 186·7 $units^2$ **d** 1400 $units^2$ **e** 1296 $units^2$ **f** 46·8 $units^2$

6:03 Surface Area of a Cone

1 Answers are in order: radius, height, slant height.

a 6, 8, 10 units **b** 10, 7·5, 12·5 units **c** 16, 30, 34 units

2 a i 188·5 $units^2$ **ii** 301·6 $units^2$ **b i** 392·7 $units^2$ **ii** 706·9 $units^2$

c i 1709·0 $units^2$ **ii** 2513·3 $units^2$

3 a 5 cm **b** 17 cm **c** 25 cm **d** 30 cm

4 a i 25 m **ii** 703·7 m^2 **b i** 52·5 cm **ii** 11 623·9 cm^2

c i 61 m **ii** 2488·1 m^2 **d i** 5·5 cm **ii** 91·2 cm^2

e i 13 cm **ii** 282·7 cm^2 **f i** 5·2 cm **ii** 45·2 cm^2

6:05 Volume of a Pyramid

1 a 48 $units^3$ **b** 100 $units^3$ **c** 330 $units^3$ **d** 200 $units^3$

e 75 $units^3$ **f** 320·16 $units^3$

2 a 18 $units^2$, 8, 48 $units^3$ **b** 9 $units^2$, 5, 15 $units^3$

c 240 $units^2$, 10, 800 $units^3$ **d** 70 $units^2$, 15, 350 $units^3$

e 51·84 $units^2$, 7·2, 124·416 $units^3$ **f** 300 $units^2$, 25, 2500 $units^3$

7:02 Inter-quartile Range

1 a i 4·5 **ii** 2, 6 **iii** 4 **b i** 4·5 **ii** 2·5, 7·5 **iii** 5

c i 6·5 **ii** 4, 10·5 **iii** 6·5 **d i** 14·5 **ii** 11, 19 **iii** 8

e i 10 **ii** 7·5, 14·5 **iii** 7 **f i** 20 **ii** 15, 23·5 **iii** 8·5

2 a 12 **b** 14 **c** 5·5 **d** 4 **e** 4·5 **f** 3·5

g 3·5 **h** 4 **i** 13 **j** 145

7:04 Standard Deviation

1 **a** **i** B **ii** C **b** **i** B **ii** A **c** **i** A **ii** C

2 **a** 1·4 **b** 0·9 **c** 1·8 **d** 2·4 **e** 5·0 **f** 2·4 **g** 20·4

8:03 Finding Unknown Sides in Similar Triangles

1 **a** *BC*, 1·5, 4·65 **b** *KL*, 1·7, 6·8 **c** side 8, 3, 9 **d** side 4, 1·75, 10·5

2 **a** 12·5 **b** 25·5 **c** 17·5 **d** 11·2 **e** 17·5 **f** 10

8:04 Similar Triangles Proofs

Answers given fill in the spaces, line by line.

1 $\angle E$; $\frac{5}{3}$; $\frac{20}{12}$, $\frac{5}{3}$

2 $\angle E$, given; $\angle F$, given; $\angle D$; AAA test

3 2; $\frac{NL}{KL}$, 2; SSS test

4 $\frac{MO}{PR}$, $\frac{20}{12}$; $\frac{30}{18}$, $\frac{5}{3}$; SSS test

5 90°, given; $\frac{1}{3}$; $\frac{12}{36}$, $\frac{1}{3}$; RHS test

6 common; 1·5; $\frac{18}{12}$, 1·5; SAS test

9:02 Trig Ratios of Obtuse Angles

1 **a** 0·643, 0·643 **b** 0·866, 0·940 **c** 0·985, 0·985 **d** 0·996, 0·996
e 0·423, 0·087 **f** 0·866, 0·866 **g** 0·174, 0·174 **h** 0·707, 0·707
i 0·914, 0·829

2 **a** Ratios are equal for a, c, d, f, g, h.
b The angles are supplementary (add to 180°).

3 **a** 60° **b** 45° **c** 70° **d** 30° **e** 85°
f 50° **g** 15° **h** 73°

4 **a** 100° **b** 160° **c** 140° **d** 170° **e** 105°
f 144° **g** 119° **h** 126°

5 **a** 53°, 127° **b** 120° **c** 117° **d** 96° **e** 145°
f 56°, 124° **g** 124° **h** 114° **i** 12°, 168°

9:03 The Sine Rule

1 **a** 48 **b** 3·5 **c** 61° **d** 16·3 **e** 13·7 **f** 15°

2 **a** 8·0 **b** 4·8 **c** 7·1 **d** 9·2 **e** 14·3 **f** 139·4

3 **a** $\frac{a}{\sin 75°} = \frac{8}{\sin 60°}$ **b** $\frac{\sin\theta}{12} = \frac{\sin 50°}{10}$ **c** $\frac{a}{\sin 23°} = \frac{7\cdot 3}{\sin 35°}$
d $\frac{\sin\theta}{8} = \frac{\sin 80°}{11}$ **e** $\frac{a}{\sin 55°} = \frac{39}{\sin 40°}$ **f** $\frac{\sin\theta}{16\cdot 8} = \frac{\sin 52°}{20\cdot 4}$
g $\frac{a}{\sin 72°} = \frac{6\cdot 7}{\sin 34°}$ **h** $\frac{a}{\sin 55°} = \frac{11\cdot 5}{\sin 65°}$ **i** $\frac{a}{\sin 35°} = \frac{28}{\sin 70°}$

4 **a** 8·9 **b** 67° **c** 5·0 **d** 46° **e** 49·7 **f** 40°
e 11·4 **e** 10·4 **e** 17·1

9:04 Sine Rule—the Ambiguous Case

1 **a** **i** 53° **ii** 127° **b** **i** 16° **ii** 164°
c **i** 7° **ii** 173° **d** **i** 41° **ii** 139°

2 **a** 70°, 110° **b** 30° only **c** 60° only **d** 55°, 125°

3 **a** 64°, 116° **b** 74°, 106° **c** 55°, 125° **d** 63°, 117° **e** 57°, 123° **f** 84°, 96°

9:05 The Cosine Rule

1 **a** 11·1 **b** 6·5 **c** 32·9 **d** 42·3

2 **a** 60° **b** 120° **c** 45° **d** 135° **e** 58° **f** 122°
g 95° **h** 131° **i** 102°

3 **a** 67° 23′ **b** 26° 59′ **c** 11° 30′ **d** 55° 46′

4 **a** $a^2 = 10^2 + 6^2 - 2 \times 10 \times 6 \cos 60°$ **b** $a^2 = 13^2 + 15{\cdot}4^2 - 2 \times 13 \times 15{\cdot}4 \cos 40°$

c $x^2 = 12^2 + 9^2 - 2 \times 12 \times 9 \cos 75°$ **d** $\cos\theta = \dfrac{20{\cdot}4^2 + 16{\cdot}8^2 - 33{\cdot}1^2}{2 \times 20{\cdot}4 \times 16{\cdot}8}$

e $\cos\theta = \dfrac{2{\cdot}5^2 + 3^2 - 3{\cdot}5^2}{2 \times 2{\cdot}5 \times 3}$ **f** $\cos\theta = \dfrac{45^2 + 73^2 - 89^2}{2 \times 45 \times 73}$

5 **a** 8·7 **b** 10·0 **c** 13·0 **d** 125° 25′ **e** 78° 28′ **f** 94° 57′

9:07 Sine Rule or Cosine Rule?

1 **a** SSA, sine rule **b** SSS, cosine rule **c** AAS, sine rule **d** AAS, sine rule
e AAS, sine rule **f** SAS, cosine rule **g** SAS, cosine rule **h** SSS, cosine rule
i SSA, sine rule **j** SAS, cosine rule **k** SSA, sine rule **l** SSA, sine rule

2 **a** 39°40′ **b** 93°49′ **c** 14·5 **d** 58·4 **e** 10·0 **f** 9·0
g 20·0 **h** 97°26′ **i** 59°48′ **j** 180·6 **k** 31°51′ **l** 40°0′

9:08 Problems with More than One Triangle

1 **a** 7·3 **b** 12·4

2 **a** 13·9 **b** 8·0 **c** 21·9

3 **a** 14·2 **b** 50°

4 **a** 5·4 **b** 14·8

5 **a** 11·5 **b** 28·6 **c** 17·1

6 **a** 30° **b** 49° **c** 79°

10:02 Literal Equations

1 **a** $x = \dfrac{y-2}{3}$ **b** $x = \dfrac{1-2y}{y}$ **c** $x = \dfrac{y-11}{6}$ **d** $x = \pm\sqrt{y-6}$

e $x = 7y - 5$ **f** $x = y^2 - 5$ **g** $x = \dfrac{7-y}{4}$ **h** $x = \dfrac{3y+8}{y}$

i $x = \dfrac{y^2+4}{6}$ **j** $x = \dfrac{4y+1}{2}$ **k** $x = \pm\sqrt{4y+2}$ **l** $x = \dfrac{y+5}{2y}$

2 **a** no **b** yes **c** no **d** no **e** yes
f yes **g** yes **h** yes

10:03 Understanding Variables

1 **a** $6x - 5$ **b** $2x - 3$ **c** $2x^2 - 1$ **d** $14x - 9$

2 **a** $20x + 44$ **b** $24 - 10x$ **c** $5x^2 - 11$ **d** $25x^2 + 9$

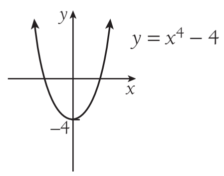
3 **a** $4x^2 + 6x - 10$ **b** $x^2 + 5x - 6$ **c** $x^2 - 11x + 18$ **d** $x^4 + 3x^2 - 10$

4 **a** ±1, ±3 **b** 9, 25 **c** 4, 49 **d** ±4, ±1
e 4, 25 **f** $\pm 1, \pm\sqrt{10}$ **g** ±2, ±5 **h** 9, 36
i $\pm\sqrt{3}, \pm 3$ **j** 49, 9

12:01 Curves of the Form $y = ax^n$ and $y = ax^n + d$

1 **a** B **b** D **c** C **d** A **e** B **f** A
g C **h** D **i** A **j** B **k** D **l** C

2 a



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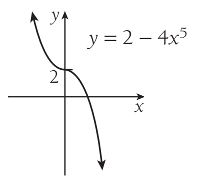

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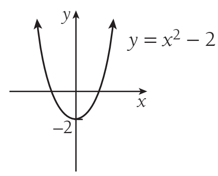

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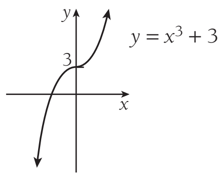

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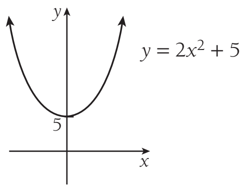

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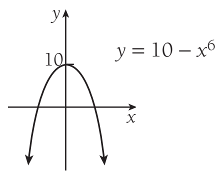

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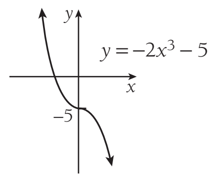

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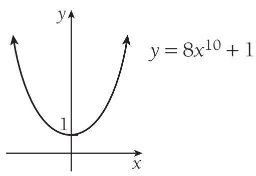

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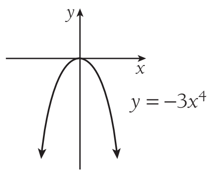

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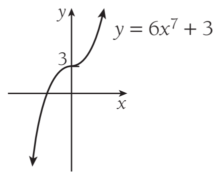

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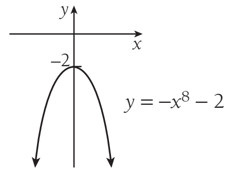

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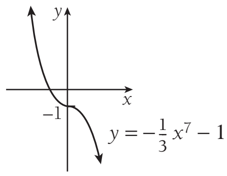


12:02 Curves of the Form $y = ax^n$ and $y = a(x - r)^n$

1 a $y = (x + 4)^6$ **b** $y = (x - 3)^6$ **c** $y = 3(x + 2)^4$ **d** $y = 3(x - 5)^4$
e $y = 2(x + 2)^5$ **f** $y = 2(x - 3)^5$ **g** $y = -(x + 3)^3$ **h** $y = -(x - 13)^3$

2 a $y = (x + 2)^3$ **b** $y = (x - 1)^3$

3 a $y = 3(x + 3)^8$ **b** $y = 3(x - 2)^8$

4 a $y = -3(x + 1)^5$ **b** $y = -3(x - 1)^5$

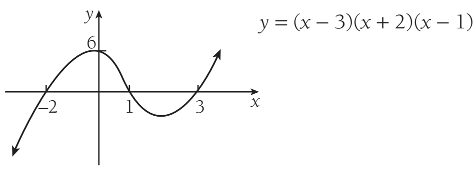
5 a $y = -(x + 2)^4$ **b** $y = -(x - 5)^4$

6 a $y = 3(x + 1)^6$ **b** $y = 3(x - 5)^6$

12:03 Curves of the Form $y = a(x - r)(x - s)(x - t)$

1 a B **b** C **c** A **d** D

2 a



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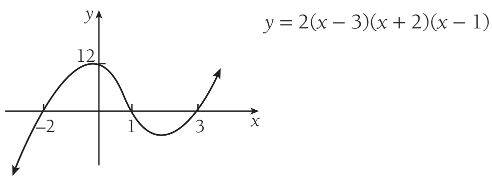

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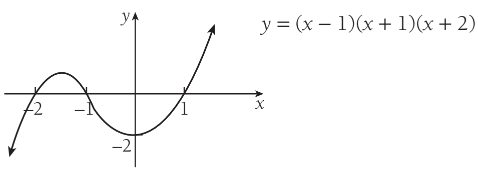

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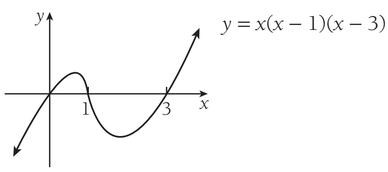

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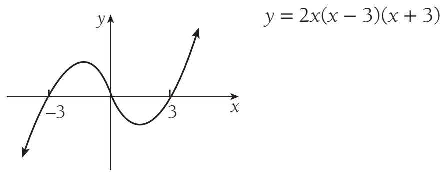

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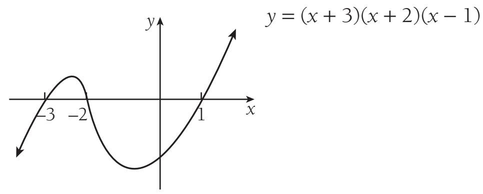

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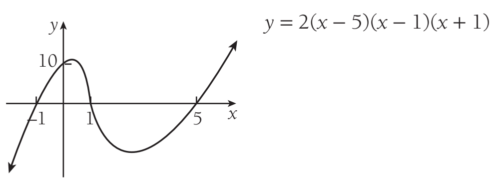

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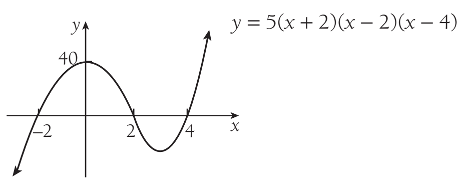

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